DERIVED WITT GROUPS OF A SCHEME

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Abstract. The Witt group of a triangulated category with duality is the quotient of the monoid of symmetric spaces by the submonoid of neutral forms. Neutral forms are defined in a traditional way, using lagrangians. To any noetherian scheme X is associated a derived category with duality, denoted by $\mathrm{K}(X)$. The Witt group of $\mathrm{K}(X)$ will be called the (derived) Witt group of X.

There is an isomorphism between the usual Witt group of a ring in which 2 is a unit and its derived Witt group. This approach allows us to compute the kernel of $\mathrm{W}(A) \to \mathrm{W}(Q)$, where A is a domain and Q its field of fractions. This kernel turns out to be the Witt group of some suitable triangulated category with duality.

The point of view of derived categories seems particularly useful for localization. Let U be an open subscheme of a regular scheme X. It is not hard to establish that $\mathrm{K}(U)$ is a localization of $\mathrm{K}(X)$ with respect to a suitable multiplicative system. Denote by J the full subcategory of $\mathrm{K}(X)$ on the objects vanishing in $\mathrm{K}(U)$. We construct a connecting homomorphism from the Witt group of $\mathrm{K}(U)$ to some Witt group of J, associating skew-symmetric forms to symmetric ones. We prove that the kernel of this homomorphism is precisely the part of $\mathrm{W}(\mathrm{K}(U))$ coming from X. Using these results we obtain a very simple proof of purity in dimension 3.

Introduction.

In the study of the usual Witt group (Knebusch's classical definition) of a regular ring, projective resolutions are very useful. In several cases, it would be very convenient to have forms on complexes and not only on projective modules. These considerations are not new and attempts to define analogues of the Witt group on these more general objects have been made for a long time. We could mention for instance A. Ranicki in [5], who defines, for any integer n, the Witt group of complexes of length n. It is then very easy to relate these groups to classical ones when n is zero!

We proceed in the opposite way. Allowing the complexes to be as wide as they want (bounded anyway), we gain in flexibility. Unfortunately, it becomes rather tricky to prove that this new group has something in common with the classical one. The purpose of this note is to establish the definition of these a derived" Witt groups and to relate them to the usual notions of quadratic spaces over schemes.

To define these Witt groups, we consider the obvious duality on complexes of projective modules. Actually, it is convenient to deal with a more general concept: any triangulated category with a suitable duality. In a recent paper, B. Youssin gives a definition of Witt groups of derived categories [7]. His point of view is slightly different from ours. In fact, the basic question is: which symmetric forms have to be considered trivial? Youssin uses for this the notion of cobordism. Our definition uses the good old lagrangians of our grand-mothers. This point of view has the advantage to be very conceptual: trivial forms are those with a big" isotropic part. In fact, Youssin's approach has more topological motivations and he doesn't study the case of schemes. His reduction to the heart of a triangulated category with t-structure ([7], thm 7.4), using the functor H^0, does not apply to complexes of projective objects (whose homology is not projective). Compare with our homomorphism $\Omega : \mathrm{W}_{\mathrm{der}} \to \mathrm{W}_{\mathrm{us}}$ in section 4 to understand this difficulty.

Our approach yields the framework of a more flexible theory, as it could be observed with the problem of localizing the Witt group of a scheme X to an open subscheme U. Actually, we can construct an obstruction for a element of $\mathrm{W}(U)$ to come from $\mathrm{W}(X)$. This obstruction lives in the Witt group $\mathrm{W}_1^{-1}(J)$ of a suitable triangulated category with duality J; the superscript -1 indicates skew-symmetry; the subscript 1 refers to a shifted duality. In other words, to each symmetric form φ on U, we can associate

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a skew-symmetric form in J, which is neutral if and only if φ is Witt-equivalent to the restriction of a form on X.

We adopt the notations and sign conventions of [6], where the reader may find all the material about triangulated categories necessary to understand this paper.

1. Duality, skew-duality, Witt groups.

1.1. Definition. Let K_1 and K_2 be triangulated categories and let T_i be the translation automorphism of K_i, $i = 1, 2$. Let $\delta = \pm 1$. An additive contravariant functor

$$F : K_1 \to K_2$$

is δ-*exact* if $F \circ T_1 = {T_2}^{-1} \circ F$ and if for any exact triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ the following triangle is exact:

$$\begin{array}{ccc} & F(A) & \\ {\scriptstyle \delta \cdot T_2\left(F(w)\right)} \swarrow & & \nwarrow {\scriptstyle F(u)} \\ F(C) & \xrightarrow[F(v)]{} & F(B) \end{array} \quad .$$

1.2. Remark. By *exact* (resp. *skew-exact*) we mean 1-exact (resp. (-1)-exact). Observe that F is skew-exact if and only if $T_2 \circ F$ (or $F \circ T_1$) is exact.

1.3. Remark. We say that a covariant functor commuting with translation is *exact* if it maps exact triangles to exact triangles and *skew-exact* if it maps exact triangles to skew-exact triangles (i.e. triangles with one sign changed). In fact, definition 1.1 endows the opposite category with a triangulation.

1.4. Definition. Consider a triangulated category K and denote by T its translation automorphism. Let $\delta = \pm 1$. A δ-*duality* is a δ-exact contravariant functor $\# : K \longrightarrow K$ such that there exists an isomorphism

$$\mathrm{can} : \mathrm{Id} \xrightarrow{\sim} \# \circ \#$$

satisfying the following conditions:

$$\mathrm{can}_{T(X)} = T(\mathrm{can}_X) \qquad \text{and} \qquad (\mathrm{can}_X)^{\#} \circ \mathrm{can}_{X^{\#}} = \mathrm{Id}_{X^{\#}}$$

for all $X \in K$. As before, a (-1)-duality will be called a *skew-duality*.

1.5. Remark. It is not hard to see that if $\#$ is a duality then $T \circ \#$ is a skew-duality, using the fact that $\# \circ T = T^{-1} \circ \#$. More generally, if $\#$ is a duality then $T^n \circ \#$ is a $(-1)^n$-duality for all $n \in \mathbb{Z}$. This property of ± 1-duality should not be confused with symmetry and skew-symmetry. These concepts have nothing in common *a priori* ! ± 1-duality indicates the behaviour of $\#$ with respect to exact triangles and we will see that a usual" duality (e.g. on a ring) gives rise to a (+1)-duality.

1.6. Definition. Let K be a triangulated category and $\# : K \to K$ a ± 1-duality. A *symmetric space* is a pair (X, φ) where X is an object of K and $\varphi : X \xrightarrow{\sim} X^{\#}$ is an isomorphism such that

$$\varphi^{\#} \circ \mathrm{can}_X = \varphi.$$

The morphism φ is usually called the *(bilinear symmetric) form* on X by pure nostalgia.

1.7. Classical definitions. Let (X_i, φ_i) be a symmetric space for $i = 1, 2$ and let $f : X_1 \xrightarrow{\sim} X_2$ be an isomorphism. We say that f is an *isometry* if $f^{\#} \varphi_2 f = \varphi_1$ and we denote it by $(X_1, \varphi_1) \simeq (X_2, \varphi_2)$.

It is easy to check that $\left(X_1 \oplus X_2, \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\right)$ is a symmetric space called the *orthogonal sum* of the former ones and denoted by $(X_1, \varphi_1) \perp (X_2, \varphi_2)$. Orthogonal sum is compatible with isometries.

1.8. Definitions. We will call a triple $(K, \#, \mathrm{can})$ a *triangulated category with duality* if the class of isometry classes of symmetric spaces in K is a set. Denote this set by $\mathrm{MW}(K, \#, \mathrm{can})$ or simply by $\mathrm{MW}(K)$. This set is given a structure of abelian monoid with $\perp$. We call it the *Witt monoid* of K.

1.9. Definition. Let $\#$ be a δ-duality on K. Let (X, φ) be a symmetric space. A pair (L, α), where L is an object of K and $\alpha : L \to X$ is a morphism, is called a *sublagrangian* of (X, φ) if $\alpha^{\#} \varphi\, \alpha = 0$. A triple (L, α, w) is called a *lagrangian* if the following triangle is exact :

$$T^{-1}(L^{\#}) \xrightarrow{w} L \xrightarrow{\alpha} X \xrightarrow{\alpha^{\#}\varphi} L^{\#}$$

and if w is δ-symmetric", i.e.

$$T^{-1}(w^{\#}) = \delta \cdot w.$$

1.10. Explanations.

Since $T^{-1}(w^{\#}) : T^{-1}(L^{\#}) \to T^{-1}\big(T^{-1}(L^{\#})^{\#}\big) = L^{\#\#}$, the identity $T^{-1}(w^{\#}) = \delta \cdot w$ should be understood as $T^{-1}(w^{\#}) = \mathrm{can}_L \circ \delta \cdot w$. In fact, this condition can be expressed by saying that there exists a commutative diagram in which the first line is an exact triangle containing α and the second line is the dual of the first one :

$$\begin{array}{ccccccc}
T^{-1}(L^{\#}) & \xrightarrow{w} & L & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & L^{\#} \\
\| & & \| \mathrm{can}_L & & \downarrow \varphi & & \| \\
T^{-1}(L^{\#}) & \xrightarrow[\delta \cdot T^{-1}(w^{\#})]{} & L^{\#\#} & \xrightarrow[\beta^{\#}]{} & X^{\#} & \xrightarrow[\alpha^{\#}]{} & L^{\#}
\end{array} .$$

Heuristically, a lagrangian is first of all a sublagrangian, which means that the form induced on L via α is zero. But moreover, in $\mathrm{K}_0(K)$ (see for instance [1]), we have that $[X] = [L] + [L^{\#}]$. The classical notion of lagrangian finds here its natural generalization. We want to stress the fact that α needs not to be split in this definition !

1.11. Definition. We say that a symmetric space possessing a lagrangian is *neutral*.

1.12. Proposition. Let (X, φ) be a symmetric space. Then $(X, \varphi) \perp (X, -\varphi)$ is neutral.

1.13. Proof : Denote by $\Delta : X \to X \oplus X$ the diagonal. Check that $(X, \Delta, 0)$ is a lagrangian of $(X, \varphi) \perp (X, -\varphi)$. ♯

1.14. Definition. Clearly the orthogonal sum of two neutral symmetric spaces is neutral again and neutrality is preserved by isometry. Denote by $\mathrm{NW}(K, \#, \mathrm{can})$ the submonoid of $\mathrm{MW}(K, \#, \mathrm{can})$ consisting of classes of neutral spaces. Denote by

$$\mathrm{W}(K, \#, \mathrm{can}) = \frac{\mathrm{MW}(K, \#, \mathrm{can})}{\mathrm{NW}(K, \#, \mathrm{can})}$$

the quotient of these two monoids. This is the *Witt group* of K. We denote by $[X, \varphi]$ the class of (X, φ).

1.15. Remark. Recall that if $H \subset G$ is a submonoid of an abelian monoid G, we can define a relation on G by setting $x \sim x'$ if there exist $h, h' \in H$ such that $x + h = x' + h'$. The set $G/\!\sim$ is again an abelian monoid. An immediate corollary of proposition 1.12 is that $\mathrm{W}(K, \#, \mathrm{can})$ is really a group as claimed in the above definition; the opposite of $[X, \varphi]$ is $[X, -\varphi]$.

1.16. Definition. Two symmetric spaces (X, φ) and (Y, ψ) are said to be *Witt-equivalent* if $[X, \varphi] = [Y, \mu]$ in $\mathrm{W}(K)$.

1.17. Remark. It is convenient to observe that if $\mathrm{can} : \mathrm{Id} \to \# \circ \#$ is an isomorphism of functors, then so is $-\mathrm{can} : \mathrm{Id} \to \# \circ \#$. A symmetric space in $(K, \#, -\mathrm{can})$ is simply a *skew-symmetric space*. Therefore we treat simultaneously both cases without particular distinction. Since we confess that we intend to often forget the mention of $\mathrm{can} : \mathrm{Id} \to \# \circ \#$ in the following, this sign will be recalled as superscript in our notation. For instance $\mathrm{W}^{-1}(K)$ will denote the Witt group of skew-symmetric spaces.

1.18. Notation. Let $\epsilon = \pm 1$ and $n \in \mathbb{Z}$. As $T^n \circ \#$ is a $(-1)^n$-duality, we introduce the notation:

$$\mathrm{W}^{\epsilon}_n(K) = \mathrm{W}(K, T^n \circ \#, \epsilon \cdot \mathrm{can}).$$

By *the* Witt group of K, we mean $\mathrm{W}^1_0(K) = \mathrm{W}(K, \#, \mathrm{can})$, which we simply abbreviate $\mathrm{W}(K)$.

1.19. Functoriality. Let K_i be a triangulated category with duality $\#_i$ for $i = 1, 2$. Suppose $\#_1$ and $\#_2$ are both exact or both skew-exact. A *morphism* of triangulated categories with duality is a covariant additive functor, exact or skew-exact (see remark 1.3) satisfying the following conditions:

$$F \circ \#_1 = \#_2 \circ F \qquad \text{and} \qquad F(\mathrm{can}_1) = \mathrm{can}_2.$$

In this case, if (X, φ) is a symmetric space for $\#_1$ then $(F(X), F(\varphi))$ is a symmetric space for $\#_2$. It is very easy to prove that neutral forms are mapped to neutral forms: if (L, α, w) is a lagrangian of the starting space, then $(F(L), F(\alpha), \delta \cdot F(w))$ is a lagrangian of its image, where $\delta = \pm 1$ comes from the δ-exactness of F. Hence F induces a group homomorphism

$$\mathrm{W}(F) : \mathrm{W}(K_1, \#_1, \mathrm{can}_1) \to \mathrm{W}(K_2, \#_2, \mathrm{can}_2).$$

The Witt group is a functor from triangulated categories with duality (resp. with skew-duality) to abelian groups.

1.20. Proposition. For all $n \in \mathbb{Z}$ and $\epsilon = \pm 1$ we have a canonical isomorphism:

$$\mathrm{W}^{\epsilon}_n(K) \xrightarrow{\sim} \mathrm{W}^{\epsilon}_{n+2}(K).$$

1.21. Proof: According to the above considerations, the translation functor $T : K \to K$ is a morphism of triangulated categories from $(K, \#, \epsilon\,\mathrm{can})$ to $(K, T^2 \circ \#, \epsilon\,\mathrm{can})$ and more generally from $(K, T^n \circ \#, \epsilon\,\mathrm{can})$ to $(K, T^{n+2} \circ \#, \epsilon\,\mathrm{can})$. The isomorphism is $\mathrm{W}(T)$. ♯

2. Witt groups of a scheme.

2.1. Construction. Let X be a noetherian scheme. Denote by $\mathcal{A}(X)$ the abelian category of coherent $\mathcal{O}_X$-modules and by $\mathcal{L}(X)$ the additive full subcategory of $\mathcal{A}(X)$ consisting of locally free $\mathcal{O}_X$-modules. Denote by $F : \mathcal{A}(X) \to \mathcal{A}(X)$ the functor $\mathcal{H}om_{\mathcal{O}_X}(—, \mathcal{O}_X)$ and observe that $F(\mathcal{L}(X)) \subset \mathcal{L}(X)$. We still denote by F the natural extension of F to the triangulated category of bounded complexes with morphisms up to homotopy:

$$\begin{aligned} F : \mathrm{K}^{\mathrm{b}}(\mathcal{A}(X)) &\longrightarrow \mathrm{K}^{\mathrm{b}}(\mathcal{A}(X)) \\ \big(M_i, \partial_i\big)^{\infty}_{i=-\infty} &\mapsto \big(F(M_{-i}), F(\partial_{-i+1})\big)^{\infty}_{i=-\infty} \end{aligned} \qquad \text{(See 4.4).}$$

Observe the following points:

(i) $F\big(\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))\big) \subset \mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$,

(ii) F is exact in the sense of definition 1.1,

(iii) if $M \in \mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$ is acyclic (i.e. $\mathrm{H}_i(M) = 0$ for all $i \in \mathbb{Z}$) then so is $F(M)$,

(iv) when $M \in \mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$, the canonical morphism $\mathrm{can}_M : M \to F(F(M))$ is an isomorphism.

Hence F induces an exact functor on the localization of $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$ with respect to quasi-isomorphisms, which we will denote by $\mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$.

2.2. Notation. Let X be a noetherian scheme. With the above notations, we put

$$\mathrm{K}(X) = \mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$$

and we denote by $\#_X : \mathrm{K}(X) \to \mathrm{K}(X)$ the localization of F.

2.3. Proposition. Let X be a noetherian scheme. Then $(\mathrm{K}(X), \#_X)$ is a triangulated category with duality.

2.4. Proof: On $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$ there exists an isomorphism of functors $\mathrm{can} : \mathrm{Id} \xrightarrow{\sim} F \circ F$. It is easy to see that $(\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X)), F, \mathrm{can})$ is a triangulated category with duality. One checks as well the same identities in the derived category. More generally, we can define the localization of a triangulated category with duality, with respect to a multiplicative system of morphisms compatible with the duality. Here $\mathrm{K}(X)$ turns out to be the localization of the triangulated category $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$ with respect to the multiplicative system of quasi-isomorphisms. ♯

2.5. Definition. Let X be a noetherian scheme. Let $n \in \mathbb{Z}$ and $\epsilon = \pm 1$. We put

$$\mathrm{W}^{\epsilon}_n(X) = \mathrm{W}^{\epsilon}_n(\mathrm{K}(X)).$$

These are the Witt groups of X.

2.6. Notation. *The* Witt group of X will be $\mathrm{W}(X) = \mathrm{W}^1_0(X)$. To avoid confusion, we denote the usual Witt group of X by $\mathrm{W}_{\mathrm{us}}(X)$. When $X = \mathrm{Spec}(A)$, we write $\mathrm{W}(A)$ for $\mathrm{W}(\mathrm{Spec}(A))$; and so on ...

2.7. Remark. The reader could verify the functoriality of these groups $\mathrm{W}^{\epsilon}_n(X)$ as a familiarizing exercise.

2.8. Remark. Suppose our scheme X is *regular* in the following sense: every coherent $\mathcal{O}_X$-module has a finite resolution by locally free coherent $\mathcal{O}_X$-modules. In that case, it is well known that our category $\mathrm{K}(X)$ is equivalent to the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A}(X))$ and $\#_X$ is the derived functor $\mathbf{R}^b F$ of the left exact functor F of 2.1 (see [6] Thm 10.5.9, p. 393).

2.9. Notation. Denote by $i_0 : \mathcal{L}(X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$ the functor that associates to any $\mathcal{O}_X$-module the complex concentrated in degree 0:

$$\mathcal{E} \quad \mapsto \quad i_0(\mathcal{E}) := \quad \cdots 0 \longrightarrow 0 \longrightarrow \mathcal{E} \longrightarrow 0 \longrightarrow 0 \cdots$$

2.10. Proposition. The map $(\mathcal{E}, \varphi) \mapsto (i_0(\mathcal{E}), i_0(\varphi))$ induces a group homomorphism

$$\iota_X : \mathrm{W}_{\mathrm{us}}(X) \to \mathrm{W}(X).$$

2.11. Proof: It is obvious that $\#_X(i_0(\mathcal{E})) = i_0(\mathcal{E}^*)$. Isometric forms map to isometric forms, orthogonal sum to orthogonal sum and so on. We only have to prove that if there exists an exact sequence of the form:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{E} \xrightarrow{\gamma^* \circ \varphi} \mathcal{F} \longrightarrow 0$$

with $\mathcal{F} \in \mathcal{L}(X)$ then $(M, \psi) := (i_0(\mathcal{E}), i_0(\varphi))$ is neutral. Set $L = i_0(\mathcal{F})$ and $\alpha = i_0(\gamma)$. We are going to find w such that (L, α, w) is a lagrangian of (M, ψ).

Let C denote the mapping cone of α. We have an exact triangle:

$$L \xrightarrow{\alpha} M \xrightarrow{j} C \xrightarrow{k} T(L)$$

where j and k are the usual morphisms. Here k is simply:

$$\begin{array}{ccccccccc} C = & \cdots 0 & \longrightarrow & \mathcal{F} & \xrightarrow{-\gamma} & \mathcal{E} & \longrightarrow & 0 & \longrightarrow 0 \cdots \\ {\scriptstyle k}\downarrow & & & {\scriptstyle -1}\| & & \downarrow & & \downarrow & \\ T(L) = & \cdots 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow 0 \cdots \end{array}$$

Consider the quasi-isomorphism:

$$\begin{array}{ccccccccc} C = & \cdots 0 & \longrightarrow & \mathcal{F} & \xrightarrow{-\gamma} & \mathcal{E} & \longrightarrow & 0 & \longrightarrow 0 \cdots \\ {\scriptstyle s}\downarrow & & & \downarrow & & {\scriptstyle \gamma^* \circ \varphi}\downarrow & & \downarrow & \\ L^\# = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^* & \longrightarrow & 0 & \longrightarrow 0 \cdots \end{array}$$

We now have an exact triangle in $\mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$:

$$L \xrightarrow{\alpha} M \xrightarrow{s \circ j} L^\# \xrightarrow{k \circ s^{-1}} T(L)\ .$$

Observe that this triangle would not exist in $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$.

A direct calculation gives $s \circ j = \alpha^\# \circ \psi$ and if we set $w = -T^{-1}(k \circ s^{-1})$ we have an exact triangle:

$$T^{-1}(L^\#) \xrightarrow{w} L \xrightarrow{\alpha} M \xrightarrow{\alpha^\# \circ \psi} L^\#\ .$$

It suffices to show that $T^{-1}(w^\#) = w$, which is equivalent to $T(s^\#) \circ k = T(k^\#) \circ s$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$. An immediate verification shows that these two morphisms of complexes are homotopic. Hence (L, α, w) is a lagrangian of (M, ψ). ♯

2.12. Remark. As noticed in this proof, the natural map sending locally free $\mathcal{O}_X$-modules with symmetric forms to complexes concentrated in degree 0 in $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$ would not send neutral forms (in the usual sense) to neutral ones. This is the main reason for introducing $\mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$ instead of $\mathrm{K}^{\mathrm{b}}(\mathcal{L}(X))$.

3. Techniques in a triangulated category.

As astonishing as it may seem, it is possible to proceed to some calculations even in such an abstract setting as triangulated categories. We give here a few results that we will use in other sections.

From now on K denotes a triangulated category with a 1-duality $\#$. The following results can easily be adapted to skew-dualities and we will use them in this case as well.

We wittingly omit the canonical isomorphism can : $\mathrm{Id} \xrightarrow{\sim} \# \circ \#$ to lighten notations and therefore consider $\# \circ \# = \mathrm{Id}$. The masochist reader may divert himself in restoring everywhere the mention of the canonical isomorphism.

3.1. Hypothesis. From now on we suppose that $\frac{1}{2}$ belongs to our category K. This simply means that we can divide by 2 in every group of morphisms $\mathrm{Hom}_K(A, B)$.

3.2. Lemma. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ be an exact triangle and let (f, g, h) be a morphism from this triangle to itself. Suppose that two of these endomorphisms are nilpotent. Then so is the third one.

3.3. Proof: Suppose that f and g are nilpotent. Composing (f, g, h) with itself a sufficient number of times, we may suppose $f = 0$ and $g = 0$. In that case, $w \circ h = 0$ gives $h = v \circ \tilde{h}$ for some $\tilde{h} : C \to B$ and then $h^2 = h \circ v \circ \tilde{h} = 0$ since $h \circ v = 0$. ♯

3.4. Lemma. Let $T^{-1}(L^\#) \xrightarrow{s} L \xrightarrow{s_1} X \xrightarrow{s_2} L^\#$ be an exact triangle such that $T^{-1}(s^\#) = s$. Suppose that $\varphi, \psi : X \to X^\#$ are two forms on X, both making the following diagram commute:

$$\begin{array}{ccccccc}
T^{-1}(L^\#) & \xrightarrow{s} & L & \xrightarrow{s_1} & X & \xrightarrow{s_2} & L^\# \\
{\scriptstyle 1}\| & & {\scriptstyle 1}\| & & \simeq\downarrow{\scriptstyle \varphi,\psi} & & \| \\
T^{-1}(L^\#) & \xrightarrow[T^{-1}(s^\#)]{} & L & \xrightarrow[s_2^\#]{} & X^\# & \xrightarrow[s_1^\#]{} & L^\#\ .
\end{array}$$

Then φ and ψ are isometric.

3.5. Proof: Since $(1,1,\varphi)$ and $(1,1,\psi)$ are isomorphisms, $(1,1,\psi^{-1}\varphi)$ is an automorphism of the first triangle and $(0,0,-1+\psi^{-1}\varphi)$ is an endomorphism. If we define $h := -1 + \psi^{-1}\varphi$, lemma 3.2 gives: $h^2 = 0$. We have $\psi(1+h) = \varphi$ and therefore $h^\# \psi = \psi\, h$. Hence $(1+\frac{h}{2})^\# \psi (1+\frac{h}{2}) = \psi(1+h) = \varphi$. ♯

3.6. Remark. It is well known that in a triangulated category, if (f,g,h) is a morphism of exact triangles and if two of these morphisms are isomorphisms so is the third. Nevertheless, given f and g, the h such that (f,g,h) is a morphism is not unique!

The very easy lemma 3.4 is then a key result, allowing us to pass through this classical problem in the theory of Witt groups. In particular, lemma 3.4 implies that a neutral form is characterized up to isometry by $w : T^{-1}L^\# \to L$ such that $T^{-1}w^\# = w$ (see 1.9 and 1.10).

3.7. Sublagrangian construction.

It is natural to wonder whether we could divide out any symmetric space by an isotropic subspace. This means constructing a form on a reduced space. In the classical context, *the* orthogonal subspace of a given subspace is well defined. In triangulated categories, since "cones" are not unique, this notion has to be more flexible. In fact, it turns out that under a small assumption on the sublagrangian this construction can be carried out independently of the choice of an orthogonal. Let us be more precise.

3.8. Lemma. Let (X,φ) be a symmetric space and (L,α) a sublagrangian. Choose any triangle on α, say

$$\begin{array}{ccc}
 & M^\# & \\
{\scriptstyle -T^{-1}(\alpha_0)}\swarrow & & \nwarrow{\scriptstyle \alpha_2} \\
L & \xrightarrow[\alpha]{} & X\ .
\end{array}$$

There exists $\eta_0 : L \to M$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
T^{-1}(M^\#) & \xrightarrow{\alpha_0} & L & \xrightarrow{\alpha} & X & \xrightarrow{\alpha_2} & M^\# \\
{\scriptstyle T^{-1}(\eta_0{}^\#)}\downarrow & & {\scriptstyle \eta_0}\downarrow & & \simeq\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \eta_0{}^\#} \\
T^{-1}L^\# & \xrightarrow[T^{-1}(\alpha_0{}^\#)]{} & M & \xrightarrow[\alpha_2^\#]{} & X^\# & \xrightarrow[\alpha^\#]{} & L^\#\ .
\end{array}$$

where the second line is the dual triangle of the first one. M could be understood as *an* orthogonal of L.

3.9. Proof: Since $\alpha^\# \circ (\varphi\,\alpha) = 0$, there exists $\eta : L \to M$ such that $\varphi\,\alpha = \alpha_2{}^\# \circ \eta$. Complete the diagram with a morphism (ν,η,φ). Observe that $(T^{-1}(\eta^\#), T^{-1}(\nu^\#), \varphi)$ is also a morphism between these two exact triangles. Putting

$$\eta_0 = \frac{\eta + T^{-1}\nu^\#}{2},$$

we get the desired one. ♯

3.10. Question. Choosing any exact triangle containing η_0, say

$$L \xrightarrow{\eta_0} M \xrightarrow{\eta_1} Y \xrightarrow{\eta_2} T(L)\,,$$

is there a symmetric form $\mu = \mu^{\#} : Y \xrightarrow{\sim} Y^{\#}$ such that $[X,\varphi] = [Y,\mu]$ in $\mathrm{W}(K)$? Heuristically, Y plays the role of $L^{\perp}/L$.

3.11. Theorem. Let (X,φ) be a symmetric space and $\alpha : L \to X$ such that $\alpha^{\#}\,\varphi\,\alpha = 0$. Suppose $\mathrm{Hom}_K(L, T^{-1}L^{\#}) = 0$. Then, with the above notations, there exists a symmetric form $\mu : Y \to Y^{\#}$ Witt-equivalent to (X,φ).

3.12. Proof : Keep the notations of lemma 3.8. Define $s := \eta_0\,\alpha_0 : T^{-1}(M^{\#}) \to M$ and notice that $T^{-1}(s^{\#}) = s$.

Observe that $\alpha \circ T^{-1}(\eta_2) = 0$. Applying the axiom of the octahedron to this relation, we get an exact triangle :

$$T^{-1}(M^{\#}) \xrightarrow{s} M \xrightarrow{\begin{pmatrix} x \\ \eta_1 \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} \alpha_2 & y \end{pmatrix}} M^{\#}$$

for some morphisms $x : M \to X$ and $y : Y \to M^{\#}$ satisfying:

$$x\,\eta_0 = \alpha \qquad \text{and} \qquad \eta_2 = T(\alpha_0)\,y.$$

Since $(x - \varphi^{-1}\,{\alpha_2}^{\#})\,\eta_0 = 0$ there exists $h : Y \to X$ such that $x + h\,\eta_1 = \varphi^{-1}\,{\alpha_2}^{\#}$. Using the automorphism $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ of $X \oplus Y$, we may suppose $x = \varphi^{-1}\,{\alpha_2}^{\#}$, keeping $\eta_2 = T(\alpha_0)\,y$.

Now we construct another octahedron for the identity $\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi^{-1}\,{\alpha_2}^{\#} \\ \eta_1 \end{pmatrix} = \varphi^{-1}\,{\alpha_2}^{\#}$ and we get an exact triangle :

$$Y \xrightarrow{y} M^{\#} \xrightarrow{{\eta_1}^{\#}} L^{\#} \xrightarrow{T(\eta_1)\,{\alpha_0}^{\#}} T(Y)$$

such that $y_1\alpha_2 = \alpha^{\#}\varphi$. From this equality, we get that $(y_1 - {\eta_0}^{\#})\,\alpha_2 = 0$ and therefore $y_1 - {\eta_0}^{\#} = k \circ T(\alpha_0)$ for a morphism $k : TL \to L^{\#}$. Our hypothesis on L insures that $k = 0$. That is $y_1 = {\eta_0}^{\#}$.

After this, compare the two triangles containing ${\eta_0}^{\#}$ to get a non necessarily symmetric isomorphism $\psi : Y^{\#} \to Y$

$$\begin{array}{ccccccc}
T^{-1}L^{\#} & \xrightarrow{-{\eta_2}^{\#}} & Y^{\#} & \xrightarrow{{\eta_1}^{\#}} & M^{\#} & \xrightarrow{{\eta_0}^{\#}} & L^{\#} \\
\| & & \downarrow{\scriptstyle \exists\psi\,\wr\,\simeq} & & \| & & \| \\
T^{-1}L^{\#} & \xrightarrow[-\eta_1 \circ T^{-1}({\alpha_0}^{\#})]{} & Y & \xrightarrow[y]{} & M^{\#} & \xrightarrow[{\eta_0}^{\#}]{} & L^{\#}
\end{array} \qquad (1)$$

There exists a morphism $\delta : M \to M$ such that the following diagram commutes :

$$\begin{array}{ccccccc}
T^{-1}(M^{\#}) & \xrightarrow{s} & M & \xrightarrow{\begin{pmatrix} \varphi^{-1}\,{\alpha_2}^{\#} \\ \eta_1 \end{pmatrix}} & X \oplus Y & \xrightarrow{\begin{pmatrix} \alpha_2 & y \end{pmatrix}} & M^{\#} \\
\| & & \uparrow{\scriptstyle \wr\,\exists\,1+\delta} & & \uparrow{\scriptstyle \begin{pmatrix} \varphi^{-1} & 0 \\ 0 & \psi \end{pmatrix}} & & \| \\
T^{-1}(M^{\#}) & \xrightarrow[T^{-1}s^{\#} = s]{} & M & \xrightarrow[\begin{pmatrix} {\alpha_2}^{\#} \\ y^{\#} \end{pmatrix}]{} & X^{\#} \oplus Y^{\#} & \xrightarrow[\begin{pmatrix} \alpha_2\,\varphi^{-1} & {\eta_1}^{\#} \end{pmatrix}]{} & M^{\#}
\end{array} \qquad (2)$$

We are going to show that $\delta^3 = 0$. From diagram (2) we get that ${\alpha_2}^{\#}\,\delta = 0$ and then $\delta = T^{-1}({\alpha_0}^{\#})\,\tilde{\delta}$ for some $\tilde{\delta} : M \to T^{-1}L^{\#}$. Since $\mathrm{Hom}_K(L, T^{-1}L^{\#}) = 0$, we have that $\tilde{\delta}\,\eta_0 = 0$ and then $\delta = \bar{\delta}\,\eta_1$ for some $\bar{\delta} : Y \to M$. Hence $\delta^3 = \big(\bar{\delta}\,\eta_1\big) \circ \delta \circ \big(T^{-1}({\alpha_0}^{\#})\,\tilde{\delta}\big)$ and it suffices to show that $\eta_1 \circ \delta \circ T^{-1}({\alpha_0}^{\#}) = 0$. To

prove this remember that $T(\alpha_0)\, y = \eta_2$ and use it in the following computation: $\eta_1\, T^{-1}(\alpha_0{}^{\#}) \overset{(1)}{=} \psi\eta_2{}^{\#} = \psi\, y^{\#}\, T^{-1}(\alpha_0{}^{\#}) \overset{(2)}{=} \eta_1\,(1+\delta)\, T^{-1}(\alpha_0{}^{\#}) = \eta_1\, T^{-1}(\alpha_0{}^{\#}) + \eta_1 \circ \delta \circ T^{-1}(\alpha_0{}^{\#})$. Hence $\delta^3 = 0$.

In the end, taking the mean between the morphism of diagram (2) and its dual, we get the wanted symmetric isomorphism on Y:

$$\begin{array}{ccccccc}
T^{-1}(M^{\#}) & \xrightarrow{\ s\ } & M & \xrightarrow{\begin{pmatrix}\varphi^{-1}\alpha_2{}^{\#}\\ \eta_1\end{pmatrix}} & X\oplus Y & \xrightarrow{\begin{pmatrix}\alpha_2 & y\end{pmatrix}} & M^{\#} \\
\simeq\uparrow{\scriptstyle 1+\frac12 T^{-1}\delta^{\#}} & & \simeq\uparrow{\scriptstyle 1+\frac12\delta} & & \uparrow{\scriptstyle \begin{pmatrix}\varphi^{-1} & 0\\ 0 & \frac12(\psi+\psi^{\#})\end{pmatrix}} & & \simeq\uparrow{\scriptstyle 1+\frac12\delta^{\#}} \\
T^{-1}(M^{\#}) & \xrightarrow[T^{-1}s^{\#}]{} & M & \xrightarrow[\begin{pmatrix}\alpha_2{}^{\#}\\ y^{\#}\end{pmatrix}]{} & X^{\#}\oplus Y^{\#} & \xrightarrow[\begin{pmatrix}\alpha_2\,\varphi^{-1} & \eta_1{}^{\#}\end{pmatrix}]{} & M^{\#}
\end{array}$$

Since $\delta\, s = 0$, we have also $s \circ T^{-1}\delta^{\#} = 0$. The proof reduces then to next lemma. ♯

3.13. Lemma. Let (Z,ξ) be a symmetric space and let

$$T^{-1}M^{\#} \xrightarrow{\ s\ } M \xrightarrow{\ s_1\ } Z \xrightarrow{\ s_2\ } M^{\#}$$

be an exact triangle with $s = T^{-1}s^{\#}$. Let $h : M \xrightarrow{\sim} M$ be such that the following diagram commutes:

$$\begin{array}{ccccccc}
T^{-1}(M^{\#}) & \xrightarrow{\ s\ } & M & \xrightarrow{\ s_1\ } & Z & \xrightarrow{\ s_2\ } & M^{\#} \\
\simeq\downarrow{\scriptstyle T^{-1}h^{\#}} & & \simeq\downarrow{\scriptstyle h} & & \simeq\downarrow{\scriptstyle \xi=\xi^{\#}} & & \simeq\downarrow{\scriptstyle h^{\#}} \\
T^{-1}(M^{\#}) & \xrightarrow[T^{-1}s^{\#}]{} & M & \xrightarrow[s_2{}^{\#}]{} & Z^{\#} & \xrightarrow[s_1{}^{\#}]{} & M^{\#}
\end{array}$$

and such that $h \circ s = s$.

Then (Z,χ) is neutral.

3.14. Proof: From $h \circ s = s$, we get an isomomorphism of triangles:

$$\begin{array}{ccccccc}
T^{-1}(M^{\#}) & \xrightarrow{\ s\ } & M & \xrightarrow{\ s_1\ } & Z & \xrightarrow{\ s_2\ } & M^{\#} \\
\| & & \simeq\downarrow{\scriptstyle h} & & \| & & \| \\
T^{-1}(M^{\#}) & \xrightarrow[s]{} & M & \xrightarrow[s_1\circ h^{-1}]{} & Z^{\#} & \xrightarrow[s_2]{} & M^{\#}
\end{array}$$

Since the first triangle is exact, so is the second. We check that $(M, s_1 \circ h^{-1}, s)$ is a lagrangian of (Z,ξ). It remains to see that $(s_1 \circ h^{-1})^{\#} \circ \xi = s_2$ which is immediate:

$$(s_1 \circ h^{-1})^{\#} \circ \xi = (h^{-1})^{\#} \circ s_1{}^{\#} \circ \xi = (h^{-1})^{\#} \circ h^{\#} \circ s_2 = s_2.$$

♯

4. The isomorphism.

4.1. Theorem. Let A be a noetherian ring in which 2 is a unit. Then the homomorphism

$$\iota_A : \mathrm{W}_{\mathrm{us}}(A) \to \mathrm{W}(A)$$

is surjective.

4.2. Proof: In the construction of section 2, the scheme X is $\mathrm{Spec}(A)$, the locally free $\mathcal{O}_X$-modules are the projective A-modules: $\mathcal{L}(X) = \mathcal{P}$ and it is well known that $\mathrm{D}^{\mathrm{b}}()P \simeq \mathrm{K}^{\mathrm{b}}()P$.

It is obvious that any bounded complex endowed with a symmetric form φ:

$$\begin{array}{rccccccccccccc}
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_{-n+2}} & P_{-n+1} & \xrightarrow{\partial_{-n+1}} & P_{-n} & \longrightarrow & 0 \cdots \\
\varphi \downarrow & & & \varphi_n \downarrow & & \varphi_{n-1} \downarrow & & & & \varphi_{-n+1} \downarrow & & \varphi_{-n} \downarrow & & \\
X^{\#} = & \cdots 0 & \longrightarrow & P^*_{-n} & \xrightarrow[\partial^*_{-n+1}]{} & P^*_{-n+1} & \xrightarrow[\partial^*_{-n+2}]{} & \cdots & \xrightarrow[\partial^*_{n-1}]{} & P^*_{n-1} & \xrightarrow[\partial^*_n]{} & P^*_n & \longrightarrow & 0 \cdots
\end{array}$$

admits the following sub-lagrangian when $n \geq 1$:

$$\begin{array}{rccccccccc}
L = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & P_{-n} \longrightarrow 0 \cdots \\
\alpha \downarrow & & & \downarrow & & & & \downarrow & & \| \\
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow[\partial_n]{} & \cdots & \xrightarrow[\partial_{-n+2}]{} & P^*_{n-1} & \xrightarrow[\partial^*_n]{} & P_{-n} \longrightarrow 0 \cdots
\end{array}$$

It is clear that $\mathrm{Hom}(L, T^{-1}L^{\#}) = 0$. So we may apply theorem 3.11. But with the notations of the sublagrangian construction, we may choose here for Y a complex shorter than X. To see this, choose your exact triangle containing α, observing that $X = C(\alpha_0)$ is the mapping cone of the morphism of complexes:

$$\begin{array}{rccccccccccc}
T^{-1}M^{\#} := & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{-\partial_n} & \cdots & \xrightarrow{-\partial_{-n+2}} & P_{-n+2} & \xrightarrow{-\partial_{-n+2}} & P_{-n+1} \longrightarrow 0 \cdots \\
\alpha_0 \downarrow & & & & & \downarrow & & & & \downarrow & & \xrightarrow{-\partial_{-n+1}} \downarrow \\
L = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & P_{-n} \longrightarrow 0 \cdots
\end{array}$$

Then verify that Y is of the form:

$$\cdots 0 \longrightarrow 0 \longrightarrow Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_{-n+1}} Q_{-n} \longrightarrow 0 \cdots .$$

As we have a homotopy equivalence $\psi : Y \to Y^{\#}$, Id_Y is homotopic to a map factorising as $Y \to Y^{\#} \to Y$. This map is (by the structure of Y) zero in degree $-n$. This implies that ∂'_{-n+1} is a split epimorphism. Hence Y is homotopically equivalent to a shorter complex.

By induction we come down to a complex concentrated in degree zero. This gives the result. ♯

In order to prove that ι_A is an isomorphism we are going to construct an inverse $\Omega_A : \mathrm{W}(A) \to \mathrm{W}_{\mathrm{us}}(A)$. This is the goal of the end of this section. We will proceed step by step, the first step being definition 4.4.

4.3. Remark. Given a symmetric form φ on a bounded complex of projective A-modules, since $\frac{1}{2} \in A$, we may suppose that φ is *strongly symmetric* in the sense that $\varphi^*_{-i} = \varphi_i$ for all $i \in \mathbb{Z}$.

4.4. Definition. Let $(X, \varphi, \bar{\varphi}, \epsilon)$ be as follow: X is a bounded complex of projective A-modules, $\varphi : X \to X^{\#}$, $\bar{\varphi} : X^{\#} \to X$ are such that $\varphi^{\#} = \varphi$, $\bar{\varphi}^{\#} = \bar{\varphi}$, and $\bar{\varphi} \circ \varphi \sim \mathrm{Id}$ with homotopy ϵ.

$$\begin{array}{rccccccccccccc}
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_{-n+2}} & P_{-n+1} & \xrightarrow{\partial_{-n+1}} & P_{-n} & \longrightarrow & 0 \cdots \\
\varphi \downarrow & & & \varphi_n \downarrow & & \varphi_{n-1} \downarrow & & & & \varphi_{-n+1} \downarrow & & \varphi_{-n} \downarrow & & \\
X^{\#} = & \cdots 0 & \longrightarrow & P^*_{-n} & \xrightarrow{\partial^*_{-n+1}} & P^*_{-n+1} & \xrightarrow{\partial^*_{-n+2}} & \cdots & \xrightarrow{\partial^*_{n-1}} & P^*_{n-1} & \xrightarrow{\partial^*_n} & P^*_n & \longrightarrow & 0 \cdots \\
\bar{\varphi} \downarrow & & & \bar{\varphi}_n \downarrow & & \bar{\varphi}_{n-1} \downarrow & & & & \bar{\varphi}_{-n+1} \downarrow & & \bar{\varphi}_{-n} \downarrow & & \\
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow[\partial_n]{} & P_{n-1} & \xrightarrow[\partial_{n-1}]{} & \cdots & \xrightarrow[\partial_{-n+2}]{} & P_{-n+1} & \xrightarrow[\partial_{-n+1}]{} & P_{-n} & \longrightarrow & 0 \cdots
\end{array}$$

$$\begin{array}{ccccccccccccccc}
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_{-n+2}} & P_{-n+1} & \xrightarrow{\partial_{-n+1}} & P_{-n} & \longrightarrow & 0 \cdots \\
\bar\varphi \circ \varphi \downarrow & & & \downarrow{\scriptstyle \bar\varphi_n\,\varphi_n} & \swarrow{\scriptstyle \epsilon_{n-1}} & \downarrow{\scriptstyle \bar\varphi_{n-1}\,\varphi_{n-1}} & & & & \downarrow{\scriptstyle \bar\varphi_{-n+1}\,\varphi_{-n+1}} & \swarrow{\scriptstyle \epsilon_{-n}} & \downarrow{\scriptstyle \bar\varphi_{-n}\,\varphi_{-n}} & & \\
X = & \cdots 0 & \longrightarrow & P_n & \xrightarrow[\partial_n]{} & P_{n-1} & \xrightarrow[\partial_{n-1}]{} & \cdots & \xrightarrow[\partial_{-n+2}]{} & P_{-n+1} & \xrightarrow[\partial_{-n+1}]{} & P_{-n} & \longrightarrow & 0 \cdots
\end{array}$$

Suppose φ and $\bar\varphi$ strongly symmetrical: $\varphi^*_{-i} = \varphi_i$ and $\bar\varphi^*_{-i} = \bar\varphi_i$ for all $i \in \mathbb{Z}$. We have $\mathrm{Id}_X = \bar\varphi\,\varphi + \partial_X\,\epsilon + \epsilon\,\partial_X$ and therefore $\mathrm{Id}_{X^\#} = \varphi^\#\,\bar\varphi^\# + \epsilon^\#\,\partial_{X^\#} + \partial_{X^\#}\,\epsilon^\#$, where we have to pay attention to the indices of $\epsilon^\#$ which are shifted by 1 (that is $(\epsilon^\#)_i := \epsilon^*_{-i-1}$). We set

$$\Omega(M, \varphi, \bar\varphi, \epsilon) = [Y, \psi] \in \mathrm{W}_{\mathrm{us}}(A)$$

where

$$Y = \cdots \oplus P^*_{-3} \oplus P_2 \oplus P^*_{-1} \oplus P_0 \oplus P^*_1 \oplus P_{-2} \oplus P^*_3 \oplus \cdots$$

$$\psi = \begin{pmatrix}
 & & & & & & & \ddots & \ddots & \ddots & \\
 & 0 & 0 & 0 & 0 & 0 & \partial_{-2} & \bar\varphi_{-3} & -\epsilon_{-4} & \\
 & 0 & 0 & 0 & 0 & \partial^*_2 & \varphi_2 & -\epsilon^*_2 & 0 & \\
 & 0 & 0 & 0 & \partial_0 & \bar\varphi_{-1} & -\epsilon_{-2} & 0 & 0 & \\
\cdots & 0 & 0 & 0 & \partial^*_0 & \varphi_0 & -\epsilon^*_0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & \partial_2 & \bar\varphi_1 & -\epsilon_0 & 0 & 0 & 0 & 0 & \\
 & 0 & \partial^*_{-2} & \varphi_{-2} & -\epsilon^*_{-2} & 0 & 0 & 0 & 0 & 0 & \\
 & \partial_4 & \bar\varphi_3 & -\epsilon_2 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\ddots & \ddots & \ddots & & & & & & & &
\end{pmatrix}$$

$$Y^* = \cdots \oplus P_{-3} \oplus P^*_2 \oplus P_{-1} \oplus P^*_0 \oplus P_1 \oplus P^*_{-2} \oplus P_3 \oplus \cdots$$

We also define the homomorphism $\bar\psi : Y^* \to Y$ by

$$\bar\psi = \begin{pmatrix}
 & & & & & & & & \ddots & \ddots & \ddots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^*_{-3} & \varphi_3 & -\partial^*_{-3} & \\
 & 0 & 0 & 0 & 0 & 0 & \epsilon_1 & \bar\varphi_2 & -\partial_3 & 0 & \\
 & 0 & 0 & 0 & 0 & \epsilon^*_{-1} & \varphi_1 & -\partial^*_{-1} & 0 & 0 & \\
\cdots & 0 & 0 & 0 & \epsilon_{-1} & \bar\varphi_0 & -\partial_1 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & \epsilon^*_1 & \varphi_{-1} & -\partial^*_1 & 0 & 0 & 0 & 0 & \\
 & 0 & \epsilon_{-3} & \bar\varphi_{-2} & -\partial_{-1} & 0 & 0 & 0 & 0 & 0 & \\
 & \epsilon^*_3 & \varphi_{-3} & -\partial^*_3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\ddots & \ddots & \ddots & & & & & & & &
\end{pmatrix}$$

4.5. Lemma. With the above notations, $\psi^* = \psi$, $\bar\psi^* = \bar\psi$ and $\psi\,\bar\psi = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix}$. Hence, (Y, ψ) is a symmetric space.

4.6. Proof: Direct computation using $\partial \circ \partial = 0$, $\partial\,\varphi = \varphi\,\partial$, $\mathrm{Id}_X = \bar\varphi\,\varphi + \partial_X\,\epsilon + \epsilon\,\partial_X$, and so on ... ♯

4.7. Important remark. In fact the above construction could be carried out with any homotopy equivalence $f : X \to X'$ with inverse $g : X' \to X$ and homotopies $\mathrm{Id}_X = g\,f + d\epsilon + \epsilon d$ and $\mathrm{Id}_{X'} = f\,g + d'\epsilon' + \epsilon' d'$. We construct in a similar way an isomorphism (!) from an alternate sum of terms of X and terms of X' to the alternate sum of the remaining terms. Replacing in the above definition φ by f, $\bar\varphi$ by g, ϵ by ϵ and ϵ^* by ϵ', you obtain two homomorphisms ρ and $\bar\rho$ with zeros outside the three diagonals. A direct computation gives that $\rho \cdot \bar\rho$ is lower triangular with 1 in the diagonal.

It is not the subject of this note, but this construction might induce a homomorphism from the "derived" K_1 of A to the usual $\mathrm{K}_1(A)$. Apart from this, we will use this construction in the end of the proof.

Here comes now a very useful result, which allows us to define Ω independently of any choice. It shows that we can change ψ under the three diagonals without changing the isometry class of ψ.

4.8. Lemma. With the above notations, consider $\widetilde{\psi} : Y \to Y^*$ such that $\widetilde{\psi}^* = \widetilde{\psi}$ and

$$\widetilde{\psi} - \psi = \begin{pmatrix} 0 & & & 0 \\ & & \iddots & 0 \\ & \iddots & \iddots & \\ 0 & 0 & & \star \end{pmatrix}.$$

Then $\widetilde{\psi} \simeq \psi$. In particular, $\widetilde{\psi}$ is an isomorphism.

4.9. Proof: We have $\widetilde{\psi} = \psi + a$, where a is the matrix in the statement. That is $\widetilde{\psi} = (1+b)\psi$, where $b = a \cdot \psi^{-1}$. Since $\psi\,\bar{\psi} = E$, where E is the matrix of lemma 4.5, $\psi^{-1} = \bar{\psi}\,E^{-1}$. As E^{-1} is also lower triangular with 1 in the diagonal, using the hypothesis on a it is easy to check that b is lower triangular with 0 in the diagonal. Hence b is nilpotent. Since $\frac{1}{2} \in A$, there exists a polynomial $P \in A[X]$ such that $P(b)^2 = 1 + b$. Then one verifies that $P(b)\,\psi\,P(b)^* = \widetilde{\psi}$. ♯

For the moment, $\Omega(X, \varphi, \bar{\varphi}, \epsilon)$ is well defined for strongly symmetric φ and $\bar{\varphi}$ and for a homotopy $\epsilon : \bar{\varphi}\,\varphi \sim \mathrm{Id}$. First of all, we are going to show that Ω is independent of the choice of ϵ.

4.10. Lemma. With the above notations, suppose we are given $\mu : X \to X$ of degree $+1$ such that $\partial\,\mu + \mu\,\partial = 0$. Then $\Omega(X, \varphi, \bar{\varphi}, \epsilon) = \Omega(X, \varphi, \bar{\varphi}, \epsilon + \mu)$.

4.11. Proof: Denote by $\check{\psi}$ the form induced on Y in the definition of $\Omega(X, \varphi, \bar{\varphi}, \epsilon + \mu)$. Consider $h : Y^* \to Y^*$ defined by:

$$h = \begin{pmatrix} \ddots & \ddots & \ddots & & & & & & \\ 0 & 1 & 0 & & & & 0 & & \\ -\mu_{-2}\,\epsilon_{-3} & -\mu_{-2}\,\bar{\varphi}_{-2} & 1 + \mu_{-2}\,\partial_{-1} & 0 & & & & & \\ 0 & 0 & 0 & 1 & 0 & & & & \\ & 0 & -\mu_0\,\epsilon_{-1} & -\mu_0\,\bar{\varphi}_0 & 1 + \mu_0\,\partial_1 & 0 & & & \\ 0 & & 0 & 0 & 0 & 1 & 0 & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}.$$

Direct computation gives that $h\,\psi\,h^* = \check{\psi} + a$, where a is as in lemma 4.8. Since $\check{\psi}$ is a form, lemma 4.8 implies that $h\,\psi\,h^*$ is isometric to $\check{\psi}$ and in particular is an isomorphism. Then h is onto and (as h is an endomorphism) it is an isomorphism. Thus, $\psi \simeq \check{\psi}$. ♯

4.12. Corollary. $\Omega(X, \varphi, \bar{\varphi}, \epsilon)$ does not depend on the choice of ϵ.

4.13. Proof: Two such ϵ would differ by a μ as in lemma 4.10. ♯

We now want to prove that Ω does not depend on the choice of the strongly symmetric φ representing the form.

4.14. Lemma. With the above notations, suppose we are given $\mu : X \to X^{\#}$ of degree $+1$ such that $\mu = \mu^{\#}$ (paying attention to indices, this equality means that $\mu^*_{-i+1} = \mu_i$ for all $i \in \mathbb{Z}$). Then

$$\Omega(X, \varphi, \bar{\varphi}, \epsilon) = \Omega(X, \varphi + \partial^*\mu + \mu\partial,\ \bar{\varphi}\,,\ \epsilon - \bar{\varphi}\mu\,).$$

4.15. Proof: Observe first of all that $\check{\varphi} := \varphi + \partial^* \mu + \mu\partial$ is another strongly symmetric form homotopic to φ and that $\epsilon - \bar{\varphi}\mu$ gives a homotopy $\check{\varphi}\,\bar{\varphi} \sim \mathrm{Id}$. Hence $\Omega(X, \varphi + \partial^*\mu + \mu\partial,\, \bar{\varphi}\,, \epsilon - \bar{\varphi}\mu\,)$ is well-defined. Denote by $\check{\psi}$ the form on Y induced by $\check{\varphi}$. Define $h : Y \to Y$ by

$$h = \begin{pmatrix} \ddots & \mu_2 & & & & \\ & 1 & 0 & & & 0 \\ & & 1 & \mu_0 & & \\ & & & 1 & 0 & \\ 0 & & & & 1 & \mu_{-2} \\ & & & & & \ddots \end{pmatrix}.$$

Lemma 4.8 applied to $h^{\#}\,\psi\,h$ and to $\check{\psi}$ gives the result. ♯

4.16. Corollary. $\Omega(X, \varphi, \bar{\varphi}, \epsilon)$ depends only on the space X and on the homotopy class of $\varphi : X \to X^{\#}$.

4.17. Proof: Two strongly symmetric φ differ by a null homotopic morphism with a homotopy that might be chosen symmetric because $\frac{1}{2} \in A$. Hence we apply lemma 4.14. We prove in the same way that $\Omega(X, \varphi, \bar{\varphi}, \epsilon)$ does not depend on the choice of the strongly symmetric inverse $\bar{\varphi}$. ♯

4.18. Notation. Let (X, f) be a symmetric space in $\mathrm{K}^{\mathrm{b}}(\mathcal{P})$. We put

$$\Omega(X, f) = \Omega(X, \varphi, \bar{\varphi}, \epsilon) \in \mathrm{W}_{\mathrm{us}}(A)$$

where $f = [\varphi]$, $f^{-1} = [\bar{\varphi}]$, φ and $\bar{\varphi}$ are strongly symmetric and where ϵ is any homotopy $\mathrm{Id} = \bar{\varphi}\,\varphi + \partial\epsilon + \epsilon\partial$. This is well defined by corollaries 4.12 and 4.16.

4.19. Remark. At this point, we have a well defined map Ω on symmetric spaces. It is obvious that $\Omega((X, \varphi) \perp (X', \varphi')) = \Omega(X, \varphi) \perp \Omega(X', \varphi')$. We now have to prove that Ω is invariant by isometry (isomorphisms are homotopy equivalences!) and that Ω sends neutral forms to zero.

4.20. Lemma. Let L be a bounded complex of projective A-modules and let a morphism of complexes $w : T^{-1}L^{\#} \to L$ be strongly symmetric in the sense that $T^{-1}w^{\#} = w$ as morphisms of complexes. Let $X = C(w)$ be the mapping cone of w and let $\varphi = C(\mathrm{Id}, \mathrm{can})$ the map given by the naturality of the mapping cone construction (see 1.10). Then φ is a form on X and $\Omega(X, \varphi) = 0$.

4.21. Proof: Let $w : T^{-1}L^{\#} \to L$ be given by:

$$\begin{array}{ccccccccccccccc} L := & \cdots 0 & \to & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_{-n+2}} & P_{-n+1} & \xrightarrow{\partial_{-n+1}} & P_{-n} & \to & 0 & \to 0 \cdots \\ w \downarrow & & & \downarrow & & \downarrow{\scriptstyle w_{n-1}} & & & & \downarrow{\scriptstyle w_{-n+1}} & & \downarrow{\scriptstyle w_{-n}} & & \downarrow & \\ T^{-1}(L^{\#}) = & \cdots 0 & \to & 0 & \to & P^*_{-n} & \xrightarrow[-\partial^*_{-n+1}]{} & \cdots & \xrightarrow[-\partial^*_{n-2}]{} & P^*_{n-2} & \xrightarrow[-\partial^*_{n-1}]{} & P^*_{n-1} & \xrightarrow[-\partial^*_n]{} & P^*_n & \to 0 \cdots \end{array}$$

and satisfy $w^*_{-i-1} = w_i$ for all $i \in \mathbb{Z}$. We may give φ explicitly by:

$$\varphi_i : \underbrace{P^*_{-i-1} \oplus P_{i-1}}_{C(w)_i} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \underbrace{P_{i-1} \oplus P^*_{-i-1}}_{\left(C(w)^{\#}\right)_i}$$

for all $i \in \mathbb{Z}$. It is then a quite easy exercise to compute $\Omega(X, \varphi) = 0$ and to show that it is metabolic. This is left to the reader. ♯

4.22. Remark. This does not insure that neutral forms map to zero since the form of the above lemma is very particular. A neutral form is isometric to such a form by lemma 3.4. The very point is to prove that Ω is well defined up to homotopy equivalence.

4.23. Lemma. Let (X,φ) be a form and $h : Y \to X$ be an isomorphism of complexes (that is $h_i =$ isomorphism for all $i \in \mathbb{Z}$). Then, $\Omega(X,\varphi) = \Omega(Y, h^{\#}\varphi\, h)$.

We have $\Omega(X,-\varphi) = -\Omega(X,\varphi)$ as well.

4.24. Proof: These are easy exercises with the following indications.

For the first assertion, use the obvious diagonal isometry (with h_i, h_i^{-1}, h_i^* or $(h_i^{-1})^*$ in the diagonal) and corollary 4.12.

For the second one, use the isometry $\begin{pmatrix} \ddots & & & & \\ & & & 0 & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & 0 & & & \ddots \end{pmatrix}$. ♯

4.25. Lemma. Let X and Y be bounded complexes and let $\alpha : X \xrightarrow{\sim} Y^{\#}$ be a homotopy equivalence (an isomorphism in our category $\mathrm{K}^{\mathrm{b}}()P$). Then

$$\Omega(X \oplus Y, \begin{pmatrix} 0 & \alpha^{\#} \\ \alpha & 0 \end{pmatrix}) = 0.$$

4.26. Proof: Expliciting the homomorphism of definition 4.4 and regrouping the parts coming from X and the parts coming from Y, it is not too hard to see that

$$\Omega(X \oplus Y, \begin{pmatrix} 0 & \alpha^{\#} \\ \alpha & 0 \end{pmatrix}) \simeq \begin{pmatrix} 0 & \rho^{*} \\ \rho & 0 \end{pmatrix},$$

where ρ is the isomorphism associated to α through the construction of remark 4.7. ♯

4.27. Corollary. Let (X,φ) be a form and let $h : Y \xrightarrow{\sim} X$ be now any isomorphism (h is a homotopy equivalence, compare with lemma 4.23). Then $\Omega(X,\varphi) = \Omega(Y, h^{\#}\varphi\, h)$.

4.28. Proof: Observe that :

$$\Omega(h^{\#}\varphi\, h) - \Omega(\varphi) \overset{\text{lemma 4.23}}{=} \Omega(h^{\#}\varphi\, h) + \Omega(-\varphi) \overset{\text{rem 4.19}}{=} \Omega(\begin{pmatrix} h^{\#}\varphi\, h & 0 \\ 0 & -\varphi \end{pmatrix}).$$

Using elementary transformations and the fact that $\frac{1}{2} \in A$, we can find an isomorphism of complexes g such that

$$g^{\#} \begin{pmatrix} h^{\#}\varphi\, h & 0 \\ 0 & -\varphi \end{pmatrix} g = \begin{pmatrix} 0 & -h^{\#}\varphi \\ -\varphi\, h & 0 \end{pmatrix}.$$

By lemma 4.23, it suffices to show that $\Omega(\begin{pmatrix} 0 & -h^{\#}\varphi \\ -\varphi\, h & 0 \end{pmatrix}) = 0$. In lemma 4.25, take the homotopy equivalence α to be $-\varphi\, h$ to get the result. ♯

4.29. Theorem. The group homomorphism

$$\iota_A \ : \ \mathrm{W}_{\mathrm{us}}(A) \longrightarrow \mathrm{W}(A)$$

is an isomorphism whose inverse is induced by Ω.

4.30. Proof: From corollary 4.27 we get a well defined additive map

$$\Omega : \mathrm{MW}(\mathrm{K}^{\mathrm{b}}(\mathcal{P})) \to \mathrm{W}_{\mathrm{us}}(A).$$

Corollary 4.27 again and lemma 4.20 implies that $\Omega(\mathrm{NW}(\mathrm{K}^{\mathrm{b}}(\mathcal{P}))) = 0$ (see remark 4.22). It is clear that $\Omega \circ \iota_A = \mathrm{Id}$ and since ι_A is already known to be an epimorphism (theorem 4.1), we have the conclusion. ♯

4.31. Corollary. If $\epsilon = \pm 1$

$$\mathrm{W}_0^{\epsilon}(A) = \mathrm{W_{us}}^{\epsilon}(A).$$

♯

4.32. Remark. Using proposition 1.20, we only have four distinct Witt groups, namely : $\mathrm{W}_0^1(X)$, $\mathrm{W}_0^{-1}(X)$, $\mathrm{W}_1^1(X)$ and $\mathrm{W}_1^{-1}(X)$. The first two seem to be strongly related to $\mathrm{W_{us}}(X)$ and $\mathrm{W_{us}}^{-1}(X)$. However, we have no idea of a classical interpretation of the other two, except over a field, where it is possible to show that they are trivial.

4.33. Example. Let A be a noetherian domain in which 2 is a unit. Denote by Q its field of fractions. Let J denote the full subcategory of $\mathrm{K^b}()P$ whose objects are those which become zero in $\mathrm{K^b}(Q)$. These are the complexes $X_\bullet \in \mathrm{K^b}()P$ such that $Q \otimes_A H_i(X) = 0$ for all $i \in \mathbb{Z}$.

Let $x \in \ker(\mathrm{W}(A) \to \mathrm{W}(Q))$. By theorem 4.29, we may suppose that

$$x \in \ker(\mathrm{W_{us}}(A) \to \mathrm{W_{us}}(Q)).$$

Then it is easy to see that there exists a classical form (P, φ), a free module L and a homomorphism $\alpha : L \to P$ with $x = [P, \varphi]$ and such that the following complex, say Y :

$$\cdots \longrightarrow 0 \longrightarrow L \xrightarrow[\alpha]{} P \xrightarrow[\alpha^*\,\varphi]{} L^* \longrightarrow 0 \longrightarrow \cdots$$

is exact when tensorized by Q over A. Apply theorem 3.11 to the sublagrangian (L, α) to find a form ψ on Y such that $[Y, \psi] = [X, \varphi] = x$. Note that (L, α) is a sublagrangian in our sense but not in the usual one, since α is not split. This shows an interesting aspect of the general setting of derived categories.

We have proved the following refreshing result :

4.34. Theorem. The sequence

$$\mathrm{W}(J) \longrightarrow \mathrm{W}(A) \longrightarrow \mathrm{W}(Q)$$

is exact. ♯

4.35. Remark. Moreover, for other reasons, we trust that $\mathrm{W}(J) \to \mathrm{W}(A)$ is injective. This is not proved yet, although it is possible to show that forms on J becoming neutral in A are already trivial in $\mathrm{W}(J)$. The fact that this kernel is the Witt group of a suitable triangulated (sub-) category shows that this extension of the definition of Witt groups leads us to a more complete theory.

5. Localization.

5.1. Definition. Let $(K, \#)$ be a triangulated category with duality. A multiplicative system of morphisms S is called *compatible with the duality* if $\#(s) \in S$ for all $s \in S$.

5.2. Notation. Let $S^{-1}K$ be the localization of K with respect to this system of morphisms. Denote by $q : K \to S^{-1}K$ the localization functor. Denote by $J(S)$ the full subcategory of K on the objects $X \in K$ such that $q(X) = 0$.

5.3. Hypothesis. We suppose S to be *saturated* in the following sense : if $q(t)$ is an isomorphism then $t \in S$.

5.4. Proposition. Let $(K, \#)$ be a triangulated category with duality and S saturated and compatible with $\#$. Then $J(S)$ endowed with the restriction of $\#$ and $S^{-1}K$ endowed with the localization of $\#$ are triangulated categories with duality. Moreover, the natural morphisms

$$\iota : J(S) \to K \qquad \text{et} \qquad q : K \to S^{-1}K$$

are covariant morphisms of triangulated categories with duality.

5.5. Proof: Left to the reader. Everything can be made explicit, using the construction of $S^{-1}K$ with fractions (see [6] § 10.3). ♯

5.6. Remark. By the above proposition, we have a group homomorphism, still denoted by

$$q : \mathrm{W}(K) \longrightarrow \mathrm{W}(S^{-1}K).$$

The goal of this section is to identify the part of $\mathrm{W}(S^{-1}K)$ coming from $\mathrm{W}(K)$. In order to do this, we are going to define a homomorphism $\partial : \mathrm{W}(S^{-1}K) \to \mathrm{W}_1^{-1}(J)$.

Choose a form in $S^{-1}K$, say (X, φ). Since the objects of $S^{-1}K$ are those of K, X is in K. *A priori* φ is a fraction but, up to isometry, it might be taken of the form $q(f)$ with f in K. This f might not be symmetric in K, but (since it is symmetric in $S^{-1}K$) there exists some $s \in S$ such that $f^{\#}\, s = f\, s$. Therefore, replacing f by $s^{\#}\, f\, s$, we may suppose any form of $S^{-1}K$ to be the localization $q(f)$ of some symmetric f in K. Since $q(f)$ is an isomorphism, f belongs to S.

Putting all together, we have proved that any isometry class $x \in \mathrm{MW}(S^{-1}K)$ contains a space of the form $(X, q(s))$ where $s : X \to X^{\#}$, is such that $s = s^{\#}$ and $s \in S$. We are going to construct the connecting homomorphism ∂ on these pairs (X, s) (definition 5.9). Then we will have to prove that this does not depend on the choice of (X, s) in x (theorem 5.15).

5.7. Construction. Let X be an object of K and $s : X \to X^{\#}$ be a morphism in S such that $s^{\#} = s$. Choose an exact triangle on s:

$$X \xrightarrow{\ s\ } X^{\#} \xrightarrow{\ s_1\ } E \xrightarrow{\ s_2\ } T(X)$$

and complete the following diagram (in which the second triangle is the dual of the first one):

$$\begin{array}{ccccccc}
X & \xrightarrow{\ s\ } & X^{\#} & \xrightarrow{\ s_1\ } & E & \xrightarrow{\ s_2\ } & T(X) \\
\| & & \| & & \downarrow{\scriptstyle \exists \psi \,|\, \simeq} & & \| \\
X & \xrightarrow[\ s^{\#}\ ]{} & X^{\#} & \xrightarrow[\ -Ts_2^{\#}\ ]{} & T(E^{\#}) & \xrightarrow[\ Ts_1^{\#}\ ]{} & T(X)
\end{array}$$

Observe that $-T\psi^{\#}$ also makes the diagram commute. Replacing ψ by $\frac{1}{2}(\psi - T\psi^{\#})$ we may suppose that

$$\psi = -T\psi^{\#} : E \xrightarrow{\ \sim\ } T(E^{\#}).$$

In the end notice that since $s \in S$, $E \in J(S)$. Therefore (E, ψ) is a skew-symmetric space in $J(S)$ with respect to the translated duality $T \circ \#$ (see remark 1.5).

5.8. Notation. We set $J = J(S)$ and abbreviate $* = T \circ \#$.

5.9. Definition. Let $s : X \to X^{\#}$ belong to S and satisfy $s = s^{\#}$. Define $\partial(X, s)$ to be the class of (E, ψ) in $\mathrm{W}_1^{-1}(J)$. An adapted version of lemma 3.4 insures that this is well defined.

5.10. Remark. This form (E, ψ) is not neutral, even if the diagram of construction 5.7 is precisely the one which insures that (E, ψ) is neutral *when* $X \in J$. To avoid confusion, recall what are the neutral forms for the skew-duality $*$. From definition 1.9, a neutral form in $\mathrm{MW}_1^{-1}(J)$ is in fact a pair (E, ψ) such that there exists an exact triangle in J (!):

$$D^{\#} \xrightarrow{\ w\ } D \xrightarrow{\ \alpha\ } E \xrightarrow{\ \beta\ } T(D^{\#})$$

and a commutative diagram (in which the second triangle is the dual of the first one):

$$\begin{array}{ccccccc}
D^{\#} & \xrightarrow{\ w\ } & D & \xrightarrow{\ \alpha\ } & E & \xrightarrow{\ \beta\ } & D^{*} \\
\| & & \| & & \downarrow{\scriptstyle \simeq\ \psi} & & \| \\
D^{\#} & \xrightarrow[\ -w^{\#}\ ]{} & D^{**} & \xrightarrow[\ \beta^{*}\ ]{} & E^{*} & \xrightarrow[\ \alpha^{*}\ ]{} & D^{*}
\end{array}
\qquad {\scriptstyle -\mathrm{can}_D}$$

This is equivalent to say that the following diagram commutes:

$$\begin{array}{ccccccc} D^{\#} & \xrightarrow{w} & D & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & D^{*} \\ \| & & \| & & \simeq \downarrow \psi & & \| \\ D^{\#} & \xrightarrow[w^{\#}]{} & D & \xrightarrow[-\beta^{*}]{} & E^{*} & \xrightarrow[\alpha^{*}]{} & D^{*} \end{array} .$$

5.11. Lemma. Let $f : X \to X^{\#}$ be such that $f = f^{\#}$ and let $s : X \to Y^{\#}$ be any morphism in S. Then

$$\partial(X \oplus Y, \begin{pmatrix} f & s^{\#} \\ s & 0 \end{pmatrix}) = 0.$$

5.12. Proof: The proof is left as a non-trivial exercise, the solution of which will be given in a forthcoming article of mine (namely, *Triangular Witt groups. Part I: The 12-term localization exact sequence* Theorem 4.8).

5.13. Corollary. Suppose given (X, s), (Y, t) and $u : Y \to X$ such that $s, t, u \in S$ and $t = u^{\#} s\, u$. Then

$$\partial(X, s) = \partial(Y, t).$$

5.14. Proof: It is easy to show that ∂ is additive with respect to orthogonal sums. Then it suffices to prove that $\partial(X \oplus Y, \begin{pmatrix} -s & 0 \\ 0 & t \end{pmatrix}) = 0$. The result is clear if u is an isomorphism (use again lemma 3.4). Consider $h = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : X \oplus Y \xrightarrow{\sim} X \oplus Y$. It is clear that $h^{\#} \cdot \begin{pmatrix} -s & 0 \\ 0 & t \end{pmatrix} \cdot h$ is a matrix of the form of the above lemma. This gives the result. ♯

5.15. Theorem. Let K be a triangulated category with duality and S a saturated system of morphisms compatible with $\#$. For all $x \in \mathrm{W}(S^{-1}K)$ there exists an object X in K and a morphism $s : X \to X^{\#}$ such that $x = [q(X), q(s)]$. The map sending x to $\partial(X, s)$ induces a well-defined group homomorphism:

$$\partial_{K,S} : \mathrm{W}(S^{-1}K) \longrightarrow \mathrm{W}_1^{-1}(J(S)).$$

5.16. Proof: The first assertion is proved in remark 5.6.

Define an equivalence relation on the pairs (X, s), with $s = s^{\#} : X \to X^{\#}$ in S, by:

$$(X, s) \sim (Y, t) \qquad \text{iff} \qquad (X, s) \text{ and } (Y, t) \text{ are isometric in } S^{-1}K.$$

Check that this relation is generated by:

$$(X, s) \sim (Y, t) \qquad \text{iff} \qquad \text{there exists } u : Y \to X \text{ such that } t = u^{\#} s\, u.$$

Then use corollary 5.13 to induce a monoid homomorphism:

$$\partial : \mathrm{MW}(S^{-1}K) \longrightarrow \mathrm{W}_1^{-1}(J(S)).$$

To end the proof, we only have to show that $\partial(\mathrm{NW}(S^{-1}K)) = 0$. This is very easy once we have observed that the natural homomorphism:

$$q : \mathrm{NW}(K) \longrightarrow \mathrm{NW}(S^{-1}K)$$

is surjective. This comes from the fact that a neutral form is characterized, up to isometry, by the w of definition 1.9 (see remark 3.6). Note that we may change w by $T^{-1}(s^{\#})\, w\, s$ for any $s \in S$ (use lemma 3.4 again!). Thus we may assume w is a morphism in K (and not a fraction). ♯

5.17. Theorem. The sequence

$$\mathrm{W}(K) \xrightarrow{\mathrm{W}(q)} \mathrm{W}(S^{-1}K) \xrightarrow{\partial_{K,S}} \mathrm{W}_1^{-1}(J(S))$$

is exact.

5.18. Proof: It is obvious that the composition is zero since the "cone" of an isomorphism is trivial.

Take an $x \in \ker(\partial_{K,S})$ and choose (X, s) such that $x = [q(X), q(s)]$. By construction, $\partial(X, s) = 0$. Choose an exact triangle:

$$X \xrightarrow{s} X^{\#} \xrightarrow{s_1} E \xrightarrow{s_2} T(X)$$

and a skew-symmetric form ψ on E as in the construction 5.7. In $\mathrm{MW}_1^{-1}(J)$ we have that $(E, \psi) \perp$ some neutral is neutral. Adding to the second space its own opposite, we may suppose that $(E, \psi) \perp$ some (other) neutral is hyperbolic, which means isometric to some $(L \oplus TL^{\#}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. Call the (other) neutral (D, μ). It appears in a triangle of the form

$$N \xrightarrow{w=w^{\#}} N^{\#} \xrightarrow{w_1} D \xrightarrow{w_2} T(N)$$

with $N \in J$! (and $w_2 = w_1^* \mu$). By definition of J, $q(N) = 0$, which means that our original x is equal to $[X \oplus N, \begin{pmatrix} s & 0 \\ 0 & w \end{pmatrix}]$ as well. We may now suppose that (E, ψ) is hyperbolic.

The conclusion follows from the next lemma: ♯

5.19. Lemma. Let $s : X \to X^{\#}$ be in S and such that $s = s^{\#}$. Suppose further that there exists an exact triangle of the form:

$$X \xrightarrow{s} X^{\#} \xrightarrow{s_1 = \begin{pmatrix} t \\ -w^* \end{pmatrix}} L \oplus TL^{\#} \xrightarrow{s_2 = \begin{pmatrix} w & t^* \end{pmatrix}} T(X)$$

for some morphisms $t : X^{\#} \to L$ and $w : L \to T(X)$ and for $L \in J$.

Then there exists a symmetric form (Z, φ) in $\mathrm{W}(K)$ such that $q(Z, \varphi)$ and $(q(X), q(s))$ are isometric.

5.20. Proof: Choose an exact triangle containing t, say:

$$Y \xrightarrow{t_0} X^{\#} \xrightarrow{t} L \xrightarrow{t_2} T(Y)$$

Apply the octahedron axiom to the identity $t = (1\ 0) \cdot s_1$ to get an exact triangle

$$X \xrightarrow{f_1} Y \xrightarrow{w^*\, t_0} TL^{\#} \xrightarrow{-Tt^{\#}} T(X)$$

for some morphism $f_1 : X \to Y$ such that $t_2 = T(f_1)\, w$ and $s = t_0\, f_1$. The two above exact triangles contain t. Comparing them, we find a non necessarily symmetric isomorphism:

$$\begin{array}{ccccccc}
T^{-1}L & \xrightarrow{T^{-1}t_2} & Y & \xrightarrow{t_0} & X^{\#} & \xrightarrow{-t} & L \\
\| & & \simeq \mid \exists\alpha & & \| & & \| \\
T^{-1}L & \xrightarrow[t_0^{\#}\, T^{-1}w]{} & Y^{\#} & \xrightarrow[f_1^{\#}]{} & X^{\#} & \xrightarrow[-t]{} & L
\end{array}$$

Observe that $q(\alpha)$ is already symmetric. In fact, $L \in J$ implies $t_0 \in S$. Since $t_0\, \alpha^{-1} = {f_1}^{\#}$, we have

$$t_0\, \alpha^{-1}\, {t_0}^{\#} = {f_1}^{\#}\, {t_0}^{\#} = (t_0\, f_1)^{\#} = s^{\#} = s.$$

As $q(t_0)$ is an isomorphism, $q(\alpha^{-1})$ is isometric to $q(s)$, hence symmetric. Since for all symmetric form φ, $\varphi \simeq \varphi^{-1}$, $q(\alpha)$ is isometric to $q(s)$.

We still have to improve our α (in fact, we will replace it by $\frac{1}{2}(\alpha + \alpha^{\#})$) to be sure that it will be symmetric (and remain an isomorphism!).

We deduce from the above diagram that $\alpha^{\#} f_1 = {t_0}^{\#}$. Composing on the right with $T^{-1}w$, we get $\alpha^{\#} T^{-1} t_2 = t_0^{\#} T^{-1} w$. Then construct $\gamma : X^{\#} \to X^{\#}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
T^{-1}L & \xrightarrow{T^{-1}t_2} & Y & \xrightarrow{t_0} & X^{\#} & \xrightarrow{-t} & L \\
\| & & \simeq \downarrow{\scriptstyle \alpha^{\#}} & & \simeq \downarrow{\scriptstyle 1+\gamma} & & \| \\
T^{-1}L & \xrightarrow[t_0^{\#}\, T^{-1} w]{} & Y^{\#} & \xrightarrow[f_1^{\#}]{} & X^{\#} & \xrightarrow[-t]{} & L
\end{array}$$

It remains to show that γ is nilpotent! From $t\,\gamma = 0$, we have $\gamma = t_0\,\widetilde{\gamma}$. From the first diagram we find also that $w^*\,\gamma\,t_0 = 0$. In the end, from the original property of f_1 to satisfy $s = t_0\,f_1$ and from $s = s^{\#}$, we get $\gamma\,s = 0$. This last equality insures that $\gamma = \check{\gamma} \circ \begin{pmatrix} t \\ -w^* \end{pmatrix}$.

Now compute

$$\gamma^3 = \check{\gamma} \begin{pmatrix} t \\ -w^* \end{pmatrix} \gamma\, t_0\, \widetilde{\gamma} = \check{\gamma} \begin{pmatrix} t\,\gamma\, t_0 \\ -w^* \gamma\, t_0 \end{pmatrix} \widetilde{\gamma} = 0.$$

Taking the mean of the morphisms $(1, \alpha, 1)$ and $(1, \alpha^{\#}, 1+\gamma)$, we get a morphism of exact triangles $(1, \frac{1}{2}(\alpha + \alpha^{\#}), 1 + \frac{1}{2}\gamma)$. Since $1 + \frac{1}{2}\gamma$ is an isomorphism (as well as 1), $\frac{1}{2}(\alpha + \alpha^{\#})$ is a form as claimed above. Clearly, since $q(\alpha)$ was already symmetric, $q(\frac{1}{2}(\alpha + \alpha^{\#})) = q(\alpha)$ which is isometric to s. ♯

5.21. Application to an open subscheme.

Let X be a regular scheme and $U \subset X$ be an open subscheme. With the notations of section 2, consider the category $\mathcal{A}(U)$ of coherent $\mathcal{O}_U$-modules and the category $\mathcal{A}(X)$ of coherent $\mathcal{O}_X$-modules. Denote by $i : U \to X$ the inclusion. It induces a morphism $\Lambda := i^* : \mathcal{A}(X) \to \mathcal{A}(U)$ which is simply the restriction. Since restriction is exact, this morphism induces an exact morphism on the corresponding derived categories for X and U.

5.22. Theorem. With notations of section 2, $\mathrm{K}(U)$ is a localization of $\mathrm{K}(X)$ with respect to a saturated multiplicative system of morphisms compatible with duality. Moreover, the localization of $\#_X$ to $\mathrm{K}(U)$ is $\#_U$.

5.23. Proof: Since X is regular, $\mathrm{K}(X) \overset{\text{def}}{=} \mathrm{K}^{\mathrm{b}}(\mathcal{L}(X)) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A}(X))$. Since U is regular too, $\mathrm{K}(U) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A}(U))$. Therefore, $\mathrm{K}(U)$ is the localization of $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}(U))$ with respect to quasi-isomorphisms and it suffices to prove that $\mathcal{A}(U)$ is a localization of $\mathcal{A}(X)$ and that every quasi-isomorphism in $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}(X))$ yields a quasi-isomorphism in $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}(U))$. The second assertion is trivial since Λ is exact and the second is well known even for non-regular schemes.

Thus $\mathrm{K}(U)$ is a localization of $\mathrm{K}(X)$ with respect to all morphisms in $\mathrm{K}(X)$ whose restriction to $\mathrm{K}(U)$ is an isomorphism. Such morphisms are precisely those whose restriction to U is a quasi-isomorphism. Then this system arises from a cohomological functor in the sense of [6]. Namely, this cohomological functor is here $H_0 \circ \Lambda$. Therefore S is multiplicative.

The other assertions are left to the reader. ♯

5.24. Corollary. Let X be a regular scheme in which 2 is a unit and let $U \subset X$ be an open subscheme. Let J denote the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{L}(X))$ on those complexes which are acyclic on U. Then there exists a group homomorphism:

$$\partial_{X,U} : \mathrm{W}(U) \longrightarrow \mathrm{W}_1^{-1}(J)$$

such that the sequence

$$\mathrm{W}(X) \xrightarrow{\text{restr.}} \mathrm{W}(U) \xrightarrow{\partial_{X,U}} \mathrm{W}_1^{-1}(J)$$

is exact.

5.25. Proof: This is juxtaposition of theorems 5.17 and 5.22. ♯

6. Purity in dimension 3.

6.1. Theorem. Let $(A,\mathfrak{m})$ be a regular local ring of dimension 3 in which 2 is a unit, Q its field of fractions and $U=\mathrm{Spec}(A)\setminus\{\mathfrak{m}\}$. Consider the commutative diagram:

$$\begin{array}{ccccc} \mathrm{W}_{\mathrm{us}}(A) & \longrightarrow & \mathrm{W}_{\mathrm{us}}(U) & \longrightarrow & \mathrm{W}_{\mathrm{us}}(Q) \\ \| & & \downarrow{\scriptstyle \iota_U} & & \| \\ \mathrm{W}(A) & \xrightarrow[\mathrm{W}(q)]{} & \mathrm{W}(U) & \longrightarrow & \mathrm{W}(Q) \end{array}$$

Then, for all $a\in\mathrm{W}_{\mathrm{us}}(U)$ there exists $b\in\mathrm{W}(A)$ such that $a=b$ in $\mathrm{W}(Q)$.

6.2. Proof: Using the notations of section 5, we denote by J the full subcategory of $\mathrm{K}^{\mathrm{b}}()P$ on the complexes whose restriction to U is acyclic. Abbreviate $\partial=\partial_{X,U}$. By corollary 5.24, it suffices to prove that for all $a\in\mathrm{W}_{\mathrm{us}}(U)$ we have $\partial\circ\iota_U(a)=0$ in $\mathrm{W}_1^{-1}(J)$. We apply construction 5.7 to compute $\partial\big(\iota_U(a)\big)$.

Let $a\in\mathrm{W}_{\mathrm{us}}(U)$. Let $(\mathcal{E},\varphi)$ be a symmetric form on U with $a=[\mathcal{E},\varphi]$. Then $\mathcal{E}$ is a locally free $\mathcal{O}_U$-module and $\varphi=\varphi^*:\mathcal{E}\xrightarrow{\sim}\mathcal{E}^*$. Set $E=\Gamma(\mathcal{E})$ and $f=\Gamma(\varphi)$, where Γ denotes global sections on U. We have (see [3]) that E is reflexive of projective dimension ≤ 1 and $f:E\xrightarrow{\sim}E^*$ is symmetric. Choose a projective resolution of E:

$$0\to P_1\xrightarrow{d_1}P_0\xrightarrow{d_0}E\to 0$$

Check that $\iota_U(a)=[q(X),q(s)])$ where:

$$\begin{array}{ccccccccccc} X= & \cdots 0 & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 & \longrightarrow & 0\cdots \\ {\scriptstyle s}\downarrow & & & \downarrow & & \downarrow{\scriptstyle d_0^*\,f\,d_0} & & \downarrow & & \\ X^{\#}= & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_0^* & \xrightarrow[d_1^*]{} & P_1^* & \longrightarrow & 0\cdots \end{array}$$

It is obvious that $s=s^{\#}$ and that $\Lambda(s)=$ quasi-isomorphism. Therefore (see construction 5.7) $\partial\circ\iota_U(a)=[Y,\psi]$ where:

$$\begin{array}{ccccccccccc} Y= & \cdots 0 & \longrightarrow & P_1 & \xrightarrow{-d_1} & P_0 & \xrightarrow{-d_0^*\,f\,d_0} & P_0^* & \xrightarrow{d_1^*} & P_1^* & \longrightarrow 0\cdots \\ {\scriptstyle\psi}\downarrow & & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\ T(Y^{\#})= & \cdots 0 & \longrightarrow & P_1 & \xrightarrow[-d_1]{} & P_0 & \xrightarrow[d_0^*\,f\,d_0]{} & P_0^* & \xrightarrow[d_1^*]{} & P_1^* & \longrightarrow 0\cdots \end{array}$$

Using twice the periodicity of the Witt groups, we may suppose that Y "starts" in degree 0 and that $\psi:Y\to T^{-3}(Y^{\#})$. Verify that $\mathrm{H}_0(Y)=\mathrm{H}_1(Y)=\mathrm{H}_2(Y)=0$. Put $M=\mathrm{H}_3(Y)$. We may identify $\mathrm{Ext}^3(M,A)$ with $\mathrm{H}_3\big(T^{-3}(Y^{\#})\big)$ and it is easy to show that $\rho=\mathrm{H}_3(\psi)$ is a skew-symmetric form on M with respect to the duality $\check{M}=\mathrm{Ext}^3_A(M,A)$. See [4] for precisions.

Since the Witt group of skew-symmetric forms over finite length A-modules is zero: $\mathrm{W}_{\mathrm{lf}}^{-1}(A)\simeq\mathrm{W}_{\mathrm{lf}}^{-1}(A/\mathfrak{m})=0$, this form (M,ρ) is metabolic. Use an exact sequence

$$0\longrightarrow N\xrightarrow{i}M\xrightarrow{\check{i}\,\rho}\check{N}\longrightarrow 0$$

and a resolution of N to prove that (Y,ψ) is neutral in J with duality $*=T\circ\#_A$. ♯

6.3. Remark. Details about the Witt group of skew-symmetric forms over finite length A-modules can be found in [4], where an *ad hoc* construction is used in place of our general theorem 5.17. In fact, our proof 6.2 is a "dramatic simplification" (M. Ojanguren *dixit*) of the one given there.

6.4. Corollary. The natural map

$$\mathrm{W}(A) \longrightarrow \mathrm{W}_{\mathrm{us}}(U)$$

is surjective.

6.5. Proof : It suffices to know that $\mathrm{W}_{\mathrm{us}}(U) \longrightarrow \mathrm{W}(Q)$ is injective (see [2]) and to apply theorem 6.1. ♯

6.6. Remark. This could be applied to show that, under the hypotheses of theorem 6.1, the sequence :

$$\mathrm{W}(A) \longrightarrow \mathrm{W}(Q) \xrightarrow[\text{residue}]{\text{second}} \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \text{of height } 1}} \mathrm{W}(\kappa(\mathfrak{p}))$$

is exact. In fact, it is not too hard to show that any $a \in \mathrm{W}(Q)$ which is in the kernel of the residue is in the image of $\mathrm{W}_{\mathrm{us}}(U) \to \mathrm{W}(Q)$. Theorem 6.1 gives the result.

6.7. Post-scriptum. At the moment this article was going to press, the editors kindly allowed me some final modifications. I would like to use this opportunity to mention that in the forthcoming series "Triangular Witt groups", the notations will be slightly changed into : W^1 for the present W_1^- *et cætera*. This is justified by the long exact sequence and by compatibility with existing notations.

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