COHOMOLOGICAL MACKEY 2-FUNCTORS

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Abstract. We show that the bicategory of finite groupoids and right-free permutation bimodules is a quotient of the bicategory of Mackey 2-motives introduced in [BD20], obtained by modding out the so-called cohomological relations. This categorifies Yoshida's Theorem for ordinary cohomological Mackey functors, and provides a direct connection between Mackey 2-motives and the usual blocks of representation theory.

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1. Introduction

We consider Mackey 2-functors in the sense of [BD20] and develop three themes. The first two are general. They consist of a decategorification result and a new simpler description of the Mackey 2-motives of [BD20]. The third theme is the more specific purpose of this paper, the study of *cohomological* Mackey 2-functors. Let us now explain in some detail the interplay of these three topics.

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A *Mackey 2-functor* $\mathcal{M}$ consists of additive categories $\mathcal{M}(G)$ depending on finite groupoids G and whose variance in G is reminiscent of ordinary Mackey functors [Web00]. This categorification, recalled in Section 2, involves restriction functors $u^*\colon \mathcal{M}(G) \to \mathcal{M}(K)$ for every morphism $u\colon K \to G$ and *two-sided* adjoints

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$i_! = i_* \colon \mathcal{M}(H) \to \mathcal{M}(G)$ to i^* for every faithful $i \colon H \rightarrowtail G$. The commutation of $i_!$ and u^* is governed by a *2-Mackey formula* (2.4). As discussed at length in [BD20, Ch. 4], this type of structure has been used for a long time in a variety of settings. For instance, in representation theory, the category $\mathcal{M}(G) = \mathrm{Mod}(\Bbbk G)$ of $\Bbbk G$-modules is an example of a Mackey 2-functor. In topology, the equivariant stable homotopy category $\mathcal{M}(G) = \mathrm{SH}(G)$ of genuine G-spectra is another one.

Recall [Web00, § 7] that an ordinary Mackey (1-) functor M is *cohomological* if, for every subgroup $H \le G$, the composite $I_H^G \, R_H^G \colon M(G) \to M(G)$ of the restriction homomorphism $R_H^G \colon M(G) \to M(H)$ with the induction (transfer) homomorphism $I_H^G \colon M(H) \to M(G)$ is equal to multiplication by the index $[G\,{:}\,H]$ on the abelian group $M(G)$. It might be tempting to define a cohomological Mackey 2-functor $\mathcal{M}$ as one for which the composite of $i^* \colon \mathcal{M}(G) \to \mathcal{M}(H)$ with $i_! \colon \mathcal{M}(H) \to \mathcal{M}(G)$ is some form of multiplication by $[G\,{:}\,H]$ on $\mathcal{M}(G)$. This is not a good idea, however, if only because of a lack of interesting examples. We propose here a definition that shifts the cohomological relation to the level of 2-cells, as follows:

1.1. *Definition.* A Mackey 2-functor $\mathcal{M}$ is *cohomological* if, whenever $i \colon H \rightarrowtail G$ is the inclusion of a subgroup H in a finite group G, the composite

$$\mathrm{Id}_{\mathcal{M}(G)} \xRightarrow{\ {}^r\eta\ } \mathrm{Ind}_H^G \mathrm{Res}_H^G \xRightarrow{\ {}^\ell\varepsilon\ } \mathrm{Id}_{\mathcal{M}(G)}$$

equals multiplication by the index $[G\,{:}\,H]$, where ${}^r\eta$ is the unit of the right adjunction $\mathrm{Res}_H^G = i^* \dashv i_* = \mathrm{Ind}_H^G$ and ${}^\ell\varepsilon$ the counit of the left one $\mathrm{Ind}_H^G = i_! \dashv i^* = \mathrm{Res}_H^G$.

The Mackey 2-functors arising in representation theory are often cohomological, like the above $\mathcal{M}(G) = \mathrm{Mod}(\Bbbk G)$ or the derived version $\mathcal{M}(G) = \mathrm{D}(\mathrm{Mod}(\Bbbk G))$. We present further examples of a more geometric nature in Section 5.

The reader should be warned that the standard decategorification of a Mackey 2-functor via the Grothendieck group K_0, as in [BD20, § 2.5], does *not* necessarily turn cohomological Mackey 2-functors into cohomological Mackey 1-functors (see Remarks 5.8 and 5.9). To restore such a connection we propose in Section 3 a different type of decategorification, that works as follows:

1.2. **Theorem** (Hom-Decategorification)**.** *Let $\mathcal{M} \colon \mathsf{gpd}^{\mathrm{op}} \to \mathsf{ADD}$ be a Mackey 2-functor. Let G be a finite group and let $X, Y \in \mathcal{M}(G)$ be two objects. Then there is an ordinary Mackey functor $M = M_{\mathcal{M},G,X,Y}$ for G whose value on every subgroup $H \le G$ is given by the abelian group*

$$M(H) = \mathrm{Hom}_{\mathcal{M}(H)}(\mathrm{Res}_H^G X \,,\, \mathrm{Res}_H^G Y).$$

This is Theorem 3.7 specialized to Example 3.8. We emphasize that this result works without the label 'cohomological' and is thus of interest in the generality of [BD20]. When $\mathcal{M}$ is moreover cohomological then all its Hom-decategorifications are cohomological in the classical sense (Theorem 5.6).

Several examples of Hom-decategorification are discussed in Section 5. For instance, group cohomology (the ur-example of a cohomological Mackey functor) is a Hom-decategorification of the derived category Mackey 2-functor $G \mapsto \mathrm{D}(\Bbbk G)$, itself a prime example of a cohomological Mackey 2-functor.

We also prove in Section 5 a descent result for cohomological Mackey 2-functors; see Theorem 5.10.

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Let us now turn our attention to Mackey motives. Fix a commutative ring $\Bbbk$ of coefficients, for instance $\Bbbk = \mathbb{Z}$. Suppose from now on that all our Mackey 2-functors take values in idempotent-complete $\Bbbk$-linear additive categories $\mathcal{M}(G)$ and $\Bbbk$-linear functors u^*. In [BD20], we constructed a $\Bbbk$-linear bicategory $\mathsf{Mot}_\Bbbk$ (there denoted '$\Bbbk\widehat{\mathsf{Span}}$') of *Mackey 2-motives*, together with a canonical contravariant embedding of the 2-category of finite groupoids

$$\mathsf{mot}\colon \mathsf{gpd}^{\mathrm{op}} \hookrightarrow \mathsf{Mot}_\Bbbk\,. \tag{1.3}$$

It enjoys the following universal property: Every Mackey 2-functor $\mathcal{M}$ factors as $\mathcal{M} \cong \widehat{\mathcal{M}} \circ \mathsf{mot}$ for a unique $\Bbbk$-linear pseudo-functor $\widehat{\mathcal{M}}$ on $\mathsf{Mot}_\Bbbk$. The original construction of the bicategory $\mathsf{Mot}_\Bbbk$ is pretty involved, with spans of 1-cells *and* spans of 2-cells. (See Recollection 4.1.) We prove in Section 4 that there is a simpler description of $\mathsf{Mot}_\Bbbk$. Again, this holds beyond the cohomological world.

1.4. **Theorem** (Mackey 2-motives via bisets)**.** *General $\Bbbk$-linear Mackey 2-motives are modeled by the block-completion $(\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}})^\flat$ of the bicategory $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ whose objects are finite groupoids, 1-morphisms are right-free finite bisets between them, and 2-morphisms are $\Bbbk$-linear combinations of spans of equivariant maps.*

Block-completion $(-)^\flat$ is the 2-categorical analogue of idempotent-completion, a standard feature of motivic constructions. It simply adds formal summands, both at the 0- and 1-level, in order to split idempotent 2-cells; see Recollection 2.2.

Theorem 1.4 categorifies the usual equivalence between Webb's *inflation functors* [Web93] and Bouc's *right-free biset functors* [Bou10]. See Remark 4.28.

It is natural to look for an analogous motivic construction with *cohomological* Mackey 2-functors. Since these are just Mackey 2-functors satisfying some additional relations at the level of 2-cells, one can obtain the corresponding bicategory of *cohomological Mackey 2-motives* $\mathsf{Mot}_\Bbbk^{\mathsf{coh}}$ by formally modding out in $\mathsf{Mot}_\Bbbk$ the relevant 2-cells. Thus formulated, $\mathsf{Mot}_\Bbbk^{\mathsf{coh}}$ remains rather mysterious and one of our main goals is to give a simple computable description. It comes as a categorification and generalization of Yoshida's Theorem [Yos83]; see Remark 6.17.

1.5. **Theorem** (Cohomological Mackey 2-motives)**.** *Cohomological $\Bbbk$-linear Mackey 2-motives are modeled by the block-completion $(\mathsf{biperm}_\Bbbk^{\mathsf{rf}})^\flat$ of the bicategory $\mathsf{biperm}_\Bbbk^{\mathsf{rf}}$ whose objects are finite groupoids, 1-morphisms are right-free permutation bimodules, and 2-morphisms are equivariant $\Bbbk$-linear maps (see Definition 2.13). In other words, there is a canonical pseudo-functor $\mathsf{mot}^{\mathsf{coh}}\colon \mathsf{gpd}^{\mathrm{op}} \to (\mathsf{biperm}_\Bbbk^{\mathsf{rf}})^\flat =: \mathsf{Mot}_\Bbbk^{\mathsf{coh}}$ sending a groupoid to itself and a functor $u\colon H \to G$ to the G,H-bimodule $\Bbbk[G(u-,-)]\colon\ H^{\mathrm{op}} \times G \to \mathrm{Mod}(\Bbbk)$, such that every $\Bbbk$-linear cohomological Mackey 2-functor $\mathcal{M}$ factors uniquely through this pseudo-functor up to isomorphism.*

$$\begin{tikzcd} \mathsf{gpd}^{\mathrm{op}} \arrow[rr, "\mathcal{M}"] \arrow[dr, "\mathsf{mot}^{\mathsf{coh}}"'] & & \mathsf{ADD}_\Bbbk^{\mathsf{ic}} \\ & \mathsf{Mot}_\Bbbk^{\mathsf{coh}} \arrow[ur, dashed, "\exists!\,\widehat{\mathcal{M}}"'] & \end{tikzcd}$$

A proof and more details on these constructions can be found in Section 6. See in particular the 2-categorical universal property of $\mathsf{Mot}_\Bbbk^{\mathsf{coh}}$ in Theorem 6.20.

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The organization of the paper is as follows. In the first part, we discuss general Mackey 2-functors, recalling the definitions in Section 2. We introduce Hom-decategorification in Section 3 and prove Theorem 1.2. We revisit Mackey 2-motives in Section 4 and prove Theorem 1.4.

In the second part of the paper, from Section 5 onwards, we focus on *cohomological* Mackey 2-functors, starting with easy properties and the first examples and applications. Then, Section 6 contains the construction of *cohomological Mackey 2-motives* and the proof of Theorem 1.5. In the short final Section 7, we discuss motivic decompositions in $\mathsf{Mot}_{\Bbbk}$ and $\mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}$ and we compare them in terms of certain explicit ring maps (Theorem 7.5). As another application, we establish that each value category $\mathcal{M}(G)$ of a cohomological Mackey 2-functor $\mathcal{M}$ admits a canonical decomposition in terms of the blocks of the group algebra (Theorem 7.4).

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2. Recollections

We fix a commutative ring $\Bbbk$ throughout the article.

2.1. *Terminology.* We use the language of bicategories, 2-categories (*i.e.* strict bicategories), pseudo-functors etc., in a standard way as recalled in [BD20, App. A]. For simplicity, a $\Bbbk$-*linear* category means an *additive* category enriched over $\Bbbk$-modules. Similarly, a $\Bbbk$-*linear* bicategory $\mathcal{B}$ means one in which all Hom categories $\mathcal{B}(X,Y)$ and all horizontal composition functors are $\Bbbk$-linear, and which admits finite direct sums of objects. A pseudo-functor $\mathcal{F}$ is $\Bbbk$-*linear* if each functor $\mathcal{F}_{X,Y}$ is $\Bbbk$-linear – thus automatically preserves directs sums. We call a contravariant pseudo-functor *additive* if it turns coproducts of objects into products, as in (Mack 1) below.

2.2. *Recollection.* An additive category is *idempotent-complete* if every idempotent endomorphism $e = e^2$ on an object X has an image, so that $X \cong \mathrm{Im}(e) \oplus \mathrm{Im}(1-e)$ identifying e with the matrix $\mathrm{diag}(1,0)$. Recall from [BD20, § A.7] that a $\Bbbk$-linear bicategory $\mathcal{B}$ is *block-complete* if its Hom categories are idempotent-complete and if every decomposition of an identity 2-cell $\mathrm{id}_{\mathrm{Id}_X}$ in orthogonal idempotent 2-cells induces a direct sum decomposition of the object X; so idempotent 2-cells split 1-cells and objects in directs sums. Every $\Bbbk$-linear bicategory $\mathcal{B}$ admits a *block-completion* $\mathcal{B} \hookrightarrow \mathcal{B}^\flat$, the universal $\Bbbk$-linear pseudo-functor into a block-complete bicategory $\mathcal{B}^\flat$. The precise construction of $\mathcal{B}^\flat$ is not difficult but requires some work. It can be found in [BD20, A.7.22]. Suffice it to say that it is a categorification of Karoubi's classical idea to replace objects X by pairs (X,e) where $e = e^2\colon X \to X$ is an idempotent on X, suitably adapted to the 2-categorical setting, *i.e.* applied to both 0-cells (objects) and to 1-cells (as objects of the Hom-categories).

2.3. *Definition.* We recall that a ($\Bbbk$-*linear*) *Mackey 2-functor* is the data of a 2-functor $\mathcal{M}\colon \mathsf{gpd}^{\mathrm{op}} \to \mathsf{ADD}_{\Bbbk}$ from the 2-category of finite groupoids, functors and natural transformations to the 2-category of (possibly large) $\Bbbk$-linear additive categories, additive functors and natural transformations. It inverts the direction of 1-cells, so that we have a *restriction* functor $u^* = \mathcal{M}(u)\colon \mathcal{M}(G) \to \mathcal{M}(H)$ for every morphism (functor) of groupoids $u\colon H \to G$, and we have a natural isomorphism $\alpha^*\colon u^* \Rightarrow v^*$ for every natural isomorphism $u \Rightarrow v$. This is subject to four axioms:

(Mack 1) *Additivity:* $\mathcal{M}(G_1 \sqcup G_2) \xrightarrow{\sim} \mathcal{M}(G_1) \times \mathcal{M}(G_2)$ for all $G_1, G_2 \in \mathsf{gpd}$.

(Mack 2) *Adjoints:* For every faithful morphism $i\colon H \rightarrowtail G$, the restriction functor $i^*\colon \mathcal{M}(G) \to \mathcal{M}(H)$ admits a left adjoint $i_!$ and a right adjoint i_*.

(Mack 3) *Mackey formulas:* For every *Mackey square* (a.k.a. pseudo-pullback, homotopy pullback; see [BD20, Ch. 2.1-2]) as on the left-hand side below

$$\begin{tikzcd}[column sep=small] & P \arrow[dl, "v"'] \arrow[dr, "j"] & \\ H \arrow[dr, "i"'] \arrow[rr, phantom, "\overset{\sim}{\underset{\gamma}{\Rightarrow}}"] & & K \arrow[dl, "u"] \\ & G & \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{c} j_! \circ v^* \xRightarrow[\gamma_!]{\simeq} u^* \circ i_! \\[1em] u^* \circ i_* \xRightarrow[(\gamma^{-1})_*]{\simeq} j_* \circ p^* \end{array} \tag{2.4}$$

where i and (thus) j are faithful, the two mates $\gamma_!$ and $(\gamma^{-1})_*$ for the adjunctions of (Mack 2) are both isomorphisms as displayed above.

(Mack 4) *Ambidexterity:* There exists a natural isomorphism $i_! \cong i_*$ between the left and right adjoints of every faithful i.

We may occasionally want to replace the source gpd of a Mackey 2-functor with a more general 2-category 'of groupoids', *cf.* Remark 4.5 or [BD20, Hyp. 5.1.1].

2.5. *Convention.* Unless otherwise stated, our Mackey 2-functors take values in the sub-2-category $\mathsf{ADD}^{\mathsf{ic}}_{\Bbbk} \subset \mathsf{ADD}_{\Bbbk}$ of idempotent-complete additive categories. (In any case, every Mackey 2-functor can always be idempotent-completed termwise.)

2.6. *Convention.* By the Rectification Theorem [BD20, Ch. 3], the two adjunctions $i_! \dashv i^* \dashv i_*$ can be chosen to satisfy some extra properties. For instance, we may take $i_! = i_*$ as functors (obvious from (Mack 4)). Moreover, we may choose units and counits so that the base change isomorphisms in (Mack 3) are mutual inverses

$$(\gamma_!)^{-1} = (\gamma^{-1})_* \tag{2.7}$$

and so that the pseudo-functors $(-)_!$ and $(-)_*$ on $\mathbb{J}^{\mathrm{co}}$ induced by the left, resp. the right, adjunctions ([BD20, A.2.10]) are the same. We will assume throughout all Mackey 2-functors to be *rectified*, *i.e.* to come with such (uniquely determined) superior choice of left and right adjunctions. (We will assume this even in the more general setting of Section 3, where the Rectification Theorem may not apply.) Their units and counits will be denoted ${}^{\ell}\eta\colon \mathrm{Id} \Rightarrow i^*i_!$ and ${}^{\ell}\varepsilon\colon i_!i^* \Rightarrow \mathrm{Id}$ (for the left adjunction $i_! \dashv i^*$) and ${}^{r}\eta\colon \mathrm{Id} \Rightarrow i_*i^*$ and ${}^{r}\varepsilon\colon i^*i_* \Rightarrow \mathrm{Id}$ (for the right one $i^* \dashv i_*$).

* * *

We make use of two closely related bicategories: that of *bisets* and that of *bimodules*. We briefly recall these well-known notions in order to establish notation.

2.8. *Recollection* (Bisets). Let G and H be finite groupoids. By a (*finite*) G,H*-biset* $S = {}_GS_H$ we mean a functor $S\colon H^{\mathrm{op}} \times G \to \mathrm{set}$ to the category of finite sets. We will often write

$$g \cdot s = S(\mathrm{id}, g)(s) \quad \text{and} \quad s \cdot h = S(h, \mathrm{id})(s)$$

to denote the 'left action' of a morphism $g \in G(x, x')$ and the 'right action' of a morphism $h \in H(y', y)$ on an element $s \in S(y, x)$. We denote by biset the bicategory with finite groupoids as objects, all G,H-bisets ${}_GS_H$ as 1-cells $H \to G$, and all equivariant maps (*i.e.* natural transformations) $\alpha\colon S \Rightarrow T$ as 2-cells.

The horizontal composition of bisets is provided by the tensor product of functors (a.k.a. set-theoretic coends). Concretely, the value at $(z, x) \in K^{\mathrm{op}} \times G$ of a composite biset $({}_G T_H) \circ ({}_H S_K) = T \times_H S$ is the following coequalizer of sets:

$$(2.9)\quad \big(T \times_H S\big)(z,x) = \mathrm{coeq}\left(\coprod_{\substack{y' \to y \\ \in \mathrm{Mor}\, H}} T(y,x) \times S(z,y') \rightrightarrows \coprod_{\substack{y \\ \in \mathrm{Obj}\, H}} T(y,x) \times S(z,y) \right)$$

Even more concretely, an element of $(T \times_H S)(z,x)$ is the equivalence class of a pair $(t,s) \in T(y,x) \times S(z,y)$ for some $y \in \mathrm{Obj}\, H$, and two pairs (t,s) and (t',s') are equivalent if and only if there exists a morphism $h \in H(y,y')$ such that $(t, h \cdot s) = (t' \cdot h, s')$. We will write $[t,s]$ for such a class, or sometimes $[t,s]_y$ or "$[t,s]$ at y" if we need to keep track of the object $y \in \mathrm{Obj}\, H$. The actions of G and K on $T \times_H S$ are the evident $g \cdot [t,s] = [g \cdot t, s]$ and $[t,s] \cdot k = [t, s \cdot k]$.

The identity biset of G is the Hom-functor $\mathrm{Id}_G = G(-,-) \colon G^{\mathrm{op}} \times G \to \mathrm{set}$. (By defining bisets and bimodules (below) as functors on $H^{\mathrm{op}} \times G$, rather than the perhaps more common $G \times H^{\mathrm{op}}$, we think of them as 'generalized Hom-functors'.)

A G,H-biset ${}_G S_H$ is *right-free* if the right action of H is free in the usual sense: $s \cdot h = s \Rightarrow h = \mathrm{id}$, for any $(y,x) \in H^{\mathrm{op}} \times G$, $s \in S(y,x)$ and $h \in H(y',y)$. It is a straightforward exercise to see that the tensor product $T \times_H S$ of two right-free bisets is again right-free. Thus right-free bisets form a 2-full sub-bicategory of biset which we denote $\mathsf{biset}^{\mathsf{rf}}$.

2.10. *Remark.* For finite groups, seen as one-object groupoids, the notions of bisets and their composition in Recollection 2.8 agree with those of Bouc [Bou10]. The full subcategory of groups in the 1-truncation $\tau_1(\mathsf{biset})$ is the ordinary category of bisets of *loc. cit.*, and *ditto* for right-free bisets. (*Cf.* [Del22a].)

2.11. *Recollection* (Bimodules). Let G and H be finite groupoids. A *G,H-bimodule* is a functor $M \colon H^{\mathrm{op}} \times G \to \mathrm{Mod}(\Bbbk)$. We denote by $\mathsf{Bimod}_\Bbbk$ the bicategory with finite groupoids as objects, G,H-bimodules as 1-cells $H \to G$, and equivariant maps as 2-cells. The horizontal composition of bimodules is given by the usual tensor product, *i.e.* $\Bbbk$-linearly enriched coends (as in (2.9) but taking the coequalizer in $\mathrm{Mod}(\Bbbk)$)

$$({}_G M_H) \circ ({}_H N_K) = M \otimes_{\Bbbk H} N,$$

that we simply denote $M \otimes_H N$. A (*finite*) *permutation G,H-bimodule* is a G,H-bimodule which admits a finite G,H-invariant basis: There exist finite sets $S(y,x)$ for all $(y,x) \in H^{\mathrm{op}} \times G$ which are collectively stable under the G- and H-actions and such that each $S(y,x)$ is a basis of the (free) $\Bbbk$-module $M(y,x)$.

Permutation bimodules are closed under tensor products, hence form a sub-bicategory $\mathsf{biperm}_\Bbbk$ of $\mathsf{Bimod}_\Bbbk$. Of course ordinary permutation $\Bbbk G$-modules are simply the essential image of G-sets inside $\Bbbk G$-modules, under $\Bbbk$-linearization. Extended to bicategories, this takes the following form:

2.12. **Proposition** (Linearization)**.** *There is a well-defined and canonical pseudo-functor* $\Bbbk[-] \colon \mathsf{biset} \to \mathsf{biperm}_\Bbbk$ *mapping a groupoid G to itself, a G,H-biset U to the G,H-bimodule $\Bbbk[U]$ defined by taking the free $\Bbbk$-module termwise:* $(\Bbbk[U])(y,x) = \Bbbk[U(y,x)]$, *and extending equivariant maps $\Bbbk$-linearly.*

Proof. This is a well-known phenomenon with groups and it extends to finite groupoids without a wrinkle. For horizontal functoriality, we use the canonical isomorphism $\Bbbk[U \times_H V] \xrightarrow{\sim} \Bbbk[U] \otimes_H \Bbbk[V]$ for every G,H-biset U and H,K-biset V, given on basis elements by $[u,v] \mapsto u \otimes v$. Details are left to the reader. □

2.13. *Definition.* We denote by $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ the 2-full sub-bicategory of $\mathsf{biperm}_{\Bbbk}$ with the same objects (finite groupoids) and whose 1-cells are *right-free* permutation bimodules, that is, those which belong to the essential image of the above linearization $\Bbbk[-]$ restricted to right-free bisets. In other words, linearization restricts to a canonical pseudo-functor $\Bbbk[-]\colon \mathsf{biset}^{\mathsf{rf}} \to \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$. Note that, for H a group, a 1-cell ${}_GM_H$ in $\mathsf{biperm}_{\Bbbk}$ is right-free in this sense iff it is free as a right $\Bbbk H$-module.

3. Hom-decategorification

In this section we show that any Mackey 2-functor provided with a coherent choice of a pair of objects in each of its value categories gives rise to an ordinary Mackey functor of Hom-groups. This is a sort of 'decategorification' procedure, distinct from the more usual K_0-style decategorification.

For future reference, we work here under more general hypotheses:

3.1. *Hypotheses.* In this section, $(\mathbb{G};\mathbb{J})$ denotes a *spannable pair* as in [Del22a, §3], *i.e.* an essentially small extensive (2,1)-category with sufficiently many Mackey squares and coproducts with respect to a 2-subcategory $\mathbb{J}$ closed under them and containing all equivalences. The examples to keep in mind are $(\mathsf{gpd};\mathsf{gpd}^{\mathsf{f}})$, used in the other sections of this article, and $(\mathsf{gpd}^{\mathsf{f}}_{/G};\mathsf{gpd}^{\mathsf{f}}_{/G})$, used in this section; but $\mathbb{G}$ does not necessarily consist of groupoids nor $\mathbb{J}$ of faithful 1-morphisms. The definition of a Mackey 2-functor still makes immediate sense for a general spannable pair $(\mathbb{G};\mathbb{J})$.

3.2. *Definition* (Mackey 1-functors). The point of Hypotheses 3.1 is that it allows us to define a *span category* $\mathrm{Sp}(\mathbb{G};\mathbb{J}) := \tau_1\,\mathsf{Span}(\mathbb{G};\mathbb{J})$ which is semi-additive, *i.e.* equipped with finite biproducts induced by the coproducts of objects in $\mathbb{G}$. Then we can define an (*ordinary*) *Mackey* (*1-*)*functor for* $(\mathbb{G};\mathbb{J})$ to be an additive functor

$$M\colon\ \mathrm{Sp}(\mathbb{G};\mathbb{J}) \to \mathrm{Ab}$$

to the category of abelian groups. Concretely (*cf.* [Del22a, §3]), a Mackey 1-functor M for $(\mathbb{G};\mathbb{J})$ consists of an abelian group $M(G)$ for every $G \in \mathrm{Obj}(\mathbb{G})$ together with a *restriction* homomorphism $u^\bullet\colon M(G) \to M(H)$ for every $u\colon H \to G$ in $\mathbb{G}$ and a *transfer* homomorphism $u_\bullet\colon M(H) \to M(G)$ when moreover $u \in \mathbb{J}$; this data must satisfy the following axioms:

(A) *Functoriality:* We have $\mathrm{id}^\bullet = \mathrm{id}_\bullet = \mathrm{id}$. For all $K \xrightarrow{v} H \xrightarrow{u} G$, we have $(u\circ v)^\bullet = v^\bullet \circ u^\bullet$; and also $(u\circ v)_\bullet = u_\bullet \circ v_\bullet$ if they belong to $\mathbb{J}$.

(B) *Isomorphism invariance:* For every 2-isomorphism $\alpha\colon u \overset{\sim}{\Rightarrow} v$ in $\mathbb{G}$, we have $u^\bullet = v^\bullet$; and also $u_\bullet = v_\bullet$ if they belong to $\mathbb{J}$.

(C) *Additivity:* $M(\emptyset) \cong 0$ and every coproduct $G \xrightarrow{i} G \sqcup H \xleftarrow{j} H$ in $\mathbb{G}$ yields an isomorphism $(i^\bullet, j^\bullet)^t\colon M(G\sqcup H) \overset{\sim}{\to} M(G)\oplus M(H)$ with inverse $(i_\bullet, j_\bullet)$.

(D) *Mackey formula:* For every Mackey square in $\mathbb{G}$ with i and q in $\mathbb{J}$

$$\begin{tikzcd}[column sep=small] & P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ H \arrow[dr, "i"'] \arrow[rr, phantom, "\overset{\sim}{\underset{\gamma}{\Rightarrow}}"] & & K \arrow[dl, "u"] \\ & G & \end{tikzcd} \tag{3.3}$$

we have $u^\bullet \circ i_\bullet = q_\bullet \circ p^\bullet$.

3.4. *Examples.* By specializing Definition 3.2 to various choices of $(\mathbb{G};\mathbb{J})$ we obtain several classical variations on the notion of Mackey functor. For instance,

$(\mathsf{gpd}^{\mathsf{f}};\mathsf{gpd}^{\mathsf{f}})$ provides the so-called *globally defined Mackey functors*. The choice $(\mathsf{gpd};\mathsf{gpd}^{\mathsf{f}})$ results in the (global) *inflation functors*. Taking $\mathbb{G}=\mathbb{J}=\mathsf{gpd}^{\mathsf{f}}_{/G}\cong G\text{-set}$ to be groupoids faithfully embedded in a fixed group G, we get the original 'G-local' notion of *Mackey functors for G*. See [Del22a] for explanations and details.

3.5. *Definition* (Coherent family of pairs). Let $\mathcal{M}\colon \mathbb{G}^{\mathrm{op}}\to \mathsf{ADD}$ be a Mackey 2-functor for $(\mathbb{G};\mathbb{J})$. By a *coherent family of pairs of objects in $\mathcal{M}$* we mean a 2-functor $\mathcal{M}''\colon \mathbb{G}^{\mathrm{op}}\to \mathsf{ADD}''$ which lifts $\mathcal{M}$ along the forgetful 2-functor $\mathsf{ADD}''\to\mathsf{ADD}$, where we write $\mathsf{ADD}'':=(\mathbb{Z}\text{-free}\sqcup\mathbb{Z}\text{-free})/\mathsf{ADD}$ for the (pseudo) slice 2-category of ADD under a coproduct of two copies of the free additive category on one object. Concretely, such a 2-functor $\mathcal{M}''$ amounts to the following data:

(0) two objects $X_G, Y_G\in\mathcal{M}(G)$ for every object $G\in\mathbb{G}$ and
(1) two isomorphisms $\lambda_u\colon X_H\overset{\sim}{\to} u^*X_G$ and $\rho_u\colon u^*X_G\overset{\sim}{\to} Y_H$ in the category $\mathcal{M}(H)$ for every morphism $u\colon H\to G$ in $\mathbb{G}$,

satisfying the following conditions:

(2) the triangles

$$\begin{tikzcd}[row sep=small] & u^*X_G \arrow[dd, "\alpha_{X_G}"] \\ X_H \arrow[ur, "\lambda_u"] \arrow[dr, "\lambda_v"'] & \\ & v^*X_G \end{tikzcd} \qquad\text{and}\qquad \begin{tikzcd}[row sep=small] u^*Y_G \arrow[dd, "\alpha_{Y_G}"'] \arrow[dr, "\rho_u"] & \\ & Y_H \\ v^*Y_G \arrow[ur, "\rho_v"'] & \end{tikzcd}$$

commute for every 2-morphism $\alpha\colon u\Rightarrow v\colon H\to G$ of $\mathbb{G}$,
(3) the equations $\lambda_{\mathrm{Id}_G}=\mathrm{id}_{X_G}$ and $\rho_{\mathrm{Id}_G}=\mathrm{id}_{Y_G}$ hold for every object $G\in\mathbb{G}$,
(4) and finally, the triangles

$$\begin{tikzcd} X_K \arrow[r, "\lambda_v"] \arrow[dr, bend right, "\lambda_{uv}"'] & v^*X_H \arrow[r, "v^*(\lambda_u)"] & v^*u^*X_G \arrow[d, equal] \\ & & (uv)^*X_G \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} v^*u^*Y_G \arrow[r, "v^*(\rho_u)"] \arrow[d, equal] & v^*X_H \arrow[r, "\rho_v"] & X_K \\ (uv)^*X_G \arrow[urr, bend right, "\rho_{uv}"'] & & \end{tikzcd}$$

commute for every composable pair of 1-morphisms $K\overset{v}{\to}H\overset{u}{\to}G$ of $\mathbb{G}$,

3.6. *Remark.* Any such lift $\mathcal{M}''$ is automatically additive, since the forgetful 2-functor $\mathsf{ADD}''\to\mathsf{ADD}$ creates direct sums of objects in the evident way.

3.7. **Theorem** (Hom-decategorification). *Let $\mathcal{M}$ be a Mackey 2-functor for $(\mathbb{G};\mathbb{J})$ (as in Hypotheses 3.1) and let $\mathcal{M}''$ be a coherent family of pairs of objects in $\mathcal{M}$ as in Definition 3.5, given by $\{X_G,Y_G,\lambda_u,\rho_u\}_{G,u}$. Then there exists a Mackey 1-functor for $(\mathbb{G};\mathbb{J})$ (Definition 3.2)*

$$M:=M_{\mathcal{M}''}=M_{\{X_G,Y_G,\lambda_u,\rho_u\}}\colon \tau_1(\mathsf{Span}(\mathbb{G};\mathbb{J}))\longrightarrow \mathrm{Ab}$$

whose values are given by the Hom group at the chosen pair

$$M(G):=\mathcal{M}(G)(X_G,Y_G)$$

for all objects $G\in\mathbb{G}$, with restriction maps (obviously) defined by

$$\begin{aligned} u^\bullet\colon M(G)=\mathcal{M}(G)(X_G,Y_G) &\longrightarrow \mathcal{M}(H)(X_H,Y_H)=M(H)\\ f &\longmapsto \rho_u\circ u^*(f)\circ\lambda_u \end{aligned}$$

for all $u\colon H \to G$ *and induction maps (less obviously) defined by*

$$\begin{aligned} i_\bullet\colon M(H) = \mathcal{M}(H)(X_H, Y_H) &\longrightarrow \mathcal{M}(G)(X_G, Y_G) = M(G) \\ g &\longmapsto {}^\ell\varepsilon \circ i_*(\rho_i^{-1} g \lambda_i^{-1}) \circ {}^r\eta \end{aligned}$$

for all $i\colon H \to G$ *in* $\mathbb{J}$ *where* ${}^r\eta$ *and* ${}^\ell\varepsilon$ *are the (co)units of Convention 2.6.*

3.8. *Example* (The G-local case). Suppose $\mathcal{M}$ is a G-local Mackey 2-functor, *i.e.* a Mackey 2-functor for $\mathbb{G} = \mathbb{J} = \mathsf{gpd}^{\mathsf{f}}_{/G} \cong G$-set where G is a fixed finite group. Then any pair of objects $X_0, Y_0 \in \mathcal{M}(G)$ defines a coherent choice as in the theorem, by setting $X_{(H,i_H)} := i_H^* X_0$ and $Y_{(H,i_H)} := i_H^* Y_0$ for every object $(H, i_H\colon H \rightarrowtail G)$ of $\mathsf{gpd}^{\mathsf{f}}_{/G}$, and $\lambda_{(u,\theta)} := (\theta^*)^{-1}\colon i_H^* X_0 \to u^* i_K^* X_0$ and $\rho_{(u,\theta)} := \theta^*\colon u^* i_K^* Y_0 \to i_H^* X$ for every morphism $(u\colon H \to K, \theta\colon i_K u \overset{\sim}{\Rightarrow} i_H)$. We thus obtain an ordinary Mackey functor M for G in the classical sense such that $M(H) = \mathcal{M}(H)(\mathrm{Res}^G_H X_0, \mathrm{Res}^G_H Y_0)$ for all subgroups $H \leq G$.

3.9. *Example.* Suppose that $\mathcal{M}$ is a Mackey 2-functor for $(\mathbb{G}; \mathbb{J})$ taking values in monoidal categories $\mathcal{M}(G)$ and strong monoidal functors u^* (for instance $\mathcal{M}$ could be a *Green 2-functor* in the sense of [Del21]). Then we may take $X_G = Y_G := \mathbf{1}$ to be the tensor unit of $\mathcal{M}(G)$ and λ_u, ρ_u to be the coherent isomorphisms of u^*. This produces a Mackey functor M for $(\mathbb{G}; \mathbb{J})$ with $M(G) = \mathrm{End}_{\mathcal{M}(G)}(\mathbf{1})$. In the presence of tensor-compatible gradings, *e.g.* in the case of tensor triangulated categories, we also have a graded version $G \mapsto M(G) = \mathrm{End}^*_{\mathcal{M}(G)}(\mathbf{1})$.

Proof of Theorem 3.7. The restriction and transfer maps defined in the theorem are clearly additive, and we need to show that they satisfy the axioms (A)-(D) of Definition 3.2. The additivity axiom (C) is immediate from Remark 3.6. Isomorphism invariance (B) is an easy consequence of Definition 3.5 (2).

Let us check (A). For $K \overset{j}{\to} H \overset{i}{\to} G$ in $\mathbb{J}$ and $g \in M(K) = \mathcal{M}(K)(X_K, Y_K)$, the following diagram (where $\eta = {}^r\eta$ and $\varepsilon = {}^\ell\varepsilon$) commutes:

$$\begin{tikzcd}
i_*X_H \arrow[r, "i_*\eta"] \arrow[d, "i_*\lambda_i", "\simeq"'] & i_*j_*j^*X_H \arrow[d, "i_*j_*j^*\lambda_i"] \arrow[dr, phantom, "(4)"] & i_*j_*X_K \arrow[l, "i_*j_*\lambda_j"', "\simeq"] \arrow[r, "i_*j_*g"] \arrow[d, equal] & i_*j_*Y_K \arrow[d, equal] \arrow[dr, phantom, "(4)"] & i_*j_*Y_H \arrow[l, "i_*j_*\rho_j"', "\simeq"] \arrow[r, "i_*\varepsilon"] & i_*Y_H \\
i_*i^*X_G \arrow[r, "i_*\eta i^*"] & i_*j_*j^*i^*X_G \arrow[d, "\simeq"'] & i_*j_*X_K \arrow[l, "\simeq"', "i_*j_*\lambda_{ij}"] \arrow[d, "\simeq"'] & i_*j_*Y_K \arrow[d, "\simeq"] & i_*j_*j^*i^*Y_G \arrow[l, "\simeq"', "i_*j_*\rho_{ij}"] \arrow[r, "i_*\varepsilon i^*"'] \arrow[u, "i_*j_*j^*\rho_i"] \arrow[d, "\simeq"] & i_*i^*Y_G \arrow[u, "i_*\rho_i", "\simeq"'] \arrow[d, "\varepsilon"] \\
X_G \arrow[u, "\eta"] \arrow[r, "\eta"'] & (ij)_*(ij)^*X_G & (ij)_*X_K \arrow[l, "(ij)_*\lambda_{ij}"] \arrow[r, "(ij)_*g"'] & (ij)_*Y_K \arrow[r, "\simeq", "(ij)_*\rho_{ij}"'] & (ij)_*(ij)^*Y_G \arrow[r, "\varepsilon"'] & Y_G
\end{tikzcd}$$

The squares marked (4) commute by Definition 3.5. The top-left and top-right ones commute by the naturality of ${}^r\eta$ and ${}^\ell\varepsilon$. The isomorphism $i_*j_* \overset{\sim}{\to} (ij)_*$ is the pseudo-functoriality of $(-)_* = (-)_!$ (equivalently obtained from that of $(-)^*$ by taking either right or left mates—see Convention 2.6), and the five remaining squares commute by its basic properties; see [BD20, A.2.10]. The two paths connecting X_G to Y_G around the perimeter display the identity $(i_\bullet \circ j_\bullet)(g) = (ij)_\bullet(g)$. The remaining identities are easier and left to the reader.

It only remains to check the Mackey formula (D). We want to show that

$$u^\bullet i_\bullet = q_\bullet p^\bullet\colon \mathcal{M}(H)(X_H, Y_H) \longrightarrow \mathcal{M}(K)(X_K, Y_K).$$

Let $f\colon X_H \to Y_H$ be a morphism in $\mathcal{M}(H)$. Its image under $u^\bullet i_\bullet$ is the composite map $X_K \to Y_K$ over the top row of the following diagram, whereas its image under

$q_\bullet p^\bullet$ is the composite over the bottom row:

$$\begin{tikzcd}
u^*X_G \arrow[r, "u^{*\,r}\eta"'] & u^*i_*i^*X_G & u^*i_*X_H \arrow[l, "\simeq", "u^*i_*\lambda_i"'] \arrow[r, "u^*i_*(f)"'] \arrow[dd, "(\gamma^{-1})_*"', "\simeq", shift right=2] & u^*i_*Y_H & u^*i_*i^*Y_H \arrow[l, "\simeq", "u^*i_*\rho_i"'] \arrow[r, "u^{*\,\ell}\varepsilon"'] & u^*Y_G \arrow[d, "\rho_u"', "\simeq"] \\
X_K \arrow[u, "\simeq", "\lambda_u"'] \arrow[d, "{}^r\eta"] & & & & & Y_K \\
q_*q^*X_P & q_*X_P \arrow[l, "q_*\lambda_q", "\simeq"'] \arrow[r, "q_*\lambda_p", "\simeq"'] & q_*p^*X_H \arrow[uu, "\gamma_!"', shift right=2] \arrow[r, "q_*p^*(f)"] & q_*p^*Y_H \arrow[uu, "\gamma_!"'] \arrow[r, "q_*\rho_p", "\simeq"'] & q_*Y_P & q_*q^*Y_P \arrow[l, "q_*\rho_q", "\simeq"'] \arrow[u, "{}^\ell\varepsilon"]
\end{tikzcd}$$

We insert in this diagram the 2-Mackey isomorphism $(\gamma^{-1})_* = (\gamma_!)^{-1} \colon u^*i_* \overset{\sim}{\Rightarrow} q_*p^*$ of (2.7). Since $\gamma_!$ is natural, the middle square above commutes. It remains to show that the left and right heptagons commute. The left heptagon is the perimeter of the following commutative diagram (where $\eta = {}^r\eta$ and $\varepsilon = {}^r\varepsilon$):

$$\begin{tikzcd}
X_K \arrow[r, "\lambda_u"] \arrow[d, "\eta"'] & u^*X_G \arrow[r, "u^*\eta"] \arrow[d, "\eta"'] & u^*i_*i^*X_G \arrow[d, "\eta"'] & u^*i_*X_H \arrow[l, "u^*i_*\lambda_i"', "\simeq"] \arrow[d, "\eta"'] \arrow[ddd, bend left=60, "(\gamma^{-1})_*"] \\
q_*q^*X_K \arrow[r, "q_*q^*\lambda_u"] & q_*q^*u^*X_G \arrow[r, "q_*q^*u^*\eta"] \arrow[d, "q_*(\gamma^{-1})^*"] & q_*q^*u^*i_*i^*X_G \arrow[d, "q_*(\gamma^{-1})^*i_*i^*"] & q_*q^*u^*i^*X_H \arrow[l, "q_*q^*u^*i_*\lambda_i"', "\simeq"] \arrow[d, "q_*(\gamma^{-1})^*i_*"'] \\
q_*X_P \arrow[u, "q_*\lambda_q", "\simeq"'] \arrow[ur, "q_*\lambda_{uq}"] \arrow[ur, phantom, "(4)", near start, shift left=4] \arrow[r, "q_*\lambda_{ip}"'] \arrow[dr, bend right, "q_*\lambda_p"'] \arrow[dr, phantom, "(4)"] & q_*p^*i^*X_G \arrow[r, "q_*p^*i^*\eta"] \arrow[dr, equal] \arrow[ul, phantom, "(2)", near start, shift right=3] & q_*p^*i^*i_*i^*X_G \arrow[d, "q_*p^*\varepsilon i^*"] \arrow[d, phantom, "(i^*\dashv i_*)"', shift right=8] & q_*p^*i^*i_*X_H \arrow[l, "q_*p^*i^*i^*\lambda_i"', "\simeq"] \arrow[d, "q_*p^*\varepsilon"'] \\
 & q_*p^*X_H \arrow[u, "q_*p^*\lambda_i"'] \arrow[rr, equal, bend right] & q_*p^*i^*X_G & q_*p^*X_H \arrow[l, "q_*p^*\lambda_i"', "\simeq"]
\end{tikzcd}$$

The latter commutes by hypotheses (2) and (4) in Definition 3.5, a zig-zag equation for the adjunction $i^* \dashv i_*$, the definition of the mate $(\gamma^{-1})_*$, and the naturality of various maps. The right heptagon is analogous and is left to the reader. □

4. Mackey 2-motives via bisets

In this section we prove Theorem 1.4. Let us begin by recalling the original construction of Mackey 2-motives in [BD20].

4.1. *Recollection.* Mackey 2-motives can be constructed in four steps:

$$\mathsf{mot}\colon \mathsf{gpd}^{\mathrm{op}} \longrightarrow \mathsf{Span}^{\mathsf{rf}} \longrightarrow \widehat{\mathsf{Span}}^{\mathsf{rf}} \longrightarrow \Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}} \longrightarrow (\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}})^\flat =: \mathsf{Mot}_\Bbbk \tag{4.2}$$

One begins by building the bicategory $\mathsf{Span}^{\mathsf{rf}}$, where objects are finite groupoids, 1-cells $H \to G$ are *right-faithful spans* in gpd

$$H \overset{u}{\leftarrow} P \overset{i}{\rightarrowtail} G$$

consisting of a functor $u\colon P \to H$ and a faithful functor $i\colon P \rightarrowtail G$ with common source. For 2-cells we use isomorphism classes (in the standard sense) of triples

$$[p, \alpha, \beta] = \quad \begin{tikzcd}
 & P \arrow[dl, "u"'] \arrow[dr, "i"] \arrow[dd, "p"] & \\
G \arrow[r, phantom, "\alpha\Searrow"] & & H \arrow[l, phantom, "\Nearrow\beta"] \\
 & Q \arrow[ul, "v"] \arrow[ur, "j"'] &
\end{tikzcd} \quad . \tag{4.3}$$

Horizontal composition is computed by forming Mackey squares (see [BD20, Ch. 5]). In the next step of (4.2), we enlarge $\mathsf{Span}^{\mathsf{rf}}$ to a bicategory $\widehat{\mathsf{Span}}^{\mathsf{rf}}$ by also allowing the formation of spans vertically, *i.e.* spans of 2-cells of $\mathsf{Span}^{\mathsf{rf}}$ (see [BD20, Ch. 6]). The bicategory $\widehat{\mathsf{Span}}^{\mathsf{rf}}$ is locally *semi*-additive, *i.e.* its Hom categories admit finite biproducts and thus are canonically enriched in abelian monoids. For the next step, we group-complete all Hom monoids of 2-cells and tensor them with $\Bbbk$ to obtain a $\Bbbk$-linear bicategory $\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}$. Finally, we define $\mathsf{Mot}_{\Bbbk}$ to be the block-completion $(-)^{\flat}$ of $\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}$ (see Recollection 2.2).

At each step, we have an evident canonical pseudo-functor as pictured in (4.2) above, starting with the contravariant embedding $(-)^*\colon \mathsf{gpd}^{\mathrm{op}} \to \mathsf{Span}^{\mathsf{rf}}$ sending a functor $u\colon H \to G$ to the span $G \overset{u}{\leftarrow} H \overset{\mathrm{Id}}{\longrightarrow} H$ and a natural isomorphism $\alpha\colon u \Rightarrow v$ to the morphism of spans represented by the triple $[\mathrm{Id}_H, \alpha, \mathrm{id}_{\mathrm{Id}_H}]$.

4.4. *Warning.* Our present notations differ slightly from [BD20]. There $\mathsf{Span}^{\mathsf{rf}}$ was denoted $\mathsf{Span}(\mathsf{gpd};\mathsf{gpd}^{\mathsf{f}})$ or simply Span, and similarly for $\widehat{\mathsf{Span}}^{\mathsf{rf}}$. The symbol $\Bbbk\widehat{\mathsf{Span}}$ was previously used to directly denote $\mathsf{Mot}_{\Bbbk}$, including block-completion.

4.5. *Remark.* Definition 2.3 and Recollection 4.1 work for more general '(2,1)-categories of groupoids' $\mathbb{G}$ and more general classes of faithful 1-morphisms $\mathbb{J}$, leading to variants $\mathsf{Mot}_{\Bbbk}(\mathbb{G};\mathbb{J})$ of the motivic bicategory. Everything in this article generalizes too but this will be ignored for simplicity (see Examples 5.1 and 5.4).

4.6. *Notation.* As in [BD20, § 6.3], idempotent-complete $\Bbbk$-linear Mackey 2-functors, together with 'induction preserving' morphisms and modifications, form a 2-category here denoted by $\mathsf{Mack}_{\Bbbk}^{\mathsf{ic}}$. It is contained in a 2-category $\mathsf{Mack}_{\Bbbk}$ of all, non-necessarily idempotent-complete, $\Bbbk$-linear Mackey 2-functors, which is itself contained in the 2-category $\mathsf{PsFun}_{\amalg}(\mathsf{gpd}^{\mathrm{op}}, \mathsf{ADD}_{\Bbbk})$ of all additive (*i.e.* coproduct-preserving) pseudo-functors, pseudo-natural transformations and modifications.

Here is the universal property of Mackey 2-motives $\mathsf{Mot}_{\Bbbk}$:

4.7. **Theorem** ([BD20, §§ 5.3 and 6.3])**.** *The canonical pseudo-functors of Recollection 4.1 induce by precomposition biequivalences of 2-categories*

$$\mathsf{PsFun}_{\Bbbk}(\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}, \mathsf{ADD}_{\Bbbk}) \overset{\sim}{\to} \mathsf{Mack}_{\Bbbk} \quad \textit{and} \quad \mathsf{PsFun}_{\Bbbk}(\mathsf{Mot}_{\Bbbk}, \mathsf{ADD}_{\Bbbk}^{\mathsf{ic}}) \overset{\sim}{\to} \mathsf{Mack}_{\Bbbk}^{\mathsf{ic}}$$

where $\mathsf{PsFun}_{\Bbbk}$ *denotes the 2-category of* $\Bbbk$*-linear (hence additive) pseudo-functors, pseudo-natural transformations and modifications.*

* * *

4.8. *Remark.* With these reminders behind us, Theorem 1.4 tells us that in the construction of Mackey 2-motives we may replace right-faithful spans of functors with right-free bisets (see Recollection 2.8 for the latter). A span $H \leftarrow P \rightarrowtail G$, from H to G, is *right*-faithful when $P \to G$ is faithful. However, a G,H-biset ${}_G S_H$, still from H to G, is *right*-free when the H-action is free, not the G-action. So the meticulous reader might be puzzled that we use the same decoration 'rf' both for "right-faithful" and "right-free" in apparently unrelated cases. In fact, the following key result shows that these two notions match beautifully.

4.9. **Theorem.** *There exists a canonical biequivalence of bicategories*

$$\begin{tikzcd} \mathsf{Span}^{\mathsf{rf}} \arrow[r, shift left, "\mathcal{R}"] \arrow[r, phantom, "\sim"] & \mathsf{biset}^{\mathsf{rf}} \arrow[l, shift left, "\int"] \end{tikzcd}$$

given by the realization bifunctor $\mathcal{R}$ *of spans (see Recollection 4.11) and the* Grothendieck construction $\int$ *on bisets (see Recollection 4.16). On objects, i.e. finite groupoids, both pseudo-functors are just the identity.*

4.10. *Remark.* Results closely related to the above one have long been known among some category-theorists (see *e.g.* [Bén00]) and topologists (see *e.g.* [Mil17]).

Proof of Theorem 1.4. The biequivalence between $\mathsf{Mot}_{\Bbbk} = (\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}})^\flat$ and $(\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}})^\flat$ is easily obtained from that of Theorem 4.9 by changing both sides as follows:

(1) take ordinary categories of spans $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ of all Hom categories ([BD20, A.4]);
(2) group-complete every Hom abelian monoid of 2-cells;
(3) extend scalars from $\mathbb{Z}$ to $\Bbbk$;
(4) and finally, take block-completions $(-)^\flat$.

Each operation is sufficiently bifunctorial to preserve biequivalences. $\square$

In order to prove Theorem 4.9, we first detail the constructions of $\mathcal{R}$ and $\int$.

4.11. *Recollection* (The realization pseudo-functor). Let us first consider the bicategory $\mathsf{Span} := \mathsf{Span}(\mathsf{gpd})$ of *all*, not necessarily right-faithful, spans between finite groupoids, as well as the bicategory biset of *all*, not necessarily right-free, bisets (Recollection 2.8). By [Hug19, §4.2] or [DH21], there is a pseudo-functor

$$\mathcal{R}\colon \mathsf{Span} \longrightarrow \mathsf{biset} \tag{4.12}$$

which sends a finite groupoid G to itself, a span of functors $i_! u^* := (H \xleftarrow{u} P \xrightarrow{i} G)$ to the composite biset

$$\mathcal{R}(i_! u^*) := \underbrace{G(i-,-)}_{\mathcal{R}_!(i)} \times_P \underbrace{H(-,u-)}_{\mathcal{R}^*(u)} \colon H^{\mathrm{op}} \times G \longrightarrow \mathrm{set}$$

and morphisms of spans (4.3) to the naturally induced morphism of bisets. Note that $\mathcal{R}$ is obtained by the universal property of Span (see [BD20, § 5.2] or [DH21, Thm. 5.4]) by 'gluing' the two more evident pseudo-functors

$$\mathcal{R}^*\colon \mathsf{gpd}^{\mathrm{op}} \longrightarrow \mathsf{biset} \quad \text{and} \quad \mathcal{R}_!\colon \mathsf{gpd}^{\mathrm{co}} \longrightarrow \mathsf{biset}$$

which map a functor $v\colon P \to Q$ to the biset $\mathcal{R}^*(v) = Q(-,v-)\colon Q^{\mathrm{op}} \times P \to \mathrm{set}$, respectively to the biset $\mathcal{R}_!(v) = Q(v-,-)\colon P^{\mathrm{op}} \times Q \to \mathrm{set}$. This gluing is possible because in gpd there are (well-behaved) adjunctions for every $v\colon P \to Q$

$$\begin{tikzcd} P \arrow[d, bend right, "\mathcal{R}_!(v) = Q(v-,-)"'] \arrow[d, phantom, "\dashv"] \\ Q \arrow[u, bend right, "Q(-,v-) = \mathcal{R}^*(v)"'] \end{tikzcd} \tag{4.13}$$

with unit $\eta\colon \mathrm{Id}_P \Rightarrow \mathcal{R}^*(v) \circ \mathcal{R}_!(v)$ and counit $\varepsilon\colon \mathcal{R}_!(v) \circ \mathcal{R}^*(v) \Rightarrow \mathrm{Id}_Q$ given by

$$\begin{aligned} \eta_{x,x'}\colon \quad & P(x,x') \longrightarrow Q(z, v(x')) \times_{z \in Q} Q(v(x), z)\,, \qquad && p \mapsto [\mathrm{id}_{v(x')}, v(p)] \\ \varepsilon_{y,y'}\colon \quad & Q(v(z), y') \times_{z \in P} Q(y, v(z)) \longrightarrow Q(y, y')\,, \qquad && [q_1, q_2] \mapsto q_1 q_2 \end{aligned}$$

for all $x, x' \in \mathrm{Obj}(P)$ and $y, y' \in \mathrm{Obj}(Q)$.

Note that the *data* of the pseudo-functor $\mathcal{R}$ is entirely determined by the data of the above pseudo-functors $\mathcal{R}_!$ and $\mathcal{R}^*$ and the adjunctions $(\mathcal{R}_!(v), \mathcal{R}^*(v), \eta, \varepsilon)$.

4.14. *Remark.* The realization pseudo-functor (4.12) is not a biequivalence, as can already be see at the level of truncated 1-categories. Indeed, the resulting functor $\tau_1(\mathcal{R})\colon \tau_1(\mathsf{Span}) \to \tau_1(\mathsf{biset})$ is full but not faithful, and its kernel admits a nice description due to Ganter and Nakaoka (see [Del22a, §6]). In order to get a biequivalence, we must restrict both its domain and codomain.

4.15. **Lemma.** *Let $i_!u^* = (H \xleftarrow{u} P \xrightarrow{i} G)$ be a span of finite groupoids. If the functor i is faithful, then the biset $\mathcal{R}(i_!u^*)$ is right-free,* i.e. *H acts freely on it.*

Proof. By definition, the biset $\mathcal{R}(i_!u^*)$ is right-free if and only if for every objects $(y,x) \in \mathrm{Obj}(H^{\mathrm{op}} \times G)$ and $y' \in \mathrm{Obj}(H)$, every element $[g,h] \in \mathcal{R}(i_!u^*)(y,x)$ and every morphism $t \in H(y',y)$, we have that $[g,h]\cdot t = [g,h]$ implies $t = \mathrm{id}_y$ (note that we must already have $y' = y$ for the first equation to make sense). Here $g \in G(iz,x)$ and $h \in H(y,uz)$ for some $z \in \mathrm{Obj}(P)$ and $[g,h]\cdot t = [g,ht]$ by definition. Thus the equation $[g,h]\cdot t = [g,h]$ means that there exists some map $p \in P(z,z)$ such that $g \circ i(p) = g$ and $u(p) \circ h = h \circ t$. As G is a groupoid, the first equation entails $i(p) = g^{-1}g = \mathrm{id}_{i(z)}$. If i is faithful, the latter implies that $p = \mathrm{id}_y$ and thus, by the second equation $u(p)h = ht$ we get $t = h^{-1}h = \mathrm{id}_y$ as wished. □

In the other direction, we use the following construction:

4.16. *Recollection* (Grothendieck construction)*.* Fix two groupoids H and G. For any G,H-biset $S \in \mathsf{biset}(H,G)$, we can define a groupoid denoted

$$\textstyle\int_G^H S \qquad \text{or simply} \qquad \int S$$

whose objects are triples (y,x,s) with $y \in \mathrm{Obj}\, H^{\mathrm{op}}$, $x \in \mathrm{Obj}\, G$ and $s \in S(y,x)$. A morphism $(y,x,s) \to (y',x',s')$ in $\int S$ is a pair of morphisms (h,g) with $h \in H(y,y')$ and $g \in G(x,x')$, such that $g \cdot s = s' \cdot h$ holds. This comes equipped with obvious projection functors $\mathrm{pr}_H\colon \int S \to H$ and $\mathrm{pr}_G\colon \int S \to G$, sending (y,x,s) and (h,g) to y and h, respectively to x and g. In other words, we obtain a span from H to G

$$\begin{tikzcd} & \int S \arrow[dl, "\mathrm{pr}_H"'] \arrow[dr, "\mathrm{pr}_G"] & \\ H \arrow[rr, dotted] & & G. \end{tikzcd}$$

4.17. *Remark.* In the case of a group G, the Grothendieck construction is often denoted $G \ltimes S$ and called *transport groupoid* or *action groupoid* (*cf.* Remark 4.27).

4.18. **Lemma.** *The construction in Recollection 4.16 defines a functor*

$$\textstyle\int := \int_G^H \colon \mathsf{biset}(H,G) \longrightarrow \mathsf{Span}(H,G)$$

for every pair of groupoids H,G, by mapping a natural transformation $\varphi\colon S \Rightarrow T$ of bisets $S,T\colon H^{\mathrm{op}} \times G \to \mathrm{set}$ to the morphism of spans

$$[\textstyle\int\varphi, \mathrm{id}, \mathrm{id}] = \qquad \begin{tikzcd} & \int S \arrow[dl, "\mathrm{pr}_H"'] \arrow[dr, "\mathrm{pr}_G"] \arrow[dd, "\int\varphi"'] & \\ H & & G \\ & \int T \arrow[ul, "\mathrm{pr}_H"] \arrow[ur, "\mathrm{pr}_G"'] & \end{tikzcd}$$

where both triangles commute and the functor $\int\varphi\colon \int S \to \int T$ sends an object (y,x,s) to $(y,x,\varphi(s))$ and a map (h,g) to (h,g).

Proof. Straightforward verification. □

4.19. **Lemma.** *A span $H \xleftarrow{u} P \xrightarrow{i} G$ belongs to the essential image of $\int_G^H$ if and only if it is* jointly faithful*, i.e. the functor $(u,i)\colon P \to H \times G$ is faithful. (For instance, the latter holds if $i\colon P \to G$ is faithful, i.e. if the span is right-faithful.)*

Proof. First notice that $(\mathrm{pr}_H, \mathrm{pr}_G)\colon \int S \to H \times G$ is (trivially!) jointly faithful for any biset S, and that the property of being jointly faithful is stable under taking isomorphic spans. Conversely, let $i_! u^* = (H \xleftarrow{u} P \xrightarrow{i} G)$ be any span. There is a canonical morphism of spans as follows

$$[\Phi, \mathrm{id}, \mathrm{id}] = \begin{tikzcd} & P \arrow[dl, "u"'] \arrow[dr, "i"] \arrow[d, "\Phi"] & \\ H & \int \mathcal{R}(i_! u^*) \arrow[l, "\mathrm{pr}_H"] \arrow[r, "\mathrm{pr}_G"'] & G \end{tikzcd} \tag{4.20}$$

where the functor $\Phi := \Phi_{i_! u^*}\colon P \longrightarrow \int \mathcal{R}(i_! u^*)$ sends an object z in P to the object $(u(z), i(z), [\mathrm{id}_{i(z)}, \mathrm{id}_{u(z)}])$ in $\int_G^H \big(G(i-,-) \times_P H(-,u-)\big)$, and maps a morphism $p \in P(z,z')$ to the pair $(u(p), i(p))$; the latter defines a morphism $(uz, iz, [\mathrm{id}, \mathrm{id}]) \to (uz', iz', [\mathrm{id}, \mathrm{id}])$ in $\int i_! u^*$, as required, since

$$i(p) \cdot [\mathrm{id}_{iz}, \mathrm{id}_{uz}] = [i(p), \mathrm{id}_{uz}] = [\mathrm{id}_{iz'}, u(p)] = [\mathrm{id}_{iz'}, \mathrm{id}_{uz'}] \cdot u(p).$$

Clearly Φ is a functor such that $\mathrm{pr}_G \circ \Phi = i$ and $\mathrm{pr}_H \circ \Phi = u$. It is always full: Given *any* morphism $(h,g)\colon (u(z), i(z), [\mathrm{id}, \mathrm{id}]) \to (u(z'), i(z'), [\mathrm{id}, \mathrm{id}])$ in the target groupoid, that is a $g \in G(iz, iz')$ and an $h \in H(uz, uz')$ such that

$$g \cdot [\mathrm{id}_{iz}, \mathrm{id}_{uz}] \overset{\text{def.}}{=} [g, \mathrm{id}_{uz}] = [\mathrm{id}_{iz'}, h] \overset{\text{def.}}{=} [\mathrm{id}_{iz'}, \mathrm{id}_{uz'}] \cdot h$$

in $\mathcal{R}(i_! u^*)(z,z') = G(i-, iz') \times_P H(uz, u-)$, by definition this means that there exists some $p \in P(z,z')$ such that $\mathrm{id} \circ i(p) = g$ and $u(p) \circ \mathrm{id} = h$, that is: $(h,g) = (u(p), i(p)) = \Phi(p)$. The functor Φ is also always essentially surjective: Given *any* object (y,x,s) with $y \in \mathrm{Obj}\, H$, $x \in \mathrm{Obj}\, G$ and $s = [g \in G(iz,x), h \in H(y,uz)]$ at some $z \in \mathrm{Obj}\, Z$, the pair (h^{-1}, g) defines an isomorphism

$$\Phi(z) = \big(u(z), i(z), [\mathrm{id}_{i(z)}, \mathrm{id}_{u(z)}]\big) \overset{\sim}{\longrightarrow} \big(y, x, [g,h]\big)$$

because $g \cdot [\mathrm{id}, \mathrm{id}] = [g,h] \cdot h^{-1}$. Finally, it is easy to see that Φ is faithful precisely when $(u,i)\colon P \to H \times G$ is faithful.

In short, $i_! u^*$ is jointly faithful if and only if Φ is an equivalence of groupoids, if and only if the morphism $[\Phi, \mathrm{id}, \mathrm{id}]$ in (4.20) defines an isomorphism of spans $i_! u^* \overset{\sim}{\to} \int \mathcal{R}(i_! u^*)$, by [BD20, Lem. 5.1.12]. The statement follows. □

4.21. **Lemma.** *There is a canonical isomorphism $\varphi_S\colon S \overset{\sim}{\Rightarrow} \mathcal{R}\big(\int_G^H S\big)$ for every G,H-biset $S\colon H^{\mathrm{op}} \times G \to \mathrm{set}$.*

Proof. Define φ_S by setting its component at $(y,x) \in H^{\mathrm{op}} \times G$ to be the map

$$\begin{aligned} \varphi_{S,y,x}\colon S(y,x) &\longrightarrow G(\mathrm{pr}_G -, x) \times_{\int S} H(y, \mathrm{pr}_H -) \\ s \quad &\mapsto \quad [\mathrm{id}_x, \mathrm{id}_y] \text{ at } (y,x,s) \in \mathrm{Obj}(\textstyle\int S) \end{aligned}$$

Its inverse, say ψ_S, has components given at each (y,x) as follows:

$$\begin{aligned} \psi_{S,y,x}\colon G(\mathrm{pr}_G -, x) \times_{\int S} H(y, \mathrm{pr}_H -) &\longrightarrow S(y,x) \\ [u_1 \in G(x_1, x), v_1 \in H(y, y_1)] \text{ at } (y_1, x_1, s_1) \quad &\mapsto \quad u_1 \cdot s_1 \cdot v_1 \end{aligned}$$

We leave to the reader the straightforward verifications that φ_S and ψ_S are well-defined, mutually inverse natural transformations. □

4.22. **Proposition.** *For every pair H, G of finite groupoids, the realization pseudo-functor of Recollection 4.11 and the Grothendieck construction of Lemma 4.18 restrict to an equivalence of Hom categories*

$$\mathsf{Span}^{\mathsf{rf}}(H,G) \underset{\int_G^H}{\overset{\mathcal{R}_{H,G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{biset}^{\mathsf{rf}}(H,G)\ .$$

Proof. Just combine Lemma 4.15, Lemma 4.18, Lemma 4.19 and Lemma 4.21. $\square$

Proof of Theorem 4.9. By Lemma 4.15, we may restrict $\mathcal{R}$ to a pseudo-functor

$$\mathcal{R}\colon\ \mathsf{Span}^{\mathsf{rf}} \longrightarrow \mathsf{biset}^{\mathsf{rf}}$$

between the 2-full sub-bicategories, restricting to right-faithful in Span and to right-free in biset. By Proposition 4.22, this pseudo-functor is an equivalence at each Hom category. As $\mathcal{R}$ is a bijection on objects by construction, we may already conclude that it is a biequivalence of its source and target bicategories. It also follows that the Grothendieck construction functors of Proposition 4.22 inherit from $\mathcal{R}$ through the Hom-equivalences the structure of a pseudo-functor $\int$, quasi-inverse to $\mathcal{R}$. $\square$

Let us compute the image of some elementary 1-cells and 2-cells under the equivalence $\mathcal{R}\colon \Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}} \xrightarrow{\sim} \Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$. These are all easy computations from the definitions.

4.23. *Example.* Let $u\colon H \to G$ be a group homomorphism and consider the 1-cell $u^*\colon G \overset{u}{\leftarrow} H \overset{\mathrm{Id}}{\longrightarrow} H$ in $\widehat{\mathsf{Span}}^{\mathsf{rf}}(G,H)$. Then $\mathcal{R}(u^*)$ is the H,G-biset ${}_H G_G$ with action $h\cdot x\cdot g = u(h)\,x\,g$.

4.24. *Example.* Let $i\colon H \rightarrowtail G$ be an injective group homomorphism and consider $i_! = i_*\colon H \overset{\mathrm{Id}}{\longleftarrow} H \overset{i}{\to} G$ in $\widehat{\mathsf{Span}}^{\mathsf{rf}}(H,G)$. Then $\mathcal{R}(i_!)$ is the G,H-biset ${}_G G_H$ with action $g{\cdot}x{\cdot}h = g\,x\,i(h)$. In that case, we can combine this 1-cell with the 1-cell ${}_H G_G$ of Example 4.23 and consider the units and counits of the adjunctions $i_! \dashv i^* \dashv i_*$. Their images under $\mathcal{R}$ are as follows. Note that ${}_H G_G \circ {}_G G_H \cong {}_H G_H$ whereas ${}_G G_H \circ {}_H G_G \cong {}_G (G\times_H G)_G$. Of course, $\mathrm{Id}_H = {}_H H_H$ and $\mathrm{Id}_G = {}_G G_G$. We have

$$\begin{aligned}
{}^{\ell}\eta_i &= \big[H \overset{\mathrm{id}}{\Leftarrow} H \overset{i}{\Rightarrow} G\big]\colon && \mathrm{Id}_H = {}_H H_H \Rightarrow {}_H G_H = i^* i_!\\
{}^{\ell}\varepsilon_i &= \big[G\times_H G \overset{\mathrm{id}}{\Leftarrow} G\times_H G \overset{\mu}{\Rightarrow} G\big]\colon && i_! i^* = {}_G(G\times_H G)_G \Rightarrow {}_G G_G = \mathrm{Id}_G\\
{}^{r}\eta_i &= \big[G \overset{\mu}{\Leftarrow} G\times_H G \overset{\mathrm{id}}{\Rightarrow} G\times_H G\big]\colon && \mathrm{Id}_G = {}_G G_G \Rightarrow {}_G G\times_H G_G = i_* i^*\\
{}^{r}\varepsilon_i &= \big[G \overset{i}{\Leftarrow} H \overset{\mathrm{id}}{\Rightarrow} H\big]\colon && i^* i_* = {}_H G_H \Rightarrow {}_H H_H = \mathrm{Id}_H
\end{aligned} \tag{4.25}$$

where the map marked μ is the multiplication of G. We recognize the Frobenius relation ${}^{r}\varepsilon \circ {}^{\ell}\eta = \big[H = H = H\big] = \mathrm{id}_H$. On the other hand, the composite ${}^{\ell}\varepsilon \circ {}^{r}\eta$ of Definition 1.1 is given by the span of G,G-bisets $\big[G \overset{\mu}{\Leftarrow} G\times_H G \overset{\mu}{\Rightarrow} G\big]$.

4.26. *Remark.* In view of Lemma 4.19, it is tempting to believe that $\mathcal{R}$ should give a biequivalence between biset and the 2-full subcategory of Span of all jointly faithful spans. Unfortunately the latter spans are not stable under composition, hence such a bicategory does not exist. To see why, just observe the diagram

$$\begin{tikzcd}
& & G \arrow[ddll, bend right] \arrow[ddrr, bend left] \arrow[dl, equal] \arrow[dr, equal] & & \\
& G \arrow[dl] \arrow[dr, equal] & = & G \arrow[dr] \arrow[dl, equal] & \\
1 \arrow[rr, dotted] & & G \arrow[rr, dotted] & & 1
\end{tikzcd}$$

displaying a (very) non-jointly-faithful composite of two jointly faithful spans.

4.27. *Remark.* For $H = 1$ the trivial group, Proposition 4.22 yields an equivalence

$$\tau_1(\mathsf{gpd}^{\mathsf{f}}_{/G}) \cong \mathsf{Span}^{\mathsf{rf}}(1,G) \underset{\int^1_G}{\overset{\mathcal{R}_{1,G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{biset}^{\mathsf{rf}}(1,G) \cong G\text{-set}\,.$$

The two identifications, at the left with the truncated comma 2-category $\tau_1(\mathsf{gpd}^{\mathsf{f}}_{/G})$ of groupoids faithfully embedded in G, and on the right with the category of left G-sets, are isomorphisms of 1-categories simply obtained by suppressing the data over the trivial group. When G is a group, this is the equivalence of categories used in [BD20, App. B] and [Del22a] to reformulate Mackey functors for a fixed group G in terms of groupoids. Indeed, the above equivalence $G\text{-set} \overset{\sim}{\to} \tau_1(\mathsf{gpd}^{\mathsf{f}}_{/G})$ is precisely the crossed product functor $X \mapsto (\pi_X \colon G \ltimes X \rightarrowtail G)$ of [BD20, Prop. B.0.8], for which we also have now (even for G any finite groupoid) a nice canonical pseudo-inverse $\tau_1(\mathsf{gpd}^{\mathsf{f}}_{/G}) \overset{\sim}{\to} G\text{-set}$. Explicitly, the latter sends an object $(P, i_P \colon P \rightarrowtail G)$ to the G-set $(i_P/-)_{\simeq}$ which maps each object $x \in \operatorname{Obj} G$ to the set $(i_P/x)_{\simeq}$ of isomorphism classes of objects $(y, i_P(y) \to x)$ in the ordinary slice category (i_P/x).

4.28. *Remark.* By 1-truncating the biequivalence $\mathcal{R} \colon \mathsf{Mot}_{\Bbbk} \overset{\sim}{\to} (\Bbbk\widehat{\mathsf{biset}^{\mathsf{rf}}})^{\flat}$ and forming categories of $\Bbbk$-linear functors on both sides (see Remark 2.10 and [Del22a, Cor. 6.22]), we obtain the well-known equivalence between inflation functors [Web93] and right-free biset functors [Bou10].

5. Cohomological Mackey 2-functors

Recall from Definition 1.1 that a (rectified) Mackey 2-functor is *cohomological* if ${}^{\ell}\varepsilon\, {}^{r}\eta = [G : H]\, \mathrm{id}_{\mathcal{M}(G)}$ for every subgroup inclusion $i \colon H \rightarrowtail G$. In this section, we provide some familiar examples as well as the first applications of our definition.

5.1. *Example* (Representation theory). There are Mackey 2-functors whose value $\mathcal{M}(G)$ at a group G is either the category of linear representations $\mathrm{Mod}(\Bbbk G)$, or its derived category $\mathrm{D}(\Bbbk G)$, or its stable module category $\mathrm{Stab}(\Bbbk G)$ (for the third example, one limits the domain to $\mathbb{G} = \mathsf{gpd}^{\mathsf{f}}$; see [BD20, Ch. 4.1-2] for details). An easy computation with the usual adjunctions shows they are all cohomological.

5.2. *Example* (Permutation modules). The full subcategories $\mathrm{perm}_{\Bbbk}(G) \subseteq \mathrm{Mod}(\Bbbk G)$ of finitely generated permutation $\Bbbk G$-modules form a Mackey sub-2-functor, since inductions and restrictions preserve permutation modules. It is still cohomological. The same holds for the subcategory $\mathrm{perm}_{\Bbbk}(G)^{\natural} \subseteq \mathrm{Mod}(\Bbbk G)$ of direct summands of permutation modules. (In characteristic p, these are known as *p-permutation* modules or *trivial source* modules. But the above makes sense for any ring $\Bbbk$.)

5.3. *Example* (Represented 2-functors). Once we develop our motivic theory, we will see that every cohomological Mackey 2-motive represents a cohomological Mackey 2-functor (Corollary 6.21). The trivial motive $(1, \mathrm{id})$ represents the Mackey 2-functor $\mathrm{perm}_{\Bbbk}(G)^{\natural}$ of Example 5.2.

5.4. *Example* (Equivariant objects). All G-local examples of Mackey 2-functors of equivariant objects from [BD20, Ch. 4.4] are cohomological, by an easy computation using the concrete adjunctions provided there. (Here as before, 'G-local' means defined over $\mathsf{gpd}^{\mathsf{f}}_{/G} \cong G$-set rather than gpd; *cf.* Examples 3.4.) These include many geometric examples, such as equivariant coherent sheaves on a noetherian G-scheme.

5.5. *Example* (Cohomological Mackey functors). There is a cohomological Mackey 2-functor whose value $\mathcal{M}(G) = \mathrm{CohMack}_{\Bbbk}(G)$ is the category of (ordinary) cohomological Mackey functors for G. To see this, we can apply the construction $\mathcal{S} \mapsto \mathcal{M}$ of [BD20, Prop. 7.3.2], as we can easily check that if the Mackey 2-functor $\mathcal{S}$ is cohomological then so is $\mathcal{M}$. Taking $\mathcal{S}$ the (pointwise dual of) the Mackey 2-functor $G \mapsto \mathrm{perm}_{\Bbbk}(G)^{\mathrm{op}}$ of Example 5.2, we obtain $\mathcal{M}\colon G \mapsto \mathrm{Fun}_{\Bbbk}(\mathrm{perm}_{\Bbbk}(G), \mathrm{Mod}\,\Bbbk) \cong \mathrm{CohMack}_{\Bbbk}(G)$ in accordance with Yoshida's Theorem (*cf.* Remark 6.17).

The next result is a first justification for the adjective 'cohomological'.

5.6. **Theorem.** *Let $\mathcal{M}$ be any cohomological Mackey 2-functor, and suppose that M is an ordinary Mackey functor obtained from $\mathcal{M}$ by the Hom-decategorification procedure as in Theorem 3.7. Then M is cohomological in the classical sense: $I^H_K R^H_K = [H\!:\!K] \cdot \mathrm{id}_{M(H)}$ for all subgroups $K \le H \le G$.*

Proof. This is a direct verification from the definition of the restrictions and transfers in Theorem 3.7, in fact for any of the classical choices of $(\mathbb{G}; \mathbb{J})$ as in Examples 3.4 where we can view subgroup inclusions as 1-morphisms in $\mathbb{J}$.

Namely, let $(i\colon H \to G) \in \mathbb{J}$ and $f \in M(G) = \mathcal{M}(G)(X_G, Y_G)$ for any coherent choice $\{X_G, Y_G, \lambda_u, \rho_u\}$ of pairs of objects in $\mathcal{M}$. Applying the transfer $i_\bullet$ to the restricted map $i^\bullet(f)$, the λ_i's and ρ_i's cancel out, leaving us with the composite

$$X_G \xrightarrow{{}^r\eta_i} i_* i^* X_G \xrightarrow{i_* i^*(f)} i_* i^* Y_G \xrightarrow{{}^\ell\varepsilon_i} Y_G.$$

By naturality of ${}^r\eta_i$ or ${}^\ell\varepsilon_i$, this is f composed with ${}^\ell\varepsilon_i \circ {}^r\eta_i = [H\!:\!K]\,\mathrm{id}$. ☐

5.7. *Example* (Group cohomology). Specializing Example 3.9 to the (global or local) cohomological Mackey functor $G \mapsto \mathcal{M}(G) = \mathrm{D}(\Bbbk G)$ of Example 5.1 yields the motivating example of a cohomological Mackey functor, namely group cohomology $G \mapsto \mathrm{H}^*(G;\Bbbk) = \mathcal{M}(G)^*(\mathbf{1},\mathbf{1})$. For a fixed group G, and any $V \in \mathrm{Mod}(\Bbbk G)$, we also get the variant $H \mapsto \mathrm{H}^*(H; V|_H)$ $(H \le G)$ with twisted coefficients by setting $X_0 = \Bbbk$ and $Y_0 = V$ in Example 3.8. Similarly, by taking stable module categories instead of derived categories we obtain Tate cohomology of finite groups.

5.8. *Remark.* The analogue of Theorem 5.6 does *not* hold for the perhaps more familiar K_0-style of decategorification (see [BD20, §2.5]). For instance, the Mackey 2-functor $G \mapsto \mathrm{mod}(\Bbbk G)$ of finitely generated representations is cohomological (*cf.* Example 5.1) but the ordinary Mackey functor $G \mapsto K_0(\mathrm{mod}\,\Bbbk G) = \mathrm{R}_{\Bbbk}(G)$, say for $\Bbbk = \mathbb{C}$, is the usual representation ring, which is not cohomological.

5.9. *Remark.* Following up on the previous remark, suppose that $\mathcal{M}$ is a *Green 2-functor* in the sense of [Del21]. In particular, this means that the categories $\mathcal{M}(G)$ are monoidal, the restriction functors are strong monoidal, and there is a projection formula $i_*(i^* X \otimes Y) \cong X \otimes i_* Y$ for faithful $i\colon H \rightarrowtail G$. After applying K_0, one gets an ordinary Mackey functor satisfying $I^G_H R^G_H = [i_*(\mathbf{1})]\cdot\mathrm{id}$, where $\mathbf{1} \in \mathcal{M}(H)$ denotes the tensor unit. We can view this as a kind of 'generalized cohomological relation' for a 'generalized index' $[i_*(\mathbf{1})] \in K_0(\mathcal{M}(G))$. This applies to the representation ring (Remark 5.8), and in fact to any Green functor arising from a Green 2-functor by K_0-decategorification. If $\mathcal{M}$ is a Green 2-functor, moreover, the projection formula $i_*i^* \cong i_*(\mathbf{1}) \otimes -$ identifies the composite ${}^{\ell}\varepsilon \circ {}^{r}\eta$ of Definition 1.1 with multiplication by the Euler characteristic $\chi(i_*(\mathbf{1})) \in \mathrm{End}_{\mathcal{M}(G)}(\mathbf{1})$ (*i.e.* the monoidal trace of the identity) of the dualizable object $i_*(\mathbf{1})$. This is a consequence of the special Frobenius structure of $i_*(\mathbf{1})$; see [Del21, §§ 7-8]. Hence such an $\mathcal{M}$ is cohomological precisely when $\chi(i_*(\mathbf{1}))$ is multiplication by $[G\!:\!H]$.

Thus cohomological Mackey 2-functors are a source of classical cohomological Mackey 1-functors, via Hom-decategorification. For the remainder of the section, we further validate our definition by sketching a couple of applications.

5.10. **Theorem** (p-local separable monadicity)**.** *Let $\mathcal{M}$ be a cohomological Mackey 2-functor, and let G be a finite group such that $\mathcal{M}(G)$ is a $\mathbb{Z}_{(p)}$-linear category for a prime number p (e.g. the base ring $\Bbbk$ is a $\mathbb{Z}_{(p)}$-algebra, e.g. it is a field of characteristic p). Let $i\colon H \rightarrowtail G$ denote the inclusion of a subgroup of index prime to p (for instance a p-Sylow). Then the monad $\mathbb{A} := i^*i_!$ on $\mathcal{M}(H)$ induced by the adjunction $i_! \dashv i^*$ is* separable*, that is its multiplication $\mu := i^*({}^{\ell}\varepsilon)i_*\colon \mathbb{A}^2 \Rightarrow \mathbb{A}$ admits an $\mathbb{A}$-bilinear section. In particular, it follows that restriction $i^*\colon \mathcal{M}(G) \to \mathcal{M}(H)$ satisfies descent, in that the canonical comparison functor $\mathcal{M}(G) \xrightarrow{\sim} \mathrm{Mod}(\mathbb{A})_{\mathcal{M}(H)}$ into the Eilenberg-Moore category of $\mathbb{A}$-modules in $\mathcal{M}(H)$ is an equivalence.*

Proof. Since $\mathcal{M}$ is cohomological, the composite ${}^{\ell}\varepsilon\,{}^{r}\eta$ acts on $\mathcal{M}(G)$ as multiplication by $[G:H]$, which is invertible by the $\mathbb{Z}_{(p)}$-linearity of $\mathcal{M}(G)$. In particular, the counit ${}^{\ell}\varepsilon$ of the adjunction $i_! \dashv i^*$ admits a natural section. As $\mathcal{M}(G)$ is assumed idempotent-complete, we may conclude with [Bal15, Lemma 2.10]. □

5.11. *Remark.* Note that Theorem 5.10 goes in the 'opposite' direction of the deceptively similar result of [BD20, Theorem 2.4.1], which says that the *other* adjunction $i^* \dashv i_*$ is separably monadic, and which holds for *any* Mackey 2-functor $\mathcal{M}$ and *any* faithful $i\colon H \rightarrowtail G$. In particular, one can always reconstruct $\mathcal{M}(H)$ from the monad i_*i^* on $\mathcal{M}(G)$, but in order to recover $\mathcal{M}(G)$ from the monad $i^*i_!$ on $\mathcal{M}(H)$ special circumstances are required, such as those in Theorem 5.10.

5.12. *Remark.* Theorem 5.10 can be reformulated as a categorification of the classical Cartan-Eilenberg stable elements formula for mod-p group cohomology. To wit, for $\mathcal{M}$ and $\Bbbk$ as in the theorem, the above monadic reconstruction of $\mathcal{M}(G)$ can be replaced by a pseudo-limit $\mathcal{M}(G) \simeq \lim_{G/P} \mathcal{M}(P)$ in ADD, where P ranges through the p-subgroups of G and the limit is taken over the corresponding orbit category. This leads to a generalization of the main results of [Bal15] to arbitrary cohomological Mackey 2-functors; see [Mai22] for details and explanations.

5.13. *Remark.* To outline another application, let us simply mention that the general Green correspondence of [BD21] is most useful for Mackey 2-functors which are cohomological, for which it gives rise to a 'p-local theory' as in modular representation theory. See [BD21, §§ 6-7] for details.

6. Cohomological Mackey 2-motives

We now turn to Theorem 1.5 and the description of cohomological Mackey 2-motives in simple terms. Here we use the results of Section 4.

Since cohomological Mackey 2-functors are Mackey 2-functors that send some special 2-cells to zero, there is a tautological approach to the bicategory of cohomological Mackey 2-motives: It is the quotient

$$\mathcal{Q}\colon\ \mathsf{Mot}_{\Bbbk} \twoheadrightarrow \mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}$$

obtained by modding out the 2-cells corresponding to the cohomological conditions:

$$(6.1)\qquad {}^{\ell}\varepsilon_i \circ {}^{r}\eta_i - [G\,{:}\,H]\cdot \mathrm{id} \qquad \in \mathrm{End}_{\mathsf{Mot}_{\Bbbk}}(\mathrm{Id}_G)$$

for every inclusion $i\colon H \rightarrowtail G$ of a subgroup H in a group G. But this definition is rather sprawling: We need to consider the closure of the above class of 2-cells inside $\mathsf{Mot}_{\Bbbk}$ under composition and $\Bbbk$-linear combination inside each Hom category; plus we need to take into account horizontal composition with arbitrary 2-cells, including whiskering. So the tautological definition is unwieldy.

Our Theorem 1.5 gives a concrete realization of $\mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}$ as $(\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}})^{\flat}$, the bicategory obtained by block-completing the bicategory of right-free permutation bimodules $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ (Definition 2.13). In view of Proposition 2.12, it is more convenient to use the model of $\mathsf{Mot}_{\Bbbk}$ via bisets, as described in Section 4. To do so, we need to translate the 2-cell (6.1) under the equivalence of Theorem 4.9. In view of Example 4.24, the image of ${}^{\ell}\varepsilon_i \circ {}^{r}\eta_i - [G\,{:}\,H]\cdot \mathrm{id}$ in the category $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}(G,G)$ is simply the following linear combination of spans of equivariant maps between right-free G,G-bisets (with μ induced by multiplication):

$$(6.2)\qquad \big[G \overset{\mu}{\Leftarrow} G\times_H G \overset{\mu}{\Rightarrow} G\big] - [G\,{:}\,H]\cdot \mathrm{id}_{\mathrm{Id}_G}.$$

For simplicity, we call this the *cohomological 2-cell* corresponding to $H \le G$.

6.3. *Remark.* Let G_1, G_2 be two groups. We can view G_1,G_2-bisets X as left $(G_1\times G_2)$-sets via $(g_1,g_2)\cdot x = g_1 x g_2^{-1}$ for every $x\in X$. Decomposing into orbits, every G_1,G_2-biset is a coproduct of transitive $G_1\times G_2$-sets of the form $(G_1\times G_2)/M$ for subgroups $M\le G_1\times G_2$. Translating back, the G_1,G_2-biset corresponding to such an orbit $(G_1{\times}G_2)/M$ is the same set with action $g_1{\cdot}[x,y]{\cdot}g_2 = [g_1\,x, g_2^{-1}y]$. Since we focus on right-free bisets, we note that $(G_1\times G_2)/M$ is right-free (over G_2) if and only if the first projection $G_1\times G_2\to G_1$ is injective on M, that is, $(\mathrm{pr}_1)|_M\colon M \rightarrowtail G_1\times G_2 \twoheadrightarrow G_1$ is faithful. Indeed, the right-action of $g_2\in G_2$ fixes a class $[x,y]\in (G_1\times G_2)/M$ if and only if $(1,(g_2)^y)\in M$.

6.4. *Example.* Let $G_1=G_2=G$ and consider the G,G-biset $G\times_H G$ of (6.2). As left $(G\times G)$-set, it is isomorphic to a single orbit $(G\times G)/M$ for the subgroup $M=\Delta(H)=\big\{\,(h,h)\,\big|\,h\in H\,\big\}$, via $(G\times G)/M \overset{\sim}{\to} G\times_H G$, $(g_1,g_2)\mapsto (g_1,g_2^{-1})$.

We now consider a class of 2-cells in $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ that will play a role later on.

6.5. *Construction.* Let G_1,G_2 be finite groups and $M\le N\le G_1\times G_2$ be subgroups such that pr_1 is injective on N. As in Remark 6.3, we view the linear combination

$$(6.6)\quad \delta(G_1,G_2,M,N) := \quad \Big[\tfrac{G_1\times G_2}{N} \Leftarrow \tfrac{G_1\times G_2}{M} \Rightarrow \tfrac{G_1\times G_2}{N}\Big] - [N\colon M]\cdot \mathrm{id}_{(G_1\times G_2)/N}$$

as an endomorphism of the G_1,G_2-biset $(G_1{\times}G_2)/N$ in the category $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}(G_2,G_1)$. The two equivariant maps denoted '$\Rightarrow$' are simply the quotient maps.

6.7. *Example.* Let $H\le G$. Take again $G_1=G_2=G$ as in Example 6.4. Taking the subgroups $M=\Delta(H)$ and $N=\Delta(G)$ in Construction 6.5, the expression (6.6) boils down to our cohomological 2-cell (6.2). Conversely, we now prove that every $\delta(G_1,G_2,M,N)$ belongs to the ideal generated by the cohomological 2-cells.

6.8. **Lemma.** *Let G_1 and G_2 be finite groups and $M \le N \le G_1 \times G_2$ subgroups such that the first projection $\mathrm{pr}_1\colon G_1 \times G_2 \to G_1$ is injective on N (and thus on M). Then the 2-cell $\delta(G_1, G_2, M, N)$ given in (6.6) belongs to the (bicategorical) ideal of 2-cells in $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ generated by the cohomological 2-cells (6.2).*

Proof. Consider the cohomological 2-cell (6.2) for the subgroup $H := M$ of $G := N$, that is, $\delta_0 := \big[N \Leftarrow N \times_M N \Rightarrow N\big] - [N\colon M]\cdot \mathrm{id}_{\mathrm{Id}_N}$. We claim that $\delta(G_1, G_2, M, N)$ is simply the 2-cell obtained from (pre-)whiskering δ_0 by the 1-cell $G_2 \to N$ given by the N, G_2-biset G_2 and (post-)whiskering it by the 1-cell $N \to G_1$ given by the G_1, N-biset G_1. In both cases, N acts on G_i via $(\mathrm{pr}_i)|_N\colon N \rightarrowtail G_1 \times G_2 \twoheadrightarrow G_i$. Note that the resulting 1-cell is as wanted:

$$G_1 \times_N N \times_N G_2 = G_1 \times_N G_2 \leftrightarrow (G_1 \times G_2)/N$$

where $\leftrightarrow$ indicates the dictionary between G_1,G_2-bisets and left $(G_1 \times G_2)$-sets of Remark 6.3. Since whiskering $\mathrm{id}_{\mathrm{Id}_N}$ in the same way gives $\mathrm{id}_{(G_1\times G_2)/N}$, it suffices to see what happens to the span $\big[N \Leftarrow N \times_M N \Rightarrow N\big]$ under these whiskerings. The result is indeed

$$\big[\ (G_1 \times G_2)/N \Longleftarrow (G_1 \times G_2)/M \Longrightarrow (G_1 \times G_2)/N\ \big]$$

since the G_1, G_2-biset $G_1 \times_N (N \times_M N) \times_N G_2 \cong G_1 \times_M G_2$ translates into the orbit $(G_1 \times G_2)/M$ as a left $(G_1 \times G_2)$-set, via Remark 6.3. Direct verification shows that the above maps $\Leftarrow$ and $\Rightarrow$ are indeed the canonical projections. $\square$

In Definition 2.13, we encountered the $\Bbbk$-linearization $\Bbbk[-]\colon \mathsf{biset}^{\mathsf{rf}} \to \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$. Interestingly, $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ accommodates spans, that is $\Bbbk[-]$ can be nicely extended along the inclusion $\mathsf{biset}^{\mathsf{rf}} \subset \widehat{\mathsf{biset}}^{\mathsf{rf}}$ to a pseudo-functor defined on *spans* of 2-cells:

6.9. **Proposition.** *There is a well-defined pseudo-functor*

$$\mathcal{P}\colon \widehat{\mathsf{biset}}^{\mathsf{rf}} \longrightarrow \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$$

which maps a finite groupoid to itself and a G, H-biset U to the permutation G,H-bimodule $\Bbbk[U]$. It maps a 2-cell given by a span of equivariant maps of G, H-bisets

$$\big[\ U \overset{\beta}{\Longleftarrow} W \overset{\alpha}{\Longrightarrow} V\ \big]$$

to the sum-over-fibers *natural transformation $\alpha_\star\beta^\star\colon \Bbbk[U] \Rightarrow \Bbbk[V]$, whose component at the object $(y, x) \in H^{\mathrm{op}} \times G$ is the $\Bbbk$-linear map defined on basis elements by*

$$(\alpha_\star\beta^\star)_{y,x}\colon \Bbbk[U(y, x)] \longrightarrow \Bbbk[V(y, x)], \quad u \longmapsto \sum_{w\in\beta_{y,x}^{-1}(u)} \alpha_{y,x}(w)\,. \tag{6.10}$$

Proof. This is a lengthy verification that we only outline. Local functoriality of $\mathcal{P}\colon \widehat{\mathsf{biset}}^{\mathsf{rf}}(H, G) \to \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}(H, G)$ entails that given a pullback of G, H-bisets

$$\begin{tikzcd} & U \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\beta"] & \\ V \arrow[dr, Rightarrow, "\gamma"'] & & W \arrow[dl, Rightarrow, "\delta"] \\ & X & \end{tikzcd}$$

we have $\beta_\star\alpha^\star = \delta^\star\gamma_\star\colon \Bbbk[V] \to \Bbbk[W]$. This is a direct verification on the bases, by definition of the cartesian product: β restricts to a bijection $\alpha^{-1}(v) \overset{\sim}{\to} \delta^{-1}(\gamma(v))$.

To show that $\mathcal{P}$ preserves horizontal composition, consider for every G,H-biset U and H,K-biset V the canonical isomorphism $\Bbbk[U\times_H V]\xrightarrow{\sim}\Bbbk[U]\otimes_H\Bbbk[V]$ that we already used in the proof of Proposition 2.12. It provides the compatibility isomorphism between $\mathcal{P}(U\circ V)$ and $\mathcal{P}(U)\circ\mathcal{P}(V)$, on the condition that it is also natural with respect to the 'backwards' morphisms $U\Leftarrow U':\beta$ and the associated $\beta^\star$. This is again a direct verification on the bases. $\square$

6.11. *Remark.* When $i\colon H\rightarrowtail G$ is the inclusion of a subgroup, we spelled out the adjunctions $i_!\dashv i^*\dashv i_*$ in $\widehat{\mathsf{biset}}^{\mathsf{rf}}$ in Example 4.24. These adjunctions now have an image in $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ under $\mathcal{P}$. Of course, $\mathcal{P}(i_!)=\mathcal{P}(i_*)=\Bbbk[{}_GG_H]={}_G\Bbbk G_H$ and $\mathcal{P}(i^*)=\Bbbk[{}_HG_G]={}_H\Bbbk G_G$. The units and counits ${}^\ell\eta,{}^\ell\varepsilon$ of ${}_G\Bbbk G_H\dashv{}_H\Bbbk G_G$ and those ${}^r\eta,{}^r\varepsilon$ of ${}_H\Bbbk G_G\dashv{}_G\Bbbk G_H$ are given in $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ by the familiar formulas:

$$\begin{array}{ccc} 1 & \longmapsto & \sum_{[y]\in G/H} y\otimes y^{-1} \\ {}_G\Bbbk G_G & \underset{{}^\ell\varepsilon}{\overset{{}^r\eta}{\rightleftarrows}} & {}_G\Bbbk G\otimes_H\Bbbk G_G \\ g'g & \longleftarrow\!\shortmid & g'\otimes g \end{array}\qquad\qquad \begin{array}{c} g\longmapsto\begin{cases} g & (g\in H)\\ 0 & (g\notin H)\end{cases} \\ {}_H\Bbbk G\otimes_G\Bbbk G_H\cong{}_HG_H \underset{{}^\ell\eta\,=\,\mathrm{incl}}{\overset{{}^r\varepsilon}{\rightleftarrows}} {}_H\Bbbk H_H \end{array}$$

Indeed, it is very easy to follow the images of the spans (4.25) under the 'sum-over-fibers' recipe of Proposition 6.9. For instance, ${}^r\varepsilon$ is given in $\widehat{\mathsf{biset}}^{\mathsf{rf}}$ by the span $\big[G\overset{i}{\Leftarrow}H\overset{\mathrm{id}}{\Rightarrow}H\big]$. Its image under $\mathcal{P}$ is the morphism $\Bbbk G\to\Bbbk H$ mapping a basis element $g\in G$ to $\sum_{h\in i^{-1}(g)}h$ which is g if $g\in H$ and zero otherwise.

6.12. *Example.* Let us compute the image under $\mathcal{P}$ of the cohomological 2-cell (6.2), for $H\le G$. Clearly, the underlying 1-cell Id_G goes to Id_G which is the bimodule $\Bbbk G$. Also clearly $[G\!:\!H]\cdot\mathrm{id}_{\mathrm{Id}_G}$ goes to $[G\!:\!H]\cdot\mathrm{id}_{\Bbbk G}$. The $\Bbbk$-linear $\mathcal{P}\big(\big[G\overset{\mu}{\Leftarrow}G\times_HG\overset{\mu}{\Rightarrow}G\big]\big)\colon$ $\Bbbk G\to\Bbbk G$ maps a basis element $g\in G$ to $\sum_{[x,y]\in\mu^{-1}(g)}xy=|\mu^{-1}(g)|\cdot g$ where $\mu\colon G\times_H G\to G$ is multiplication. That fiber $\mu^{-1}(g)$ has $[G:H]$ elements. In conclusion, $\mathcal{P}$ maps the cohomological 2-cell (6.2) to zero.

We are now ready to describe $\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ as a 2-quotient of $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$.

6.13. **Theorem.** *Consider the unique $\Bbbk$-linear extension $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}\to\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ of the pseudo-functor $\mathcal{P}$ of Proposition 6.9, which we again denote by $\mathcal{P}$. It is the identity on objects, essentially surjective on 1-cells, and full on 2-cells. Its kernel on 2-cells* $\ker(\mathcal{P}):=\{\varphi\in\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}_2\mid\mathcal{P}(\varphi)=0\}$ *is generated by the* cohomological 2-cells (6.2) *for all inclusions $i\colon H\rightarrowtail G$ of a subgroup H of a finite group G.*

Proof. The pseudo-functor $\mathcal{P}$ is the identity on objects by definition. It is also essentially surjective on 1-cells since permutation modules are $\Bbbk$-linearizations of bisets. The point is the behavior on 2-cells. To formalize the statement, consider the ($\Bbbk$*-linear, additive, bicategorical*) *ideal* $\mathcal{J}$ of $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ generated by (6.2), meaning the smallest class of 2-cells containing those and stable under horizontal composition with arbitrary 2-cells, and such that its restriction to each Hom category is closed under taking linear combinations and (vertical) composites with arbitrary maps.

By Example 6.12, we know that $\mathcal{P}(\mathcal{J}) = 0$. So we have a factorization

$$\begin{tikzcd}
& \Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}} \arrow[dl, "\text{quot.}"'] \arrow[dr, "\mathcal{P}"] & \\
\mathcal{B} := \dfrac{\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}}{\mathcal{J}} \arrow[rr, "\overline{\mathcal{P}}"] & & \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}
\end{tikzcd}$$

where the left-hand quotient bicategory $\mathcal{B}$ has the same objects and 1-cells as $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ and the obvious quotient by $\mathcal{J}$ as 2-cells. The claim of the theorem is that $\overline{\mathcal{P}}$ is a biequivalence. Since $\overline{\mathcal{P}}$ is, like $\mathcal{P}$, the identity on objects and essentially surjective on 1-cells, the claim is that $\overline{\mathcal{P}}$ is fully faithful on each Hom category.

Using additivity, we easily reduce to connected groupoids as objects and transitive bisets as 1-cells. So we need to prove the following. Let G_1 and G_2 be finite groups, U and V two transitive G_1, G_2-bisets, then the $\Bbbk$-linear homomorphism

$$\overline{\mathcal{P}}\colon \operatorname{Hom}_{\mathcal{B}(G_1,G_2)}(U,V) \to \operatorname{Hom}_{\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}(G_1,G_2)}(\Bbbk[U], \Bbbk[V]) \tag{6.14}$$

is bijective. To prove this, we shall find a certain number n of generators of the left-hand $\Bbbk$-module $\operatorname{Hom}_{\mathcal{B}(G_1,G_2)}(U,V)$, prove that their images under $\overline{\mathcal{P}}$ form a $\Bbbk$-basis of $\operatorname{Hom}_{\mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}(G_1,G_2)}(\Bbbk[U], \Bbbk[V])$ and prove that the latter is a free $\Bbbk$-module of the same rank n. In particular those generators are $\Bbbk$-linearly independent in $\operatorname{Hom}_{\mathcal{B}(G_1,G_2)}(U,V)$ already (as their images are) and thus $\overline{\mathcal{P}}$ is an isomorphism.

So let us produce those n generators. As in Remark 6.3, we view bisets as left Γ-sets for $\Gamma = G_1 \times G_2$ and we are assuming that $U = \Gamma/K$ and $V = \Gamma/L$ for subgroups $K, L \le \Gamma$. The number n mentioned above will be $n = |K\backslash\Gamma/L|$.

By construction of $\mathcal{B}$ as a quotient of $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ and by additivity (for disjoint unions) in the middle object of spans, it is easy to see that $\operatorname{Hom}_{\mathcal{B}(G_1,G_2)}(\Gamma/K\,,\,\Gamma/L)$ is generated $\Bbbk$-linearly by equivalence classes of the form

$$[\ \Gamma/K \xLeftarrow{\cdot\beta} \Gamma/M \xRightarrow{\cdot\gamma} \Gamma/L\ ] \tag{6.15}$$

for subgroups $M \le \Gamma$. Both maps in (6.15) are necessarily given by (inner) right-multiplication $[x] \mapsto [x\cdot\beta]$ and $[x] \mapsto [x\cdot\gamma]$ by elements $\beta, \gamma \in \Gamma$ such that $M^{\beta} \le K$, respectively $M^{\gamma} \le L$. Replacing M by a conjugate subgroup of Γ, we can assume that $\beta = 1$. In that case, this span (6.15) is equal to the composite of spans

$$\begin{tikzcd}[column sep=small]
& \Gamma/N \arrow[dl, Rightarrow, "\mathrm{pr}"'] \arrow[dr, Rightarrow, "\mathrm{id}"] & & \Gamma/M \arrow[dl, Rightarrow, "\mathrm{pr}"'] \arrow[dr, Rightarrow, "\mathrm{pr}"] & & \Gamma/N \arrow[dl, Rightarrow, "\mathrm{id}"'] \arrow[dr, Rightarrow, "\cdot\gamma"] & \\
\Gamma/K \arrow[rr, dotted] & & \Gamma/N \arrow[rr, dotted] & & \Gamma/N \arrow[rr, dotted] & & \Gamma/L
\end{tikzcd}$$

where N is short for $K \cap {}^{\gamma}L$ and all 'pr' denote canonical projections. By Lemma 6.8, the middle span becomes multiplication by the index $[N\!:\!M]$ in the quotient bicategory $\mathcal{B}$. Our generator (6.15) is therefore equal to $[N\!:\!M]$ times the element

$$[\ \Gamma/K \xLeftarrow{\mathrm{pr}} \Gamma/(K \cap {}^{\gamma}L) \xRightarrow{\cdot\gamma} \Gamma/L\ ] \tag{6.16}$$

where $\gamma \in \Gamma$. Thus the spans (6.16), for $\gamma \in \Gamma$, are generators of $\operatorname{Hom}_{\mathcal{B}(G_1,G_2)}(U,V)$. Note that the generator (6.16) only depends on the class $[\gamma] \in K\backslash\Gamma/L$, since for every $k \in K$ and $\ell \in L$ the following diagram of 2-cells commutes in $\mathsf{biset}^{\mathsf{rf}}$:

$$\begin{tikzcd}
& \Gamma/K\cap{}^{\gamma}L \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "\cdot\gamma"] \arrow[dd, Rightarrow, "\cdot k^{-1}", "\simeq"'] & \\
\Gamma/K & & \Gamma/L \\
& \Gamma/K\cap{}^{k\gamma\ell}L \arrow[ul, Rightarrow] \arrow[ur, Rightarrow, "\cdot k\gamma\ell"'] &
\end{tikzcd}$$

So it suffices to take a span (6.16) for each class $[\gamma]$ in $K\backslash\Gamma/L$ to obtain our set of n generators of $\mathrm{Hom}_{\mathcal{B}(G_1,G_2)}(U,V)$ in (6.14), where $n = |K\backslash\Gamma/L|$.

It only remains to check that the images under $\overline{\mathcal{P}}$ of the spans (6.16) for $[\gamma] \in K\backslash\Gamma/L$ form a $\Bbbk$-basis of the $\Bbbk$-module $\mathrm{Hom}_{\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}(G_1,G_2)}(\Bbbk[U],\Bbbk[V])$ of (6.14). In terms of left Γ-modules, this $\Bbbk$-module is $\mathrm{Hom}_\Gamma(\Bbbk(\Gamma/K),\Bbbk(\Gamma/L))$. We compute

$$\begin{aligned}
&\mathrm{Hom}_\Gamma(\Bbbk[\Gamma/K],\Bbbk[\Gamma/L]) \;\cong\; \mathrm{Hom}_\Gamma(\Bbbk,\Bbbk[\Gamma/K]\otimes_\Bbbk \Bbbk[\Gamma/L]) \;\cong \\
&\cong \bigoplus_{[\gamma]\in K\backslash\Gamma/L} \mathrm{Hom}_\Gamma(\Bbbk,\Bbbk[\Gamma/K\cap{}^{\gamma}L]) \;\cong \bigoplus_{[\gamma]\in K\backslash\Gamma/L} \mathrm{Hom}_{K\cap{}^{\gamma}L}(\Bbbk,\Bbbk) \;\cong \bigoplus_{[\gamma]\in K\backslash\Gamma/L} \Bbbk \;\cong\; \Bbbk^n
\end{aligned}$$

by combining the self-duality of $\Bbbk[\Gamma/K]$ for the tensor product, the Mackey formula, and the adjunction between restriction and induction along $K\cap{}^{\gamma}L \le \Gamma$. By explicitly retracing the element $1\in\Bbbk$ in the summand for $[\gamma]\in K\backslash\Gamma/L$ in the target $\Bbbk^n$, we find its preimage in $\mathrm{Hom}_\Gamma(\Bbbk[\Gamma/K],\Bbbk[\Gamma/L])$ to be the $\Bbbk\Gamma$-linear morphism $[x]\mapsto\sum_{[y]}[y\gamma]$ with $[y]$ running through those cosets in $\Gamma/K\cap{}^{\gamma}L$ such that $[y]=[x]$ in Γ/K. The latter map is precisely the image of the generator (6.16) under $\overline{\mathcal{P}}$, by a direct application of Proposition 6.9. Indeed, the image under $\mathcal{P}$ of the span $\big[\Gamma/K \overset{\mathrm{pr}}{\Leftarrow} \Gamma/(K\cap{}^{\gamma}L) \overset{\cdot\gamma}{\Rightarrow} \Gamma/L\big]$ is by definition $(\cdot\gamma)_\star\,\mathrm{pr}^\star\colon \Bbbk[U]\to\Bbbk[V]$, that is, it maps a generator $[x]\in U=\Gamma/K$ to the sum $\sum_{[y]\in\mathrm{pr}^{-1}([x])}[y\gamma]$ over its preimages $[y]\in\Gamma/(K\cap{}^{\gamma}L)$, which are precisely those $[y]$ such that $[y]=[x]$ in Γ/K. ☐

6.17. *Remark* (Yoshida's Theorem; see [Web00, §7]). Fix a finite group G. Consider the functor $\mathcal{P}_{1,G}\colon \Bbbk\widehat{\mathsf{biset}^{\mathsf{rf}}}(1,G)\to\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}(1,G)$, the component functor of our pseudo-functor $\mathcal{P}$ at the pair $(1,G)$. By Remark 4.27, its source 1-category is

$$\mathrm{Sp}_\Bbbk(G) := \Bbbk(\widehat{G\text{-set}}),$$

the $\Bbbk$-linear span category of finite left G-sets, *i.e.* the classical Burnside category for G. Its target is just $\mathrm{perm}_\Bbbk(G)$, the category of finitely generated left permutation $\Bbbk G$-modules. Thus $\mathcal{P}_{1,P}$ identifies with 'Yoshida's functor'

$$Y_G\colon \mathrm{Sp}_\Bbbk(G)\to\mathrm{perm}_\Bbbk(G)$$

sending a left G-set X to $\Bbbk[X]$ and a span of G-maps to the associated sum-over-fibers $\Bbbk G$-linear map. As we know, this functor is $\Bbbk$-linear, essentially surjective and full, and Yoshida's Theorem says that an ordinary Mackey functor for G (*i.e.* an additive functor $\mathrm{Sp}_\Bbbk(G)\to\mathrm{Mod}(\Bbbk)$) is cohomological if and only if it factors through Y_G. Equivalently, this says that the kernel of Y_G is generated as a $\Bbbk$-linear categorical ideal by the differences $I^K_L R^K_L - [K:L]\cdot\mathrm{id}_{K/L}$ for all $L\le K\le G$. Thus we can view the results of this section as a categorification of Yoshida's Theorem.

We now derive from Theorem 6.13 the 2-universal property for $\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$ and thus a proof of Theorem 1.5. The moment has also come to define our model for the bicategory of cohomological Mackey 2-motives.

6.18. *Definition.* We define the *bicategory of cohomological Mackey 2-motives* to be $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk} := (\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk})^{\flat}$, the block-completion of $\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$ in the sense of [BD20, Constr. A.7.22]. Its objects are pairs (G,ϕ) with G a finite groupoid and ϕ an idempotent element of the ring $\mathrm{End}_{\Bbbk\,\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}}(\mathrm{Id}_G)$; a 1-cell $(H,\psi)\to(G,\phi)$ is a pair (M,μ) with M a right-free permutation H,G-module and $\mu=\mu^2$ an idempotent equivariant map $M\Rightarrow M$ absorbing ϕ and ψ. In particular, the Hom category at (the cohomological motives of) two groups $G:=(G,\mathrm{id})$ and $H:=(H,\mathrm{id})$ is

$$\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}((H,\mathrm{id}),(G,\mathrm{id}))=\big(\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}(H,G)\big)^{\natural},$$

the usual idempotent-completion of the additive Hom category in $\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$. Its objects (M,μ) can be identified with the images $\mathrm{Im}(\mu)$ taken in the abelian category $\mathsf{Bimod}_{\Bbbk}(H,G)$ of all G,H-bimodules, and the latter are summands of (right-free) permutation bimodules in the usual sense. (See Example 5.2.)

We have a pseudo-functor $\mathsf{mot}^{\mathsf{coh}}\colon \mathsf{gpd}^{\mathrm{op}}\to\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$ composed of the pseudo-functors $\mathsf{mot}\colon\mathsf{gpd}^{\mathrm{op}}\to\mathsf{Mot}_{\Bbbk}$ of (4.2) and $\mathcal{P}\colon\mathsf{Mot}_{\Bbbk}\twoheadrightarrow\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$ of Theorem 6.13. As with $\mathsf{Mot}_{\Bbbk}$, for any finite groupoid G, we still write G for the object (G,id) in $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$.

Theorem 6.20 will justify the motivic terminology.

6.19. *Definition.* Recall the definition of a *cohomological Mackey 2-functor* (Definition 1.1). Extending Notation 4.6, in the following we write

$$\mathsf{CohMack}_{\Bbbk}\subset\mathsf{Mack}_{\Bbbk}\qquad\text{and}\qquad\mathsf{CohMack}^{\mathsf{ic}}_{\Bbbk}\subset\mathsf{Mack}^{\mathsf{ic}}_{\Bbbk}$$

for the 1- and 2-full sub-bicategories of those (idempotent-complete or not) Mackey 2-functors which are *cohomological*.

6.20. **Theorem** (Universal property)**.** *The pseudo-functor* $\mathsf{mot}^{\mathsf{coh}}\colon\mathsf{gpd}^{\mathrm{op}}\to\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$ *induces by precomposition biequivalences of 2-categories*

$$\mathsf{PsFun}_{\Bbbk}(\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk},\mathsf{ADD}_{\Bbbk})\xrightarrow{\sim}\mathsf{CohMack}_{\Bbbk}\quad\textit{and}\quad\mathsf{PsFun}_{\Bbbk}(\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk},\mathsf{ADD}^{\mathsf{ic}}_{\Bbbk})\xrightarrow{\sim}\mathsf{CohMack}^{\mathsf{ic}}_{\Bbbk}$$

where $\mathsf{PsFun}_{\Bbbk}$ *denotes 2-categories of* $\Bbbk$*-linear (hence additive) pseudo-functors, pseudo-natural transformations and modifications. In particular, every idempotent-complete* $\Bbbk$*-linear cohomological Mackey 2-functor factors uniquely up to isomorphism through* $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$ *as claimed in Theorem 1.5.*

Proof. By the universal property of Mackey 2-motives (see Theorem 4.7), the canonical embedding $\mathsf{mot}\colon\mathsf{gpd}^{\mathrm{op}}\to\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$ induces a biequivalence

$$\mathsf{PsFun}_{\Bbbk}(\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}},\mathsf{ADD}_{\Bbbk})\xrightarrow{\sim}\mathsf{Mack}_{\Bbbk}\,.$$

This restricts to a biequivalence between, on the left, $\Bbbk$-linear pseudo-functors $\widehat{\mathcal{M}}$ annihilating the cohomological 2-cells (6.1), and, on the right, (rectified) Mackey 2-functors $\mathcal{M}$ which are cohomological. Combined with the (evident) biequivalences arising from the 2-universal property of the quotient $\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}\twoheadrightarrow\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}/\ker(\mathcal{P})$ and from the biequivalence $\overline{\mathcal{P}}\colon\Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}/\ker(\mathcal{P})\xrightarrow{\sim}\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$ of Theorem 6.13, this yields the first claimed biequivalence. Since $\mathsf{ADD}^{\mathsf{ic}}_{\Bbbk}$ is block-complete, the second claimed biequivalence follows readily from the first one by the universal property [BD20, Thm. A.7.23] of the block-completion $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}=(\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk})^{\flat}$. □

6.21. **Corollary.** *For every cohomological Mackey 2-motive X, the composite*

$$\mathsf{gpd}^{\mathrm{op}}\xrightarrow{\ \mathsf{mot}^{\mathsf{coh}}\ }\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}\xrightarrow{\ \mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}(X,-)\ }\mathsf{ADD}^{\mathsf{ic}}_{\Bbbk}$$

is a cohomological Mackey 2-functor. (*Cf.* [BD20, Ch. 7.2].)

Proof. Immediate from the second biequivalence in Theorem 6.20. □

7. Motivic decompositions

In this section, we take a closer look at the bicategory of $\Bbbk$-linear cohomological Mackey 2-motives $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk} := (\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk})^{\flat}$ of Definition 6.18, in order to compare cohomological and general motives.

Let us start with a few words about motivic decompositions. Any decomposition of the Mackey 2-motive $\mathsf{mot}(G) \simeq X_1 \oplus \ldots \oplus X_n$ in $\mathsf{Mot}_{\Bbbk}$ can be realized for *any* Mackey 2-functor $\mathcal{M}$ as a decomposition of the additive category $\mathcal{M}(G)$:

$$\mathcal{M}(G) \simeq \widehat{\mathcal{M}}(X_1) \oplus \ldots \oplus \widehat{\mathcal{M}}(X_n)$$

where $\widehat{\mathcal{M}}$ is the pseudo-functor extending $\mathcal{M}$ to Mackey 2-motives. See after (1.3). Motivic decompositions are universal, in that they only depend on the group G and not on the particular Mackey 2-functor $\mathcal{M}$. See [BD20, §7.4-5] for details.

7.1. *Remark.* In [BD20, Ch. 7.4], we found an explicit isomorphism of $\Bbbk$-algebras between the motivic algebra of 2-cells $\mathrm{End}_{\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}}(\mathrm{Id}_G)$ of a group G and the socalled *crossed Burnside algebra* $\mathrm{B}^{\mathsf{c}}_{\Bbbk}(G)$ first studied by Yoshida [Yos97]. Concretely, $\mathrm{B}^{\mathsf{c}}_{\Bbbk}(G)$ is a finite free $\Bbbk$-module generated by the set of G-conjugacy classes $[H,a]_G$ of pairs (H,a) with $H \le G$ a subgroup and $a \in \mathrm{C}_G(H)$ an element of the centralizer of H, equipped with the (commutative!) multiplication induced by the formula

$$[K,b]_G \cdot [H,a]_G = \sum_{[g]\in K\backslash G/H} [K \cap {}^{g}H, bgag^{-1}]_G.$$

Thus every general Mackey 2-motive is equivalent to a direct sum of pairs (G,e) with G a finite group and $e = e^2 \in \mathrm{B}^{\mathsf{c}}_{\Bbbk}(G)$ an idempotent.

An analogous discussion can be done for *cohomological* motivic decompositions of $\mathsf{mot}^{\mathsf{coh}}(G)$ in $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$. To understand the analogue of the crossed Burnside algebra in the cohomological setting, we need to understand $\mathrm{End}_{\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}}(\mathrm{Id}_G)$.

7.2. *Remark.* For a finite group G, the 2-cell endomorphism $\Bbbk$-algebra in $\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}$ can be easily identified with the center of the group algebra $\Bbbk G$:

$$\mathrm{End}_{\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}}(\mathrm{Id}_G) \cong \mathrm{Z}(\Bbbk G).$$

Indeed, if φ is an equivariant endomorphisms of the bimodule $\mathrm{Id}_G = {}_G\Bbbk G_G$, the image $\varphi(1_G)$ determines φ and belongs to the center, since $g\varphi(1_G) = \varphi(g) = \varphi(1_G)g$ for all $g \in G$. Conversely, it is clear that any element of the center may serve as $\varphi(1_G)$. Hence, by the definition of the block-completion, every cohomological Mackey 2-motive is equivalent to a direct sum of pairs (G,f) with G a group and $f = f^2 \in \Bbbk G$ a central idempotent. It is indecomposable if and only if f is primitive.

7.3. *Remark.* The decomposition of a Mackey 2-motive into a direct sum of indecomposable ones, both in $\mathsf{Mot}_{\Bbbk}$ (as in Remark 7.1) and in $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$ (as in Remark 7.2), is unique up to permutation and equivalence of the factors; see [Del22b, Cor. 7.9].

Following the above pattern, we now obtain:

7.4. **Theorem.** *If $\mathcal{M}$ is any cohomological Mackey 2-functor, its value category at each finite group G admits a canonical decomposition into direct factors*

$$\mathcal{M}(G) \cong \bigoplus_{b} \mathcal{M}(G; b)$$

indexed by the blocks (primitive central idempotents) b of the group algebra $\Bbbk G$.

Proof. Just apply the reasoning of [BD20, § 7.5] to the motivic decompositions of Remark 7.2, as explained above. ☐

Recall the linearization pseudo-functor $\mathcal{P}\colon \Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}} \to \mathsf{biperm}_{\Bbbk}^{\mathsf{rf}}$ of Section 6. By applying block-completion $(-)^\flat$ to both sides, it extends to a pseudo-functor

$$\mathcal{P}\colon\ \mathsf{Mot}_{\Bbbk} \longrightarrow \mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}$$

comparing general and cohomological Mackey 2-motives. The following result gives a very concrete description of the effect of $\mathcal{P}$ on equivalence classes of motives:

7.5. **Theorem.** *For every finite group G, there is a well-defined surjective morphism of commutative rings $\rho_G\colon \mathrm{B}^{\mathsf{c}}_{\Bbbk}(G) \to \mathrm{Z}(\Bbbk G)$ sending a basis element $[H,a]_H$ to $\sum_{[x]\in G/H} xax^{-1} = \sum_{[y]\in H\backslash G} y^{-1}xy$. Collectively, they govern the behavior of $\mathcal{P}$ on equivalence classes of 2-motives, meaning that $\mathcal{P}$ maps the general Mackey 2-motive $\oplus_i(G_i, e_i)$ (see Remark 7.1) to the cohomological Mackey 2-motive $\oplus_i(G_i, \rho_{G_i}(e_i))$ (see Remark 7.2), where of course $(G,0)\cong 0$ in both bicategories.*

Proof. Define ρ_G by the following diagram:

$$\begin{tikzcd}
\mathrm{B}^{\mathsf{c}}_{\Bbbk}(G) \arrow[r, dashed, "\rho_G"] \arrow[d, "\cong", "{[\text{BD20, Thm. 7.4.5}]}"'] & Z(\Bbbk G) \\
\mathrm{End}_{\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}}(\mathrm{Id}_G) \arrow[r, two heads, "\mathcal{P}"] & \mathrm{End}_{\mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}}(\mathrm{Id}_G) \arrow[u, "\cong"', "\text{Remark 7.2}"]
\end{tikzcd}$$

A direct inspection of the definitions reveals that ρ_G is indeed given by the claimed formula. (We use here the notation ρ_G because this map is essentially a special case of the homonymous one studied in [BD20, Ch. 7.5]; indeed $\rho_G([H,a]_G)$ corresponds in $\mathrm{End}_{\mathsf{Mot}_{\Bbbk}^{\mathsf{coh}}}(\mathrm{Id}_G)$ to the composite equivariant map

$${}_G\Bbbk G_G \overset{{}^r\eta}{\Longrightarrow} {}_G\Bbbk G \otimes_H \Bbbk G_G \overset{\mathrm{Ind}(\gamma_a)}{\Longrightarrow} {}_G\Bbbk G \otimes_H \Bbbk G_G \overset{{}^\ell\varepsilon}{\Longrightarrow} {}_G\Bbbk G_G$$

with ${}^r\eta$ and ${}^\ell\varepsilon$ as in Remark 6.11 and $\gamma_a\colon {}_H\Bbbk G_G \Rightarrow {}_H\Bbbk G_G$ defined by $g \mapsto ag$.) Each ρ_G is a surjective map of commutative rings because it is a composite of such; in particular, $\mathcal{P}$ is 2-full by Theorem 6.13. The remaining claims are immediate from Remark 7.1 and Remark 7.2. ☐

7.6. *Remark.* Note that $\rho_G([H,1]_G) = [G\!:\!H] \in \Bbbk \subset \Bbbk G$ (of course!). Recall that the ordinary *Burnside ring* $\mathrm{B}_{\Bbbk}(G)$ identifies with the subalgebra of $\mathrm{B}^{\mathsf{c}}_{\Bbbk}(G)$ generated by such basis elements, hence is sent to $\Bbbk$ by ρ_G. It follows that, on *cohomological* Mackey 2-functors, the idempotents of $\mathrm{B}_{\Bbbk}(G)$ do not produce any interesting factor. This is wrong for *non*-cohomological Mackey 2-functors, *e.g.* for equivariant stable homotopy theory (see [BD20, Ex. 4.3.8] and [GM95, App. A]).

Very recently, Oda, Takegahara and Yoshida [OTY22] have proved explicit refinements of Theorem 7.5 for some important cases of the base ring $\Bbbk$, by describing the primitive idempotents of $\mathrm{B}^{\mathrm{c}}_{\Bbbk}(G)$ and their behavior under ρ_G in terms of the structure of G. Perhaps not surprisingly, this problem appears to be subtle even for $\Bbbk$ a well-behaved local ring, where there are connections with character theory; *cf.* [Bou03]. On our part, we can offer the following general lifting result:

7.7. **Corollary.** *Assume the commutative ring $\Bbbk$ is a complete local noetherian ring, for instance, a field. In this case, the pseudo-functor $\mathcal{P}\colon \mathsf{Mot}_{\Bbbk} \to \mathsf{Mot}^{\mathsf{coh}}_{\Bbbk}$ is essentially surjective on objects and 1-cells. In particular, it is genuinely a quotient pseudo-functor of $\Bbbk$-linear bicategories – not just up to retracts.*

Proof. By Definition 6.18 and Theorem 6.13, we already know that each object or 1-cell of the block-completion $\mathsf{Mot}^{\mathsf{coh}}_{\Bbbk} = (\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk})^{\flat}$ is a retract of an object or 1-cell in the image of $\mathcal{P}$. Hence it suffices to show that arbitrary idempotents can be lifted along the algebra morphisms $\mathcal{P}\colon \mathrm{End}_{\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}}}(S) \to \mathrm{End}_{\mathsf{biperm}^{\mathsf{rf}}_{\Bbbk}}(\mathcal{P}(S))$ for all 1-cells S of $\Bbbk\widehat{\mathsf{Span}}^{\mathsf{rf}} \cong \Bbbk\widehat{\mathsf{biset}}^{\mathsf{rf}}$. By construction, the latter are surjective (by Theorem 6.13) morphisms of (noncommutative) finite dimensional $\Bbbk$-algebras. For $\Bbbk$ as above, the result now follows from the general lifting theorem [Lin18, Thm. 4.7.1]. □

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