PATCH-DENSITY IN TENSOR-TRIANGULAR GEOMETRY

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Abstract. The spectrum of a tensor-triangulated category carries a compact Hausdorff topology, called the constructible topology, also known as the patch topology. We prove that patch-dense subsets detect tt-ideals and we prove that any infinite family of tt-functors that detects nilpotence provides such a patch-dense subset. We review several applications and examples in algebra, in topology and in the representation theory of profinite groups.

1. Introduction

In this short note we sharpen our understanding of two fundamental themes of tensor-triangular geometry: the classification of thick tensor-ideals and the detection of tensor-nilpotence. The origins of the subject can be traced back to the Nilpotence Theorem of Devinatz–Hopkins–Smith [DHS88] in topology. Using Morava K-theories $K(n)$ at a prime p, they prove that a morphism $f\colon k \to L$ in the p-local stable homotopy category $\mathrm{SH}_{(p)}$, with finite source k, must be $\otimes$-nilpotent if $K(n)(f) = 0$ for all $0 \le n \le \infty$; here, $K(\infty)$ means mod-p homology. This theorem led to a classification of the thick subcategories of finite p-local spectra in [HS98], as being exactly the so-called 'chromatic' tower:

$$\mathrm{SH}^c_{(p)} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \cdots \supsetneq \ \mathcal{C}_n = \mathrm{Ker}\left(K(n-1)_*\right) \ \supsetneq \cdots \supsetneq \mathcal{C}_\infty = 0 \tag{1.1}$$

Already in this initial example, we can point to the germ of what we wish to discuss. On the one hand, in the Nilpotence Theorem, if the morphism $f\colon k \to \ell$ also has finite target ℓ then we do not need to know that $K(\infty)(f) = 0$ to conclude that f is $\otimes$-nilpotent. On the other hand, in the chromatic tower (1.1), the smallest subcategory $\mathcal{C}_\infty$ is not really 'seen' by any finite object: If $k \in \mathrm{SH}^c_{(p)}$ belongs to all $\mathcal{C}_n$ for $n < \infty$ then it belongs to $\mathcal{C}_\infty$ as well. In other words, there is no finite object that has infinite chromatic level. In both cases, the 'stuff at ∞' seems somewhat irrelevant. To explain the parallels between these two phenomena let us remind the reader of some elementary tt-geometry.

Let $\mathcal{K}$ be an essentially small rigid tt-category, such as $\mathrm{SH}^c_{(p)}$ above. To this, we can associate a space $\mathrm{Spc}(\mathcal{K})$, called the spectrum, that affords the universal support theory $\mathrm{supp}(k) \subseteq \mathrm{Spc}(\mathcal{K})$ for $\mathcal{K}$, see [Bal05]. A support theory for $\mathcal{K}$ on a space X consists of closed subsets $\sigma(k) \subseteq X$ for each object $k \in \mathcal{K}$ that behave in a predictable manner as one operates on k through the tensor triangulated structure. The supports in $\mathrm{Spc}(\mathcal{K})$ induce a classification of thick tensor-ideals:

$$\{\,\text{tt-ideals in } \mathcal{K}\,\} \xrightarrow[\sim]{\ \mathrm{supp}\ } \{\,\text{Thomason subsets of } \mathrm{Spc}(\mathcal{K})\,\}.$$

Date: 2025 March 19.

2020 *Mathematics Subject Classification.* 18F99.

Key words and phrases. constructible topology, patch topology, tensor-triangular geometry, density, detection of nilpotence.

Being universal means that any support theory (X, σ) is classified by a continuous map $\phi\colon X \to \operatorname{Spc}(\mathcal{K})$ such that $\sigma(k) = \phi^{-1}(\operatorname{supp}(k))$ for all $k \in \mathcal{K}$. Of particular importance among all support theories (X, σ) are those that *distinguish supports*, in the sense that $\sigma(k) = \sigma(\ell)$ forces $\operatorname{supp}(k) = \operatorname{supp}(\ell)$. This means that the classifying map $\phi\colon X \to \operatorname{Spc}(\mathcal{K})$ might not be a homeomorphism but the supports in X are a fine enough invariant to distinguish tt-ideals. Let us rephrase this property using point-set topology.

Recall that the *constructible* topology on $\operatorname{Spc}(\mathcal{K})$ – a. k. a. the *patch* topology – is generated by the quasi-compact opens and their complements. (Recollection 2.2.) It is easy to see that a support theory (X, σ) distinguishes supports if and only if the image of the classifying map $\phi\colon X \to \operatorname{Spc}(\mathcal{K})$ is *patch-dense*, *i.e.* dense for the constructible topology. For example, any patch-dense subset $X \subseteq \operatorname{Spc}(\mathcal{K})$ equipped with $\sigma(k) := X \cap \operatorname{supp}(k)$ distinguishes supports. We can now state the main result of this paper, proved in Section 3:

1.2. **Theorem.** *Let $\mathcal{K}$ be an essentially small rigid tt-category and consider a family $\{F_i\colon \mathcal{K} \to \mathcal{L}_i\}_{i \in I}$ of tt-functors. The following are equivalent:*

(i) *The tt-functors $F_i\colon \mathcal{K} \to \mathcal{L}_i$ jointly detect $\otimes$-nilpotence of morphisms that are $\otimes$-nilpotent on their cones.*

(ii) *The subset $\cup_i \operatorname{Im}(\operatorname{Spc}(F_i)) \subseteq \operatorname{Spc}(\mathcal{K})$ is patch-dense.*

(ii') *The maps $\operatorname{Spc}(F_i)\colon \operatorname{Spc}(\mathcal{L}_i) \to \operatorname{Spc}(\mathcal{K})$ jointly distinguish supports.*

For a single tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ this theorem recovers the surjectivity result of [Bal18]; see Remark 3.3. Our proof is a generalization of that proof.

Let us go back to the example at the start of this introduction. The classification in the chromatic tower (1.1) translates into $\operatorname{Spc}(\mathrm{SH}^c_{(p)})$ being the space

$$\mathcal{C}_1 \rightsquigarrow \mathcal{C}_2 \rightsquigarrow \cdots \rightsquigarrow \mathcal{C}_n \rightsquigarrow \mathcal{C}_{n+1} \rightsquigarrow \cdots \rightsquigarrow \mathcal{C}_\infty \tag{1.3}$$

in which the closed subsets are precisely the subsets closed under specialization ($\rightsquigarrow$ going towards the right in the above picture). The observations made earlier about the irrelevance of $K(\infty)$ and of $\mathcal{C}_\infty$ are equivalent to the patch-density of the complement $X = \operatorname{Spc}(\mathrm{SH}^c_{(p)}) \smallsetminus \{\mathcal{C}_\infty\} = \{\, \mathcal{C}_n \,|\, 1 \le n < \infty \,\}$ of the closed point at infinity. In fact, this X is the only patch-dense proper subset of $\operatorname{Spc}(\mathrm{SH}^c_{(p)})$.

A second goal of this article is to identify situations where patch-density occurs. We emphasize the following implication that is particularly interesting:

1.4. **Corollary.** *If $\mathcal{K}$ is rigid and a family of tt-functors $\{F_i\colon \mathcal{K} \to \mathcal{L}_i\}_{i \in I}$ jointly detects $\otimes$-nilpotence then the subset $\cup_i \operatorname{Im}(\operatorname{Spc}(F_i)) \subseteq \operatorname{Spc}(\mathcal{K})$ is patch-dense.*

There are numerous instances in the recent literature where the space $\operatorname{Spc}(\mathcal{K})$ is difficult to describe explicitly but where it is much easier to describe some natural patch-dense subset. We explain in Theorem 2.8 how to recover $\operatorname{Spc}(\mathcal{K})$ from such a patch-dense subset X together with the restricted support theory supp_X on X, defined by $\operatorname{supp}_X(k) = X \cap \operatorname{supp}(k)$ for all $k \in \mathcal{K}$. Several examples of these phenomena will be given in Section 4. We also point out the recent paper by Gómez [Gó25] where families of tt-functors as above are used to study stratification of 'big' tt-categories.

Acknowledgements. We thank Scott Balchin and Tobias Barthel for interesting discussions during the preparation of this work.

2. Patch-density

2.1. *Notation.* We denote by Spec the category of spectral spaces in the sense of Hochster [Hoc69] with continuous spectral maps as morphisms.

2.2. *Recollection.* Let X be a spectral space, like for instance $X = \mathrm{Spc}(\mathcal{K})$. The collection $\mathcal{QO}(X)$ of all quasi-compact open subsets is an open basis of X. Their complements form an open (!) basis of the so-called dual topology X^*. We also speak of *Thomason subset* for an open in X^*. The smallest topology on the set X that contains both the original and the dual topology is called the *constructible* (or *patch*) topology, and is denoted X_{con}. It is generated by quasi-compact opens in X and their complements, and is always a Boolean space (hence compact Hausdorff). To be precise, a subset of X is called *constructible* if it can be obtained by finite union and finite intersection from the quasi-compact opens of X and their complements. A constructible-open is an arbitrary union of constructibles; a constructible-closed (or *proconstructible*) is an arbitrary intersection of constructibles. For more on this see [DST19, § 1.3]. Note that a subset $D \subset X$ is *patch-dense*, *i.e.* dense in X_{con}, if and only if D meets every non-empty constructible in X. Our textbook reference [DST19] uses 'patch dense' and 'constructibly dense' interchangeably.

2.3. *Remark.* Let $\mathcal{K}$ be an essentially small rigid tt-category and consider the spectral space $\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} \subsetneq \mathcal{K} \text{ prime tt-ideal}\}$. Each object $k \in \mathcal{K}$ comes with a closed subset $\mathrm{supp}(k) = \left\{ \mathcal{P} \,\middle|\, k \notin \mathcal{P} \right\}$. Its complement $U(k) = \mathrm{supp}(k)^c$ is quasi-compact, and all quasi-compact opens are of this form. See [Bal05].

Let us write $C(k, \ell) = \mathrm{supp}(k) \cap \mathrm{supp}(\ell)^c = \left\{ \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\middle|\, k \notin \mathcal{P} \text{ and } \ell \in \mathcal{P} \right\}$ for every $k, \ell \in \mathcal{K}$. It follows from $C(k, \ell) \cap C(k', \ell') = C(k \otimes k', \ell \oplus \ell')$ that these constructible subsets $C(k, \ell)$ form a basis of the constructible topology on $\mathrm{Spc}(\mathcal{K})$. Thus a subset $D \subseteq \mathrm{Spc}(\mathcal{K})$ is patch-dense if and only if D meets $C(k, \ell)$ for every $k, \ell \in \mathcal{K}$ such that $\mathrm{supp}(k) \nsubseteq \mathrm{supp}(\ell)$.

2.4. *Definition.* A family of maps $\phi_i \colon Y_i \to \mathrm{Spc}(\mathcal{K})$ is said to *(jointly) distinguish supports* if the following implication holds, for any $k, \ell \in \mathcal{K}$:

$$\phi_i^{-1}(\mathrm{supp}(k)) = \phi_i^{-1}(\mathrm{supp}(\ell)) \text{ for all i} \quad \Rightarrow \quad \mathrm{supp}(k) = \mathrm{supp}(\ell).$$

In view of Remark 2.3, this is equivalent to the subset $\cup_i \mathrm{Im}(\phi_i)$ intersecting non-trivially every non-empty $C(k, \ell)$, that is, to the purely topological condition that the subset $\cup_i \mathrm{Im}(\phi_i)$ be patch-dense in $\mathrm{Spc}(\mathcal{K})$.

2.5. *Example.* Let $D \subseteq \mathrm{Spc}(\mathcal{K})$ be a subspace. Consider on D the restricted support theory for $\mathcal{K}$, defined by $\mathrm{supp}_D(k) = D \cap \mathrm{supp}(k)$ for every object $k \in \mathcal{K}$. Then D is patch-dense if and only if the map $\{\mathcal{J} \subseteq \mathcal{K} \text{ radical tt-ideal}\} \longrightarrow \mathrm{Subsets}(D)$ mapping $\mathcal{J}$ to $\cup_{k \in \mathcal{J}} \mathrm{supp}_D(k)$ is injective. (Recall that $\mathcal{J}$ radical means $k^{\otimes s} \in \mathcal{J} \Rightarrow k \in \mathcal{J}$; this condition is automatic if $\mathcal{K}$ is rigid.) Indeed, distinguishing supports means distinguishing principal (radical) tt-ideals $\langle k \rangle$, and the latter characterize all (radical) tt-ideals since $\mathcal{J} = \cup_{k \in \mathcal{J}} \langle k \rangle$.

2.6. **Lemma.** *Let X be a spectral space. Let D be a set and $i \colon D \to X$ a function (e.g. the inclusion of a subset). Then the following are equivalent:*

(i) *The image $i(D)$ is patch-dense in X.*
(ii) *For every quasi-compact open $U, V \subseteq X$, if $i^{-1}(U) \subseteq i^{-1}(V)$ then $U \subseteq V$.*
(iii) *For every pair of spectral maps $\alpha, \beta \colon X \to Z$ to a spectral space Z, if $\alpha\, i = \beta\, i$ then $\alpha = \beta$.*

Proof. Note that (ii) is simply a reformulation of patch-density (i): The constructible subsets $U \cap V^c$ form a basis of the constructible topology and (ii) says that such $U \cap V^c$ can only avoid $i(D)$ when $U \cap V^c$ is empty.

(i)$\Rightarrow$(iii) is easy. The spectral maps $\alpha, \beta \colon X \to Z$ are also continuous for the constructible topologies, which are Hausdorff. Hence if α and β agree on the dense subset $i(D)$ they are equal.

For (iii)$\Rightarrow$(ii), let $U, V \subseteq X$ be two quasi-compact open such that $U \not\subseteq V$ and let us show that $i^{-1}(U) \not\subseteq i^{-1}(V)$. Consider the Sierpiński space $Z = \{0 \rightsquigarrow 1\}$ with 1 the closed point and define maps $\alpha, \beta \colon X \to Z$ by asking that α sends $U \cup V$ to 0 and the rest to 1, whereas β sends V to 0 and the rest to 1. These are continuous spectral maps. The maps α and β are different on the subset $U \cap V^c \neq \varnothing$. By (iii), $\alpha\, i(x) \neq \beta\, i(x)$ for some $x \in D$. Hence the element $i(x)$ belongs to $U \cap V^c$, showing that $i^{-1}(U) \ni x$ is not contained in $i^{-1}(V) \not\ni x$. $\square$

We can reconstruct a spectral space from a patch-dense subset together with the trace of the quasi-compact opens of the ambient space.

2.7. *Hypothesis.* Let X be a set with a chosen non-empty collection $\mathcal{U}$ of subsets, closed under finite intersections and finite unions. (In particular, $\mathcal{U}$ contains X and $\varnothing$.)

Let us say that a map $f \colon X \to X'$ to a spectral space X' is *spectral* (with respect to $\mathcal{U}$) if $f^{-1}(U) \in \mathcal{U}$ for every quasi-compact open $U \in \mathcal{QO}(X')$. Note that if X is spectral and $\mathcal{U} = \mathcal{QO}(X)$, then f being spectral is the usual definition (including continuity). The *spectral closure* $\overline{X}^{\mathcal{U}}$ of X with respect to $\mathcal{U}$ is the initial spectral map $f \colon X \to \overline{X}^{\mathcal{U}}$ to a spectral space $\overline{X}^{\mathcal{U}}$; in other words, any other spectral map $f' \colon X \to X'$ factors as $f' = \bar{f}' \circ f$ for a unique spectral map $\bar{f}' \colon \overline{X}^{\mathcal{U}} \to X'$.

2.8. **Theorem.** *Let $(X, \mathcal{U})$ be a pair as in Hypothesis 2.7.*

(a) *The spectral closure $\overline{X}^{\mathcal{U}}$ of X with respect to $\mathcal{U}$ does exist (and is unique up to unique homeomorphism compatible with $X \to \overline{X}^{\mathcal{U}}$, as usual).*

(b) *Let $i \colon X \to X'$ be a (set-)function to a spectral space X' (e.g. an inclusion) and suppose that $\mathcal{U} = \left\{ i^{-1}(U) \,\middle|\, U \in \mathcal{QO}(X') \right\}$ is exactly the 'restriction' to X of the quasi-compact opens of X'. Then the spectral closure $\overline{X}^{\mathcal{U}}$ with respect to $\mathcal{U}$ can be realized as the constructible-closure $\overline{\mathrm{Im}(i)}^{\mathrm{con}}$ of the image of X in X'.*

Proof. Part (a) is standard but we could not locate a precise reference in the literature. Let Spec' be the category of pairs $(X, \mathcal{U})$ as in Hypothesis 2.7, with obvious morphisms (functions pulling back chosen subsets to chosen subsets). There is an obvious functor $\mathcal{Q} \colon \mathsf{Spec} \to \mathsf{Spec}'$ sending a spectral space X to $(X, \mathcal{QO}(X))$, which is moreover fully faithful, by definition of spectral maps. We prove that $\mathcal{Q}$ admits a left adjoint $\mathcal{P}$ thus proving (a) with $\overline{X}^{\mathcal{U}} := \mathcal{P}(X, \mathcal{U})$. Observe that Spec is locally small, well-powered [DST19, Corollary 5.2.8] and has a cogenerator [DST19, Corollary 1.2.8] and that Spec' is locally small. In both Spec and Spec', limits are computed by taking limits in Sets and adding the obvious structure. It follows easily that the functor $\mathcal{Q}$ preserves limits. By the Special Adjoint Functor Theorem [ML98, Theorem V.8.2] this functor has a left adjoint $\mathcal{P}$.

For (b), let $Y = \overline{\mathrm{Im}(i)}^{\mathrm{con}}$ be the constructible-closure of X in X' and $j \colon X \to Y$ the function $i \colon X \to X'$ corestricted. By [DST19, Theorem 2.1.3] the subspace Y of X is a spectral subspace. The map $j^* \colon \mathcal{QO}(Y) \to \mathcal{U}$ that maps U to $j^{-1}(U)$ is surjective by hypothesis on $\mathcal{U}$ and the fact that $Y \subseteq X'$ is a spectral subspace. By Lemma 2.6 (i)⇒(ii) this map $j^*\colon \mathcal{QO}(Y) \to \mathcal{U}$ is also injective. So j^* is bijective. On the other hand, let $\eta\colon X \to \overline{X}^{\mathcal{U}}$ be the unit of the $\mathcal{P} \dashv \mathcal{Q}$ adjunction. For every spectral space Z, precomposing with η defines the bijection of the adjunction $\mathsf{Spec}(\overline{X}^{\mathcal{U}}, Z) \xrightarrow{\sim} \mathsf{Spec}'(X, \mathcal{Q}Z)$. We can therefore apply Lemma 2.6 (iii)⇒(ii) to the function $\eta\colon X \to \overline{X}^{\mathcal{U}}$ to see that $\eta^*\colon \mathcal{QO}(\overline{X}^{\mathcal{U}}) \to \mathcal{U}$, defined by $U \mapsto \eta^{-1}(U)$, is injective. Finally, by the universal property applied to $j\colon X \to Y$, there exists a spectral map $\bar{j}\colon \overline{X}^{\mathcal{U}} \to Y$ such that $\bar{j} \circ \eta = j$. We can again consider $\bar{j}^*\colon \mathcal{QO}(Y) \to \mathcal{QO}(\overline{X}^{\mathcal{U}})$ for this third map, still defined by $U \mapsto \bar{j}^{-1}(U)$, and we get a commutative diagram of lattices (for inclusion):

$$\begin{tikzcd}
\mathcal{QO}(Y) \arrow[rr, "\bar{j}^*"] \arrow[dr, "j^*"'] & & \mathcal{QO}(\overline{X}^{\mathcal{U}}) \arrow[dl, "\eta^*"] \\
& \mathcal{U} &
\end{tikzcd}$$

We have shown that j^* is bijective and that η^* is injective. Hence $\bar{j}^*$ is bijective. It follows from Stone duality [DST19, 3.2.10] that $\bar{j}$ is a homeomorphism. □

2.9. *Remark.* One can also give an explicit formula for $\overline{X}^{\mathcal{U}}$ as the constructible-closure of the image of the spectral map $\mathrm{ev}_X\colon X \to \prod_{U\in\mathcal{U}}\{0 \rightsquigarrow 1\}$ that is defined by $(\mathrm{ev}_X(x))(U) = 0$ if and only if $x \in U$. Indeed, that product provides a spectral space X' as in Theorem 2.8 (b).

A direct consequence of Theorem 2.8 (b) in tt-geometry is the following:

2.10. **Corollary.** *Let $\mathcal{K}$ be an essentially small tt-category and let $X \subseteq \mathrm{Spc}(\mathcal{K})$ be a patch-dense subset. Then the space $\mathrm{Spc}(\mathcal{K})$ can be reconstructed as the spectral closure $\overline{X}^{\mathcal{U}}$ of the pair $(X,\mathcal{U})$ for $\mathcal{U} = \{\, X \cap \mathrm{supp}(k)^c \,|\, k \in \mathcal{K}\,\}$. In particular, $\mathrm{Spc}(\mathcal{K})$ can be reconstructed from the restricted support theory (X, supp_X), where $\mathrm{supp}_X(k) = X \cap \mathrm{supp}(k)$ for every $k \in \mathcal{K}$.* □

3. Detecting tensor-nilpotence

3.1. *Recollection.* (a) A morphism $f\colon k \to \ell$ in a tensor category is said to be *⊗-nilpotent* if $f^{\otimes s}\colon k^{\otimes s} \to \ell^{\otimes s}$ is zero for $s \gg 0$.
(b) More generally f is *⊗-nilpotent on an object* m if $f^{\otimes s} \otimes m\colon k^{\otimes s} \otimes m \to \ell^{\otimes s} \otimes m$ is zero for $s \gg 0$. This explains the meaning of f being *⊗-nilpotent on its cone.*
(c) A family of tensor functors $\{F_i\colon \mathcal{K} \to \mathcal{L}_i\}_{i\in I}$ is said to *(jointly) detect ⊗-nilpotence* (on a class of morphisms) if for each morphism $f\colon k \to \ell$ (in that class), we have the following implication:

$$F_i(f) \text{ is } \otimes\text{-nilpotent for all } i \in I \quad \Rightarrow \quad f \text{ is } \otimes\text{-nilpotent.}$$

3.2. **Theorem.** *Let $\mathcal{K}$ be an essentially small rigid tt-category and let $\{F_i\colon \mathcal{K} \to \mathcal{L}_i\}_{i\in I}$ be a family of tt-functors. The following are equivalent:*

(i) *The tt-functors $F_i\colon \mathcal{K} \to \mathcal{L}_i$ jointly detect ⊗-nilpotence of morphisms that are ⊗-nilpotent on their cones.*
(ii) *The maps $\mathrm{Spc}(F_i)\colon \mathrm{Spc}(\mathcal{L}_i) \to \mathrm{Spc}(\mathcal{K})$ jointly distinguish supports.*
(iii) *The maps $\mathrm{Spc}(F_i)\colon \mathrm{Spc}(\mathcal{L}_i) \to \mathrm{Spc}(\mathcal{K})$ are jointly epimorphic in the category of spectral spaces.*
(iv) *The subset $\cup_{i\in I} \mathrm{Im}(\mathrm{Spc}(F_i)) \subseteq \mathrm{Spc}(\mathcal{K})$ is patch-dense.*

3.3. *Remark.* In [Bal18, Theorem 1.3], a special case of Theorem 3.2 was proved, namely when the family consists of a single tt-functor $F\colon \mathcal{K} \to \mathcal{L}$; in that case, the map $\mathrm{Spc}(F)$ is surjective. Indeed, the image of a single spectral map is always a closed subset for the constructible topology [DST19, Corollary 1.3.23].

Let us do a little preparation for the proof of Theorem 3.2.

3.4. *Remark.* Let $\mathcal{L}$ be an essentially small tt-category, that is not assumed to be rigid. For every rigid object k in $\mathcal{L}$, we consider $k^\vee$ the dual of k and we denote by A_k the ring object $k\otimes k^\vee \cong \mathsf{hom}(k,k)$, by $\eta_k\colon \mathbb{1} \to A_k$ its unit (a. k. a. coevaluation) and $\xi_k\colon J_k \to \mathbb{1}$ its homotopy fiber, so that we have an exact triangle in $\mathcal{L}$,

$$J_k \xrightarrow{\xi_k} \mathbb{1} \xrightarrow{\eta_k} A_k \longrightarrow \Sigma J_k\,. \tag{3.5}$$

Let us recall a few standard observations from [Bal10].

(a) By the unit-counit relation in the adjunction $(k\otimes -) \dashv (k^\vee \otimes -)$ the map $\eta_k \otimes k$ is a split monomorphism, hence k is a summand of $k\otimes k^\vee \otimes k$ and $\xi_k \otimes k = 0$.
(b) The tt-ideal generated by k, or equivalently by A_k, is also the nilpotence locus of ξ_k, that is, $\langle k\rangle = \langle A_k\rangle = \big\{\, m\in\mathcal{L} \,\big|\, \exists\, s \gg 1 \text{ s.t. } \xi_k^{\otimes s}\otimes m = 0 \,\big\}$.
(c) Any tt-functor preserves rigidity, duals, η_k, etc. So once Construction (3.5) is performed in one category (*e.g.* the $\mathcal{K}$ in the theorem), its properties get transported by any tt-functor $F\colon\mathcal{K}\to\mathcal{L}$ (*e.g.* the F_i), even if $\mathcal{L}$ is not assumed rigid.

Let us improve on (b):

(d) Let $\ell\in\mathcal{L}$ be another rigid object and consider the map $f = \xi_k\otimes\ell\colon J_k\otimes\ell\to\ell$. Then f is $\otimes$-nilpotent (in fact zero) on its cone, which is $A_k\otimes\ell$, because of (a). Furthermore, we have

$$f \text{ is } \otimes\text{-nilpotent if and only if } \ell\in\langle k\rangle.$$

Indeed, if $0 = f^{\otimes s}\colon J_k^{\otimes s}\otimes\ell^{\otimes s}\to\ell^{\otimes s}$, we see that $\ell^{\otimes s}\in\langle\mathrm{cone}(f^{\otimes s})\rangle\subseteq\langle\mathrm{cone}(f)\rangle\subseteq\langle\mathrm{cone}(\xi_k)\rangle = \langle A_k\rangle = \langle k\rangle$ and therefore $\ell\in\langle k\rangle$, since ℓ is rigid. Conversely, if $\ell\in\langle k\rangle = \langle A_k\rangle$ we already know that ξ_k is nilpotent on ℓ by (b). This proves the claim. By rigidity again, the condition $\ell\in\langle k\rangle$ is equivalent to $\mathrm{supp}(\ell)\subseteq\mathrm{supp}(k)$.

Proof of Theorem 3.2. We abbreviate $Y_i = \mathrm{Spc}(\mathcal{L}_i)$ and $X = \mathrm{Spc}(\mathcal{K})$ with $\phi_i = \mathrm{Spc}(F_i)\colon Y_i\to X$. The equivalence between (ii) and (iv) was already explained in Definition 2.4. The equivalence between (iii) and (iv) follows easily from Lemma 2.6 applied to $D = \cup_i \mathrm{Im}(\phi_i)$. It suffices to prove that (i) is equivalent to (ii).

For (i)$\Rightarrow$(ii), let $k,\ell\in\mathcal{K}$ such that $\mathrm{supp}(\ell)\not\subseteq\mathrm{supp}(k)$. Let $f = \xi_k\otimes\ell$ as in Remark 3.4 (d) which tells us that f is not $\otimes$-nilpotent, although it is $\otimes$-nilpotent on its cone. By our assumption (i), there must exist some $i\in I$ such that $F_i(f)$ is not $\otimes$-nilpotent. Let $k_i = F_i(k)$ and $\ell_i = F_i(\ell)$ in $\mathcal{L}_i$, which are rigid objects, with $k_i^\vee\cong F_i(k^\vee)$ and $\ell_i^\vee\cong F_i(\ell^\vee)$. Under these identifications, $\xi_{k_i} = F_i(\xi_k)$ and $F_i(f) = \xi_{k_i}\otimes\ell_i$. We can thus apply Remark 3.4 (d) to k_i and ℓ_i in $\mathcal{L}_i$ and deduce that $\phi_i^{-1}(\mathrm{supp}(\ell)) = \mathrm{supp}(\ell_i)\not\subseteq\mathrm{supp}(k_i) = \phi_i^{-1}(\mathrm{supp}(k))$. This shows that the family $(\phi_i)_{i\in I}$ jointly distinguishes supports.

The proof of (ii)$\Rightarrow$(i) is a straightforward adaptation of the proof in [Bal18, Theorem 1.4]. Indeed, if $f\colon k\to\ell$ is a morphism in $\mathcal{K}$ that is $\otimes$-nilpotent on its cone and $F_i(f)$ is $\otimes$-nilpotent, say $F_i(f^{\otimes s_i}) = 0$, then as in *loc. cit.* we deduce that

$$\phi_i^{-1}(\mathrm{supp}(\mathrm{cone}(f^{\otimes s_i}))) = \phi_i^{-1}\left(\mathrm{supp}(k^{\otimes s_i})\cup\mathrm{supp}(\ell^{\otimes s_i})\right).$$

We now observe that these supports do not change if we replace s_i by 1; for the left-hand side see [Bal18, Proposition 2.10]. In other words, we have, for all $i \in I$:

$$\phi_i^{-1}(\mathrm{supp}(\mathrm{cone}(f))) = \phi_i^{-1}(\mathrm{supp}(k) \cup \mathrm{supp}(\ell)) = \phi_i^{-1}(\mathrm{supp}(k \oplus \ell)).$$

By our assumption we get

$$\mathrm{supp}(\mathrm{cone}(f)) = \mathrm{supp}(k \oplus \ell) = \mathrm{supp}(k) \cup \mathrm{supp}(\ell)$$

and one concludes that f is $\otimes$-nilpotent by a standard argument: $k, \ell \in \langle \mathrm{cone}(f) \rangle$ forces $f \colon k \to \ell$ to be $\otimes$-nilpotent on k (and ℓ) hence to be $\otimes$-nilpotent. □

4. Examples

Let us show that patch-dense subsets commonly arise in nature.

4.A. **Visible locus.**

4.1. *Example.* Consider the tt-category of finite p-local spectra $\mathrm{SH}^c_{(p)}$ for some prime p. This example was already discussed in the introduction so we will be brief and content ourselves with providing references. A morphism $f \colon k \to \ell$ in $\mathrm{SH}^c_{(p)}$ is $\otimes$-nilpotent if and only if $K(n)_*(f)$ is for all $\infty > n \geq 0$, where $K(n)$ denotes the nth Morava K-theory (at the prime p). This follows from the Nilpotence Theorem of Devinatz, Hopkins, Smith in the form of [HS98, Theorem 3.iii), Corollary 2.2]. This corresponds to the fact that the 'finite' points in the spectrum are patch-dense, see (1.3).

We turn this into a more general observation, using that the 'finite' points are precisely the weakly visible ones:

4.2. *Recollection.* Recall that a subspace $Y \subseteq X$ of a spectral space X is called *weakly visible* if $Y = V \cap W^c$ for two Thomason subsets $V, W \subseteq X$. If $\mathcal{T}$ is a rigidly-compactly generated tt-category, then the weakly visible subsets in $\mathrm{Spc}(\mathcal{T}^c)$ are those that can be described in terms of $\otimes$-idempotents in $\mathcal{T}$, as in [BF11]. For a theory of stratification of $\mathcal{T}$ based on this the reader can consult [BHS23].

4.3. **Proposition.** *Let X be a spectral space. The set of weakly visible points in X is patch-dense.*

Proof. Let $W = U \cap V^c$ be a non-empty basic open for the constructible topology, with U, V quasi-compact open in X and let $x \in W$. Since the closed subspace V^c is itself a spectral space there is a point $y \in V^c$ with $y \rightsquigarrow x$ and which has no further generalizations in V^c [DST19, Corollary 4.1.4]. As U is open we have $y \in W$ so it suffices to show that y is weakly visible. By construction, $\{y\} = V^c \cap \mathrm{gen}\{y\}$, where gen denotes the set of generalizations. We conclude since V^c is Thomason and the set $\mathrm{gen}\{y\} = \cap_{O \in \mathcal{Q}\mathcal{O}(X) | y \in O} O$ is the complement of a Thomason. □

4.4. *Example.* For an algebraic situation similar to Example 4.1, let R be a local commutative ring whose maximal ideal $\mathfrak{m}$ is not finitely generated, not even up to radicals. (For example, the local ring at a closed point in infinite affine space.) This is equivalent to the closed point $\{\mathfrak{m}\}$ not being (weakly) visible in $\mathrm{Spec}(R)$ in the sense of Recollection 4.2. By Proposition 4.3, the punctured spectrum $\mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$ is patch-dense. This corresponds to the fact that $\otimes$-nilpotence of maps in $\mathrm{D}_{\mathrm{perf}}(R)$ is detected by the residue field functors $- \otimes_R \kappa(\mathfrak{p})$ for the non-maximal primes $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$.

4.5. *Remark.* In both Examples 4.1 and 4.4, the complement of the closed point(s) was patch-dense. This is not possible if the space X is noetherian. Indeed, when X is noetherian, the open complement $X \smallsetminus \{x\}$ of every closed point x is quasi-compact, as every open is; this implies that $\{x\}$ is open for the constructible topology. Hence any patch-dense subset must contain all the closed points.

In fact, for noetherian spaces there is a *smallest* patch-dense subset. It consists precisely of the locally closed points [DST19, Proposition 4.5.21, Corollary 8.1.19].

4.6. *Example.* A (spectral) space X is called *Jacobson* if its closed points are dense in every closed subset. For instance, the underlying space of every scheme locally of finite type over a field, or locally of finite type over $\mathbb{Z}$, is a Jacobson space.

In a Jacobson space, a subset $D \subseteq X$ that contains all closed points is necessarily patch-dense. Indeed, let $Y^c \cap Z \neq \varnothing$ be a non-trivial basic constructible, with Y and Z closed subsets with quasi-compact complement. Since the open $Y^c \cap Z$ of Z is non-empty, it must contain a closed point of X, which is in D by assumption.

In view of Remark 4.5, in 'usual' algebraic geometry, say, when dealing with schemes of finite type over a field, or of finite type over $\mathbb{Z}$, a subset is patch-dense if and only if it contains all the closed points.

4.B. **Retractable limits.**

4.7. *Notation.* Let $\mathcal{K} = \operatorname{colim} \mathcal{K}_i$ be the directed colimit of essentially small tt-categories. Let $X_i := \operatorname{Spc}(\mathcal{K}_i)$ and $X = \operatorname{Spc}(\mathcal{K})$. We denote by $\pi_i^*\colon \mathcal{K}_i \to \mathcal{K}$ the canonical tt-functor, and by $\pi_i\colon X \to X_i$ the induced map on spectra. Recall from [Gal18, Proposition 8.2] that the maps π_i induce a homeomorphism

$$X \xrightarrow{\sim} \lim_i X_i. \tag{4.8}$$

Let us assume that each π_i^* admits a tt-retraction

$$\sigma_i^*\colon \mathcal{K} \to \mathcal{K}_i,$$

so that $\pi_i \circ \sigma_i = \mathrm{id}\colon X_i \to X_i$ for all i, where $\sigma_i := \operatorname{Spc}(\pi_i^*)$.

4.9. **Corollary.** *The family $\{\sigma_i^*\colon \mathcal{K} \to \mathcal{K}_i\}$ jointly detects $\otimes$-nilpotence. In particular, if $\mathcal{K}$ is rigid then the subset $\cup_i \operatorname{Im}(\sigma_i) \subseteq \operatorname{Spc}(\mathcal{K})$ is patch-dense.*

Proof. Let $f\colon k \to \ell$ be a morphism in $\mathcal{K}$ such that $\sigma_i^*(f)$ is $\otimes$-nilpotent for all i. We may choose i such that $f = \pi_i^*(f')$ for some $f'\colon k' \to \ell'$ in $\mathcal{K}_i$. But then $f' = \sigma_i^* \circ \pi_i^*(f') = \sigma_i^*(f)$ is $\otimes$-nilpotent hence so is f. The second statement follows from Theorem 3.2. □

We now discuss some examples of this result.

4.C. **Profinite equivariance.**

4.10. *Example.* Let G be a profinite group. The tt-category of finite genuine G-spectra is the directed colimit

$$\mathrm{SH}(G)^c = \operatorname*{colim}_N \mathrm{SH}(G/N)^c$$

where $N \trianglelefteq G$ runs through open normal subgroups, and the transition is given by inflation functors, see [BBB24]. For any such $N \trianglelefteq G$, consider the geometric fixed points functor $\Phi^N\colon \mathrm{SH}(G)^c \to \mathrm{SH}(G/N)^c$ which gives a retraction to inflation. It follows from (the easy part of) Corollary 4.9 that the geometric fixed points functors jointly detect $\otimes$-nilpotence.[1] The geometric fixed points for closed subgroups induce a bijection [BBB24, Proposition 7.4]

$$\mathrm{Sub}(G)/G \times \mathrm{Spc}(\mathrm{SH}^c) \xrightarrow{\sim} \mathrm{Spc}(\mathrm{SH}(G)^c).$$

On the other hand, we deduce from the second part of Corollary 4.9 that the subset

$$\mathrm{Sub}^{\mathrm{open}}(G)/G \times \mathrm{Spc}(\mathrm{SH}^c) \subseteq \mathrm{Spc}(\mathrm{SH}(G)^c)$$

corresponding to *open* subgroups is already patch-dense. (One could combine this with Example 4.1 to exhibit an even smaller patch-dense subset.) Note that this subset is directly linked to the tt-geometry of equivariant spectra for *finite* groups.

4.11. *Example.* We continue to denote by G a profinite group. Let k be a field of characteristic p. The tt-category of compact derived permutation modules is the directed colimit

$$\mathrm{DPerm}(G;k)^c = \operatorname*{colim}_{N \trianglelefteq G} \mathrm{DPerm}(G/N;k)^c$$

along the inflation functors, see [BG23a] or [BG25]. In this case, it is the *modular fixed points* $\Psi^N \colon \mathrm{DPerm}(G) \to \mathrm{DPerm}(G/N)$ of [BG23b] that yield retractions to inflation. The spectrum admits a set-theoretic stratification

$$\mathrm{Spc}(\mathrm{DPerm}(G)^c) = \coprod_{H \le G} \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \tag{4.12}$$

where $G/\!\!/H = N_G(H)/H$ is the Weyl group. It follows again from Corollary 4.9 that a patch-dense subset is given by the strata in (4.12) for $H \le G$ *open*.

4.D. **Support varieties.**

4.13. *Example.* Let G again be a profinite group, and k a field of characteristic p. The bounded derived category of finite-dimensional k-linear (discrete) G-representations is the directed colimit

$$\mathrm{D_b}(\mathrm{mod}(G;k)) = \operatorname*{colim}_{N \trianglelefteq G} \mathrm{D_b}(\mathrm{mod}(G/N;k))$$

along inflation. It follows easily from the case of finite groups [BCR97] that the spectrum is an invariant of the cohomology algebra:

$$\mathrm{Spc}(\mathrm{D_b}(\mathrm{mod}(G;k))) = \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^\bullet(G;k)), \tag{4.14}$$

see [Gal19, Proposition 6.5]. Here we do not have a retraction $\mathrm{D_b}(\mathrm{mod}(G;k)) \to \mathrm{D_b}(\mathrm{mod}(G/N;k))$ to inflation in general so this does not fit in the framework of Section 4.B. Nevertheless, there is another natural family of functors that jointly detects $\otimes$-nilpotence. At the level of cohomology algebras the statement is that the ring maps

$$\mathrm{Res}_E \colon \mathrm{H}^\bullet(G;k) \to \mathrm{H}^\bullet(E;k)$$

for *finite* elementary abelian p-subgroups $E \le G$ detects which elements are nilpotent, see [Sch96, Proposition 8.7] or [MS04, Theorem 1]. We now upgrade this to a categorical statement.

[1] In fact, it is known [BBB24, Corollary 8.7] that the geometric fixed points functors $\{\Phi^H\}_{H \le G}$ (running through closed subgroups H) jointly detect $\otimes$-nilpotence for morphisms $f \colon k \to L$ of genuine G-spectra in which only k is assumed compact.

4.15. **Proposition.** *Let G be a profinite group. The family of restriction functors*

$$\mathrm{Res}_E\colon\ \mathrm{D_b}(\mathrm{mod}(G;k)) \to \mathrm{D_b}(\mathrm{mod}(E;k))$$

to finite elementary abelian p-subgroups $E \le G$ jointly detects $\otimes$-nilpotence. Hence their images on spectra cover a patch-dense subset

$$\cup_E \operatorname{Im}(\mathrm{Spc}(\mathrm{Res}_E)) \subseteq \mathrm{Spc}(\mathrm{D_b}(\mathrm{mod}(G;k))).$$

Proof. Let f be a morphism in $\mathrm{D_b}(\mathrm{mod}(G;k))$ such that $\mathrm{Res}_E(f)$ is $\otimes$-nilpotent for all finite elementary abelian p-subgroups $E \le G$. There is an open normal subgroup $N \trianglelefteq G$ such that f is inflated from some g in $\mathrm{D_b}(\mathrm{mod}(G/N;k))$. By [MS04, Proposition 1], there exists $N' \trianglelefteq N$ another open normal subgroup such that for every elementary abelian p-subgroup $F' \le G/N'$ its image $F := F'N/N$ in G/N is the image of a finite elementary abelian p-subgroup E of G, that is, $F = EN/N$. For each such F' we have

$$\mathrm{Res}^{G/N'}_{F'} \mathrm{Infl}^{G/N}_{G/N'}(g) = \mathrm{Infl}^{F}_{F'} \mathrm{Res}^{G/N}_{F}(g). \tag{4.16}$$

To prove that f is $\otimes$-nilpotent it suffices to show $\mathrm{Infl}^{G/N}_{G/N'}(g)$ is, hence by (4.16) and Quillen, that $\mathrm{Res}^{G/N}_{F}(g) = \mathrm{Res}^{G/N}_{EN/N}(g)$ is (for all F'). While inflation typically is not faithful, it is along surjections between elementary abelian groups because the latter are split. Hence it suffices to show $\otimes$-nilpotence of

$$\mathrm{Infl}^{EN/N}_{E} \mathrm{Res}^{G/N}_{EN/N}(g) = \mathrm{Res}^{G}_{E} \mathrm{Infl}^{G/N}_{G}(g) = \mathrm{Res}^{G}_{E}(f),$$

which was exactly our assumption. □

4.17. *Example.* Let $G = (C_p)^{\mathbb{N}}$ be a countably infinite pro-elementary abelian group, cf. [BG25, Example 6.10]. Then $\mathrm{Spc}(\mathrm{D_b}(\mathrm{mod}(G;k))) = \overline{\mathbb{P}}^{\infty}_k$ is an infinite-dimensional projective space, together with a unique closed point attached. The patch-dense subset $\cup_E \operatorname{Im}(\mathrm{Spc}(\rho_E))$ is given by $\cup_n \overline{\mathbb{P}}^n_k$, those points with only finitely many 'non-zero coordinates'.

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