# CODESCENT THEORY I: FOUNDATIONS

PAUL BALMER AND MICHEL MATTHEY

Abstract. Consider a cofibrantly generated model category $\mathcal{S}$, a small category $\mathcal{C}$ and a subcategory $\mathcal{D}$ of $\mathcal{C}$. The category $\mathcal{S}^{\mathcal{C}}$ of functors from $\mathcal{C}$ to $\mathcal{S}$ has a model structure, with weak equivalences and fibrations defined objectwise but only on $\mathcal{D}$. Our first concern is the effect of moving $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{S}$. The main notion introduced here is the "$\mathcal{D}$-codescent" property for objects in $\mathcal{S}^{\mathcal{C}}$. Our program aims at reformulating as codescent statements the Conjectures of Baum-Connes and Farrell-Jones, and, in the long run, at tackling them with new methods. We set the grounds of a systematic theory of codescent, including pull-backs, push-forwards and various invariance properties.

## 1. Introduction

What we shall call *codescent theory* is a simple local-to-global concept, which appears all over mathematics. The first main goal of this paper is to explain this idea in elementary terms. The second one is to prove some general results which will be used in [2] to establish the reformulation as codescent statements of the Baum-Connes Conjecture and of the Farrell-Jones Isomorphism Conjectures. The gain of this reformulation will be twofold. First, it will become conceptually clear what these famous but slightly esoteric conjectures are about. Secondly, it will bring a much more flexible framework in which to study them. For both reasons, it seems important to present a rigorous and elementary explanation of what codescent precisely means and we strove to make the present notes accessible to beginners. Let us start with a heuristic motivation.

Suppose we are studying a family of topological spaces $X(c)$ depending functorially on $c \in \mathcal{C}$, where $c$ can be thought of as a "parameter" belonging to a small category $\mathcal{C}$. Suppose we are given a subset of parameters $\mathcal{D} \subset \mathcal{C}$, possibly much smaller, on which we have some "homotopical information" about $X$, *i.e.* about $X(d)$, only for $d \in \mathcal{D}$; when can we extend this information to the whole of $\mathcal{C}$? For instance, suppose we have two such families of spaces $X$ and $Y$, and suppose we are given a natural transformation $\eta\colon X \longrightarrow Y$ for which we know that $\eta(d)\colon X(d) \longrightarrow Y(d)$ is a weak homotopy equivalence (*i.e.* a $\pi_*$-isomorphism) for each $d \in \mathcal{D}$; when can we guarantee that $\eta(c)\colon X(c) \longrightarrow Y(c)$ is a weak homotopy equivalence for all $c \in \mathcal{C}$? We shall call $\eta$ a *$\mathcal{D}$-weak homotopy equivalence* in the former situation and a *$\mathcal{C}$-weak homotopy equivalence* in the latter. Proving that a $\mathcal{D}$-weak homotopy equivalence between $X$ and $Y$ is indeed a $\mathcal{C}$-weak homotopy equivalence is a prototypical codescent question.

We will give below a model-theoretic definition of codescent, but here is an equivalent formulation, which is better suited for a first introduction since it does not

---

*Date*: July 5, 2004.
Research supported by Swiss National Science Foundation, grant 620-66065.01.

involve homotopical algebra. In particular, because of this, the notion of codescent will be independent of the choice of specific model category structures. For this definition, we need two well-known facts. The first one is that there exists a category $\mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}})$ which is the category $\mathcal{T}\mathrm{op}^{\mathcal{C}}$ of functors from $\mathcal{C}$ to the category $\mathcal{T}\mathrm{op}$ of topological spaces, with the $\mathcal{C}$-weak homotopy equivalences inverted. The restriction of a $\mathcal{C}$-weak homotopy equivalence being trivially a $\mathcal{D}$-weak homotopy equivalence, there is a restriction functor

$$\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\colon\ \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}}) \longrightarrow \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{D}})\,.$$

The second fact we need is that this restriction $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}$ has a left adjoint

$$\mathcal{I}\mathrm{nd}^{\mathcal{C}}_{\mathcal{D}}\colon\ \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{D}}) \longrightarrow \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}})\,.$$

An object $X \in \mathcal{T}\mathrm{op}^{\mathcal{C}}$ *satisfies codescent with respect to* $\mathcal{D}$ exactly when $X$, viewed in $\mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}})$, belongs to the image of this functor $\mathcal{I}\mathrm{nd}^{\mathcal{C}}_{\mathcal{D}}$. This simple definition of codescent suffers from the disadvantage of the category $\mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}})$ and the functor $\mathcal{I}\mathrm{nd}^{\mathcal{C}}_{\mathcal{D}}$ not being described concretely enough. Both are unique up to isomorphism and the important fact is their existence. A concrete construction of $\mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathcal{C}})$ and of $\mathcal{I}\mathrm{nd}^{\mathcal{C}}_{\mathcal{D}}$ is one of the main reasons why model categories enter the game.

A substantial recollection of homotopical algebra is the subject of Appendix A and the reader should proceed to it now, in case of doubt. We start by proving that $\mathcal{T}\mathrm{op}^{\mathcal{C}}$ is equipped with a model category structure in which the weak equivalences are the $\mathcal{D}$-weak homotopy equivalences. Stress the absence of misprint: we really consider $\mathcal{D}$-weak homotopy equivalences on $\mathcal{T}\mathrm{op}^{\mathcal{C}}$. Then any $X \in \mathcal{T}\mathrm{op}^{\mathcal{C}}$ has a so-called *cofibrant replacement* $QX$ for this model structure:

$$QX \xrightarrow[\mathcal{D}\text{-weq}]{\xi_X} X\,.$$

We shall say that $X$ *has the codescent property with respect to* $\mathcal{D}$ (or simply $X$ *satisfies* $\mathcal{D}$*-codescent*) if the map $\xi_X$ is a $\mathcal{C}$-weak homotopy equivalence. We will prove in Theorem 13.5 that this is equivalent to the preceding formulation.

As an illustration of the codescent property, a classical argument of homotopy theory (Ken Brown's Lemma) allows us to answer the initial heuristical question, namely: if $\eta\colon X \longrightarrow Y$ is a $\mathcal{D}$-weak homotopy equivalence and if $X$ and $Y$ both satisfy $\mathcal{D}$-codescent, then $\eta$ is a $\mathcal{C}$-weak homotopy equivalence (see Corollary 6.3).

*It is then a natural and conceptually meaningful problem to determine whether a given functor* $X \in \mathcal{T}\mathrm{op}^{\mathcal{C}}$ *satisfies* $\mathcal{D}$*-codescent and we can thus start looking around in mathematics for functors having this nice property.*

For instance, given a group $G$, consider the orbit categories $\mathcal{C} = \mathrm{Or}(G)$ and $\mathcal{D} = \mathrm{Or}(G, \mathcal{F})$, for some family $\mathcal{F}$ of subgroups, and let $X(H)$ be a space whose $n$-th homotopy group is the $n$-th $K$-theory group $K_n^{\mathrm{alg}}(\mathbb{Z}[H])$, for all $H \in \mathcal{C}$. We shall see in [2] that the morphism $\xi_X$ is essentially a global assembly map and that the natural question whether $X$ satisfies codescent is strongly connected to the Farrell-Jones Isomorphism Conjecture. Namely, for a given group, we will prove that $K$-theory satisfies codescent for these suitable orbit categories if and only if the Isomorphism Conjecture holds for this group *and all its subgroups.* The homotopy theory of diagrams of spaces over an orbit category has long been known, as for instance in [5].

Of course, the terminology is inspired by the notion of *descent* for presheaves of spaces on a Grothendieck site. In algebraic geometry, it is a well-known and often-answered question whether $K$-theory satisfies descent for a given Grothendieck topology. We shall comment further on this analogy in Section 5.

In fact, the category of topological spaces could have been replaced here by any cofibrantly generated model category $\mathcal{S}$, as for example the category $\mathcal{T}\mathrm{op}_\bullet$ of pointed topological spaces, or the category s$\mathcal{S}$ets of simplicial sets, or the category $\mathcal{S}$p of spectra (of pointed simplicial sets, for instance), or even the category $\mathrm{Ch}(R\text{-mod})$ of chain complexes of left $R$-modules for a unital ring $R$. We shall naturally present everything in this generality, both for aesthetical reasons and to ensure the flexibility of the theory.

The aim of the article is *not* to produce yet another model category structure on a category of diagrams, nor to enter the discussion of how general $\mathcal{S}$ can be or how model categories could be re-axiomatized to allow any diagram category to be again a model category. These questions are important but they are not really relevant to us. We are intentionally picking and choosing the level of generality in different sections to suit the exposition. What we need model categories for is to prove theorems about moving the functor $X$ and the shape-categories $\mathcal{C}$ and $\mathcal{D}$. In practice, the category of values $\mathcal{S}$ will be rather dumb : topological spaces, chain complexes and the like. Essentially everything proven in this first part can be obtained after replacing the assumption that $\mathcal{S}$ is cofibrantly generated by the weaker assumption that the $\mathcal{D}$-relative structure on $\mathcal{S}^{\mathcal{C}}$, *i.e.* the one described above, really is a model category structure.

The book Mac Lane [15] will be our reference for general notions from category theory such as adjunctions, (co)units, (co)limits, and so (co)on. Our references for model categories are given in Appendix A.

Here is an outline of the content of the paper.

Consider the category $\mathcal{S}^{\mathcal{C}}$ of covariant functors from a small category $\mathcal{C}$ to a cofibrantly generated model category $\mathcal{S}$. As explained above, we need a *relative* model structure on $\mathcal{S}^{\mathcal{C}}$ with the weak equivalences and the fibrations tested over some given subcategory $\mathcal{D}$ of $\mathcal{C}$, that is, $\mathcal{D}$-objectwise. We denote this model category by $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$. Proving that $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ indeed is a model category is done in Section 3 and involves classical and well-known techniques. Here, we base the proof on a more general result, Theorem 2.1, which says that one can produce a model structure on a given category $\mathcal{B}$, using a set of functors $\{\varepsilon_a : \mathcal{B} \longrightarrow \mathcal{M}_a\}$ from $\mathcal{B}$ to a collection of model categories $\{\mathcal{M}_a\}$.

The notion of $\mathcal{D}$-codescent is introduced in Section 4, where the theory we are mainly concerned with really begins.

In Section 5, we explain, as a background motivation, the analogies and the main differences between codescent and the standard notion of descent in algebraic geometry and $K$-theory.

Section 6 is devoted to the liberty one can take in the definition of codescent and to the resulting flexibility of the codescent property.

In Section 7, we introduce and discuss various Quillen functors at the level of the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, induced by a functorial change of one of the categories $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{D}$. Some useful Quillen adjunctions are established, notably concerning the induction and restriction functors.

In Section 8, some slightly more subtle Quillen adjunctions, that turn out to be crucial in [2], are brought to light. For example, it is shown that under various favorable circumstances, the restriction functor is a *left* Quillen functor, whereas it is, for rather easy reasons, always a right Quillen functor.

Next, in Section 9, we discuss when the Quillen functors of Sections 7 and 8 preserve the codescent property. This constitutes a central part of the paper.

Section 10 gathers basic facts about codescent. The cofibrant replacements in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ are also briefly commented on. In Part II, see [1], we produce very explicit cofibrant approximations in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ under mild conditions on $\mathcal{S}$.

We explain in Section 11 how one can *prune away* some data (namely, some morphisms or objects) from the categories $\mathcal{C}$ and $\mathcal{D}$, without altering the codescent property of a given $X$.

Using results of the paper, we treat some elementary examples in Section 12.

In Section 13, we study the homotopy category of the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$. We describe the functors induced at the level of homotopy categories by the induction and the restriction functors. We also reformulate "at this homotopy level" the codescent property, as first defined in the Introduction. We also prove that the homotopy category of $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ and that of $\mathcal{U}_{\mathcal{S}}(\mathcal{D},\mathcal{D})$ are equivalent categories.

Finally, we introduce the *codescent locus* in Section 14. A way of describing this notion is as follows: the $\mathcal{D}$-codescent locus of a functor $X \in \mathcal{S}^{\mathcal{C}}$ is the largest full subcategory of $\mathcal{C}$ on which the restriction of $X$ satisfies $\mathcal{D}$-codescent. Most of the main results in the paper have a very convenient reformulation in this language. *This very brief section can serve as an index to the rest of the paper.*

Appendix A contains a substantial – but almost minimal for our purposes – recollection of definitions and results on model categories. Appendix B recalls the notion of right and left Kan extensions and the corresponding adjunctions. Roughly speaking, this concerns the various functorial behaviours of the category $\mathcal{S}^{\mathcal{A}}$ under a functorial change of the source-category $\mathcal{A}$.

Sections 6, 9, 10, 11, 13 and 14 are part of the theory of codescent properly speaking, the other sections rather being the necessary preparatory material. Other aspects of the theory will be the subject of forthcoming parts.

**Acknowledgements:** The authors express their gratitude to Daniel Dugger for helpful comments, and to the referee for a very careful reading of the manuscript and for suggesting many improvements regarding the exposition.

## Contents

## 2. Pulling back cofibrantly generated model structures

We start with a rather technical but quite general result on how to define a cofibrantly generated model structure on a given category, by "pulling-back" cofibrantly generated model structures via a set of functors.

Notions such as relative $I$-cell complexes or smallness are recalled in Appendix A, where the definition of a cofibrantly generated model category is also to be found (see A.5 and A.24).

**Theorem 2.1.** *Let $\mathcal{B}$ be a complete and cocomplete category, and let $A$ be a set (of "indices"). Suppose that for every "index" $a \in A$, we are given a cofibrantly generated model category $(\mathcal{M}_a, \mathcal{W}eq_a, \mathcal{C}of_a, \mathcal{F}ib_a)$ with generating sets $I_a \subset \mathcal{C}of_a$ and $J_a \subset \mathcal{W}eq_a \cap \mathcal{C}of_a$. Suppose we are also given functors*

$$\varepsilon_a : \mathcal{B} \longrightarrow \mathcal{M}_a$$

*for all $a \in A$, which fulfill the following three conditions:*

(a) *for $a \in A$, the functor $\varepsilon_a$ preserves pushouts and transfinite compositions;*
(b) *for $a \in A$, the functor $\varepsilon_a$ has a left adjoint $\iota_a : \mathcal{M}_a \longrightarrow \mathcal{B}$;*
(c) *for $a, b \in A$, the following inclusions hold:*

$$\varepsilon_b \circ \iota_a(I_a) \subset I_b\text{-cell} \qquad \textit{and} \qquad \varepsilon_b \circ \iota_a(J_a) \subset J_b\text{-cell}.$$

*Then $\mathcal{B}$ inherits the structure of a cofibrantly generated model category with weak equivalences and fibrations tested via the functors $\{\varepsilon_a\}_{a\in A}$, and with cofibrations given by the left lifting property, as follows:*

$$\begin{aligned} \underline{\mathcal{W}eq} &:= \{f \mid \varepsilon_a(f) \in \mathcal{W}eq_a, \text{ for all } a \in A\} \\ \underline{\mathcal{F}ib} &:= \{f \mid \varepsilon_a(f) \in \mathcal{F}ib_a, \text{ for all } a \in A\} \\ \underline{\mathcal{C}of} &:= \mathrm{LLP}(\underline{\mathcal{W}eq} \cap \underline{\mathcal{F}ib}). \end{aligned}$$

*Furthermore, the sets*

$$\underline{I} := \bigcup_{a\in A} \iota_a(I_a) \qquad \textit{and} \qquad \underline{J} := \bigcup_{a\in A} \iota_a(J_a)$$

*can be taken as sets of generating cofibrations. Finally, for every $a \in A$, we have $\varepsilon_a(\underline{I}\text{-cell}) \subset I_a\text{-cell}$ and $\varepsilon_a(\underline{J}\text{-cell}) \subset J_a\text{-cell}$.*

Morally and typically, functors $\varepsilon_a$ satisfying conditions (a) and (b) would simply be functors preserving small colimits and limits. Condition (c) expresses the relation between the various functors. A key device in the proof will be the following simple observation.

**Lemma 2.2.** *Let $F\colon \mathcal{D} \longrightarrow \mathcal{E}$ be a functor admitting a right adjoint $U\colon \mathcal{E} \longrightarrow \mathcal{D}$.*

(i) *Consider two morphisms $f$ in $\mathcal{D}$ and $g$ in $\mathcal{E}$. Then $g \in \mathrm{RLP}(F(f))$ if and only if $U(g) \in \mathrm{RLP}(f)$.*

(ii) *Assume that $U$ preserves transfinite compositions. Given a class of morphisms $\mathcal{K}$ in $\mathcal{E}$ and an object $d \in \mathcal{D}$ which is small relative to $U(\mathcal{K})$, then $F(d)$ is small relative to $\mathcal{K}$.*

*Proof.* Part (i) is an easy exercise on adjunctions, see if necessary [13, Lem. 2.1.8]. Part (ii) is also easy. Let $\kappa$ be a cardinal such that $d$ is $\kappa$-small relative to $U(\mathcal{K})$. Then, for any $\kappa$-filtered ordinal $\lambda$ and for every $\lambda$-sequence

$$e_0 \longrightarrow e_1 \longrightarrow \dots \longrightarrow e_\beta \longrightarrow \dots$$

in $\mathcal{K}$, its composite with $U$,

$$U(e_0) \longrightarrow U(e_1) \longrightarrow \dots \longrightarrow U(e_\beta) \longrightarrow \dots\,,$$

is a $\lambda$-sequence in $U(\mathcal{K})$ by assumption on $U$. Now, using successively adjunction, $\kappa$-smallness of $d$, the assumption on $U$ again, and adjunction again, we see that

$$\begin{aligned}\operatorname*{colim}_{\beta<\lambda} \mathrm{mor}_{\mathcal{E}}(F(d), e_\beta) &= \operatorname*{colim}_{\beta<\lambda} \mathrm{mor}_{\mathcal{D}}(d, U(e_\beta)) = \mathrm{mor}_{\mathcal{D}}\big(d, \operatorname*{colim}_{\beta<\lambda} U(e_\beta)\big)\\ &= \mathrm{mor}_{\mathcal{D}}\big(d, U\big(\operatorname*{colim}_{\beta<\lambda} e_\beta\big)\big) = \mathrm{mor}_{\mathcal{E}}\big(F(d), \operatorname*{colim}_{\beta<\lambda} e_\beta\big)\,.\end{aligned}$$

This proves that $F(d)$ is $\kappa$-small relative to $\mathcal{K}$. ☐

*Proof of Theorem 2.1.* Let us define $\underline{I}$ and $\underline{J}$ as in the "furthermore part" of the Theorem. We start by making and proving two claims.

*Claim 1*: We have $\mathrm{RLP}(\underline{I}) = \underline{\mathcal{W}eq} \cap \underline{\mathcal{F}ib}$ and $\mathrm{RLP}(\underline{J}) = \underline{\mathcal{F}ib}$.

To see this, we apply Part (i) of Lemma 2.2 for $F := \iota_a$ and $U := \varepsilon_a$:

$$\begin{aligned}\mathrm{RLP}(\underline{I}) &= \mathrm{RLP}\Big(\bigcup_{a\in A} \iota_a(I_a)\Big) = \bigcap_{a\in A} \mathrm{RLP}\big(\iota_a(I_a)\big)\\ &= \bigcap_{a\in A} \varepsilon_a^{-1}\big(\mathrm{RLP}(I_a)\big) = \bigcap_{a\in A} \varepsilon_a^{-1}(\mathcal{W}eq_a \cap \mathcal{F}ib_a) = \underline{\mathcal{W}eq} \cap \underline{\mathcal{F}ib}\,.\end{aligned}$$

A similar argument proves the other equality.

*Claim 2*: For every $b \in A$, we have $\varepsilon_b(\underline{I}\text{-cell}) \subset I_b$-cell and $\varepsilon_b(\underline{J}\text{-cell}) \subset J_b$-cell.

From hypothesis (a), we have $\varepsilon_b(\underline{I}\text{-cell}) \subset \varepsilon_b(\underline{I})$-cell and $\varepsilon_b(\underline{J}\text{-cell}) \subset \varepsilon_b(\underline{J})$-cell. Note that if $K$ is a *set* of $L$-cell complexes, then any $K$-cell complex is an $L$-cell complex. So, we deduce the claim from the inclusions $\varepsilon_b(\underline{I}) \subset I_b$-cell and $\varepsilon_b(\underline{J}) \subset J_b$-cell, which hold by hypothesis (c).

We now want to check that $\mathcal{B}$ and the classes of morphisms $\underline{\mathcal{W}eq}$, $\underline{I}$ and $\underline{J}$ satisfy conditions (K1)-(K6) of Kan's Theorem A.28.

Condition (K1) is easy. Indeed, for every $a \in A$, the condition holds for $\mathcal{W}eq_a$, and $\varepsilon_a$ is a functor. So, the result follows from the equality $\underline{\mathcal{W}eq} = \bigcap_{a\in A} \varepsilon_a^{-1}(\mathcal{W}eq_a)$.

Condition (K2) comes from applying Lemma 2.2 (ii) to $F := \iota_b$ and $U := \varepsilon_b$, with $b \in A$, to $\mathcal{K} := \underline{I}$-cell and to $d$ being the domain of an arbitrary morphism in $I_b$. The hypothesis of Lemma 2.2 (ii) that $d$ is small relative to $U(\mathcal{K})$ follows from the fact – proven in Claim 2 – that $U(\mathcal{K}) \subset I_b$-cell and from the definition of $\mathcal{M}_b$ being cofibrantly generated. This shows that the domain of every morphism in $\iota_b(I_b)$ is small relative to $\underline{I}$-cell. A similar argument applies to $\underline{J}$ and gives (K3).

For Condition (K4), note that Claim 2 implies that we have $\underline{J}$-cell $\subset \underline{\mathcal{W}eq}$ since $J_b$-cell $\subset \mathrm{cof}(J_b) \subset \mathcal{W}eq_b$. So, it suffices to see that $\underline{J}$-cell $\subset \mathrm{cof}(\underline{I})$. It is clear from Claim 1 that $\mathrm{RLP}(\underline{I}) \subset \mathrm{RLP}(\underline{J})$. Applying the obviously inclusion-reversing operation $\mathrm{LLP}(-)$ yields that $\mathrm{cof}(\underline{J}) \subset \mathrm{cof}(\underline{I})$ and *a fortiori* that $\underline{J}$-cell $\subset \mathrm{cof}(\underline{I})$.

Conditions (K5) and (K6) follow immediately from Claim 1, which guarantees, here, that $\mathrm{RLP}(\underline{I}) = \underline{\mathcal{W}eq} \cap \mathrm{RLP}(\underline{J})$. □

**Definition 2.3.** Let $\mathcal{B}$ be a category, $A$ a set, and $\{\varepsilon_a : \mathcal{B} \longrightarrow \mathcal{M}_a\}_{a\in A}$ a collection indexed by $A$ of functors to model categories $\mathcal{M}_a$. Assume that the hypotheses of Theorem 2.1 are satisfied. We shall refer to the induced model structure on $\mathcal{B}$ described in Theorem 2.1 as the *model structure on $\mathcal{B}$ pulled back from $\{\mathcal{M}_a\}_{a\in A}$ via $\{\varepsilon_a\}_{a\in A}$*.

**Proposition 2.4.** *Let $\mathcal{B}$ be a complete and cocomplete category. Consider a collection $\{\varepsilon_a : \mathcal{B} \longrightarrow \mathcal{B}_a\}_{a\in A}$ of functors to complete and cocomplete categories $\mathcal{B}_a$. Consider, for every $a \in A$, a further collection $\{\varphi_{a,b} : \mathcal{B}_a \longrightarrow \mathcal{M}_{a,b}\}_{b\in B_a}$ of functors to cofibrantly generated model categories $\mathcal{M}_{a,b}$. Assume that*

(a) *for every $a \in A$, the collection of functors $\{\varphi_{a,b}\}_{b\in B_a}$ satisfies the hypotheses of Theorem 2.1.*

*Endow each $\mathcal{B}_a$ with the model structure pulled back from $\{\mathcal{M}_{a,b}\}_{b\in B_a}$ via $\{\varphi_{a,b}\}_{b\in B_a}$. Assume further that*

(b) *the collection of functors $\{\varepsilon_a\}_{a\in A}$ satisfies the hypotheses of Theorem 2.1.*

*Then, the whole collection of composed functors $\{\varphi_{a,b} \circ \varepsilon_a\}_{a\in A, b\in B_a}$ satisfies the hypotheses of 2.1 and the model structure on $\mathcal{B}$ pulled back from $\{\mathcal{B}_a\}_{a\in A}$ via $\{\varepsilon_a\}_{a\in A}$ is the same as the model structure pulled back directly from $\{\mathcal{M}_{a,b}\}_{a\in A, b\in B_a}$ via $\{\varphi_{a,b} \circ \varepsilon_a\}_{a\in A, b\in B_a}$.*

*Proof.* We only have to check that the collection of composed functors satisfies the hypotheses (a), (b) and (c) of Theorem 2.1. Conditions (a) and (b) are clear. Condition (c) uses the last sentence of Theorem 2.1 applied to the functors $\{\varepsilon_a\}_{a\in A}$. The rest is straightforward. □

## 3. The model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ on $\mathcal{S}^{\mathcal{C}}$

Suppose given a cofibrantly generated model category $\mathcal{S}$ (see A.24), a small category $\mathcal{C}$ and a subcategory $\mathcal{D}$ of $\mathcal{C}$. As an application of the result of Section 2, we show that there is a model structure on the category $\mathcal{S}^{\mathcal{C}}$ of *covariant* functors from $\mathcal{C}$ to $\mathcal{S}$, *i.e.* of $\mathcal{S}$-valued *co*-presheaves over $\mathcal{C}$, with the weak equivalences and the fibrations defined $\mathcal{D}$-objectwise.

**Convention 3.1.** For the rest of the paper, we make the following agreements:

(i) For a (small) category $\mathcal{C}$, by a *subset of* $\mathcal{C}$, we mean a subset of $\mathrm{obj}(\mathcal{C})$.

(ii) If a subset $\mathcal{D}$ in a (small) category $\mathcal{C}$ is considered itself as a category without further mention, then we mean $\mathcal{D}$ as a *full* subcategory of $\mathcal{C}$.

**Definition 3.2.** It will be convenient to designate by a *pair of small categories* any pair $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is a small category and $\mathcal{D}$ is a subset of $\mathcal{C}$.

**Definition 3.3.** Let $\mathcal{S}$ be a category and $\mathcal{C}$ a small category. We denote by $\mathcal{S}^{\mathcal{C}}$ the category of (covariant) functors from $\mathcal{C}$ to $\mathcal{S}$, with the natural transformations as morphisms. An object in $\mathcal{S}^{\mathcal{C}}$ is sometimes called a *$\mathcal{C}$-diagram in $\mathcal{S}$*. We sometimes refer to $\mathcal{S}$ as the *category of "values"*.

**Definition 3.4.** Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. We call a morphism $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$ a *$\mathcal{D}$-weak equivalence* (respectively a *$\mathcal{D}$-fibration*) if, for every $d \in \mathcal{D}$, the morphism $\eta(d)\colon X(d) \longrightarrow Y(d)$ is a weak equivalence (respectively a fibration) in $\mathcal{S}$. We use respectively and respectfully the following notations:

$$X \xrightarrow[\mathcal{D}\text{-weq}]{\eta} Y \qquad \text{and} \qquad X \xrightarrow[\mathcal{D}\text{-fib}]{\eta} Y\,.$$

A *trivial $\mathcal{D}$-fibration* is a $\mathcal{D}$-fibration which is also a $\mathcal{D}$-weak equivalence.

As kindly pointed out to us by Peter May, the next result is already known as [16, Variant 10], when $\mathcal{S}$ stands for the category of weak Hausdorff $k$-spaces.

**Theorem 3.5.** *Let $\mathcal{S}$ be a cofibrantly generated model category and let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. Consider the category $\mathcal{S}^{\mathcal{C}}$ equipped with $\mathcal{D}$-weak equivalences, $\mathcal{D}$-fibrations and with cofibrations defined by the left lifting property with respect to trivial $\mathcal{D}$-fibrations. Then, this determines a cofibrantly generated model category structure on $\mathcal{S}^{\mathcal{C}}$.*

*Proof.* The category $\mathcal{S}^{\mathcal{C}}$ is complete and cocomplete: *small limits and colimits in $\mathcal{S}^{\mathcal{C}}$ are obtained $\mathcal{C}$-objectwise.* Consider, for any $d \in \mathcal{D}$, the evaluation functor

$$\varepsilon_d\colon \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}, \quad X \longmapsto X(d)\,.$$

This functor $\varepsilon_d$ clearly commutes with small limits and colimits. As can be seen in B.6, its left adjoint $\iota_d\colon \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$ is given by

$$\iota_d(s)\colon \mathcal{C} \longrightarrow \mathcal{S}, \quad c \longmapsto \coprod_{\mathrm{mor}_{\mathcal{C}}(d,c)} s\,,$$

for every object $s \in \mathcal{S}$, and by

$$\iota_d(\alpha)\colon \iota_d(s) \longrightarrow \iota_d(s'), \quad c \longmapsto \coprod_{\mathrm{mor}_{\mathcal{C}}(d,c)} \alpha\,,$$

for every morphism $\alpha\colon s \longrightarrow s'$ in $\mathcal{S}$. In particular, for $d$ and $b$ in $\mathcal{D}$,

$$\varepsilon_b \circ \iota_d(\alpha) = \coprod_{\mathrm{mor}_{\mathcal{C}}(d,b)} \alpha$$

is a coproduct of copies of $\alpha$. We apply Theorem 2.1 with $\mathcal{B} := \mathcal{S}^{\mathcal{C}}$, $A := \mathrm{obj}\,\mathcal{D}$, and, for every $d \in \mathcal{D}$, with $\mathcal{M}_d := \mathcal{S}$ and $\varepsilon_d$ as above. Conditions (a) and (b) are clear. To see that Condition (c) is fulfilled, observe that a coproduct of maps in $I$ is an $I$-cell. This can be found in [13, Lem. 2.1.13] for instance. $\square$

**Notation 3.6.** Let $\mathcal{S}$ be a cofibrantly generated model category and let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. The model category on $\mathcal{S}^{\mathcal{C}}$ defined in Theorem 3.5 will be denoted by

$$\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) := \mathcal{S}^{\mathcal{C}} \text{ with the model structure of Theorem 3.5.}$$

When $\mathcal{D} = \mathcal{C}$, we also write $\mathcal{U}_{\mathcal{S}}(\mathcal{C})$ for $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{C})$. If $\mathcal{S}$ is clear from the context, we drop it from the notations, writing $\mathcal{U}(\mathcal{C}, \mathcal{D})$ and $\mathcal{U}(\mathcal{C})$ respectively. This notation is inspired by the one in Dugger [6], although he writes $U\mathcal{C}$ for our $\mathcal{U}_{\mathrm{s}\mathcal{S}\mathrm{ets}}(\mathcal{C}^{\mathrm{op}})$.

**Definition 3.7.** A morphism in $\mathcal{S}^{\mathcal{C}}$ that is a cofibration in $\mathcal{U}(\mathcal{C},\mathcal{D})$ is called a *$\mathcal{D}$-cofibration*, although this can *not* be tested $\mathcal{D}$-objectwise in general; *trivial $\mathcal{D}$-cofibrations* are the trivial cofibrations of $\mathcal{U}(\mathcal{C},\mathcal{D})$. In the same spirit, an object $X \in \mathcal{S}^{\mathcal{C}}$ is called *$\mathcal{D}$-cofibrant* if it is cofibrant in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ (see A.7).

*Remark* 3.8. As the proof of Theorem 3.5 shows, the model structure on $\mathcal{U}(\mathcal{C},\mathcal{D})$ does only depend on the *set of objects* $\mathcal{D}$ and not on morphisms between those objects (hence Definition 3.2).

*Remark* 3.9. Note that the functorial factorizations for $\mathcal{U}(\mathcal{C},\mathcal{D})$ (and hence the cofibrant replacement) are given by Theorem A.28 and its proof, that is, those functorial factorizations are obtained via Quillen's small object argument with respect to $\underline{I}$ and $\underline{J}$, see [12] or [13]. For more on this topic, we refer to the final part of Section 10 below.

*Remark* 3.10. When $\mathcal{S} = \mathrm{s}\mathcal{S}\mathrm{ets}$ and $\mathcal{D} = \mathcal{C}$, Theorem 3.5 gives in particular the model structure of Dwyer-Kan [7], which is also the "left" model structure of Heller [11, § II.4]. The special case where $\mathcal{D} = \mathcal{C}$ with $\mathcal{S}$ an arbitrary cofibrantly generated model category is also to be found in Hirschhorn [12, § 11.6].

*Remark* 3.11. For a subcategory $\mathcal{D}$ of a small category $\mathcal{C}$, and for $\mathcal{S}$ equal to the category of simplicial sets or of topological spaces, the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ does not coincide with the category $\mathcal{S}^{\mathcal{C},\mathcal{D}}$ considered by Dwyer and Kan in [8] : the latter is the category of $\mathcal{D}$-restricted $\mathcal{C}$-diagrams, that is, the full subcategory of the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C})$ of those $X \in \mathcal{S}^{\mathcal{C}}$ such that $X(\alpha)$ is a weak equivalence in $\mathcal{S}$ for every morphism $\alpha$ in $\mathcal{D}$. So, this is really different from what we consider here.

* * *

For the notion of retract, used in the next definition, we refer to A.4 (i).

**Definition 3.12.** Let $\mathcal{D}$ and $\mathcal{D}'$ be two subsets of a (small) category $\mathcal{C}$. We call $\mathcal{D}$ and $\mathcal{D}'$ *essentially equivalent* in $\mathcal{C}$ if every object of $\mathcal{D}$ is isomorphic in $\mathcal{C}$ to some object of $\mathcal{D}'$ and if every object of $\mathcal{D}'$ is isomorphic in $\mathcal{C}$ to some object of $\mathcal{D}$. We say that $\mathcal{D}$ and $\mathcal{D}'$ are *retract equivalent* in $\mathcal{C}$ if every object of $\mathcal{D}$ is a retract in $\mathcal{C}$ of some object of $\mathcal{D}'$ and if every object of $\mathcal{D}'$ is a retract in $\mathcal{C}$ of some object of $\mathcal{D}$.

If $\mathcal{D}$ and $\mathcal{D}'$ are essentially equivalent, then they are retract equivalent.

**Proposition 3.13.** *Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ and $\mathcal{D}'$ be subsets of $\mathcal{C}$, that are retract equivalent in the above sense. Then, the model structures $\mathcal{U}(\mathcal{C},\mathcal{D})$ and $\mathcal{U}(\mathcal{C},\mathcal{D}')$ on the category $\mathcal{S}^{\mathcal{C}}$ are the same, up to the choice of the functorial factorizations.*

*Proof.* If an object $d$ is a retract of some object $d'$ and if a morphism $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$ is a weak equivalence or a fibration at $d'$ then the same is true at $d$, by Axiom **(MC 3)** for the model category $\mathcal{S}$. Thus $\mathcal{U}(\mathcal{C},\mathcal{D})$ and $\mathcal{U}(\mathcal{C},\mathcal{D}')$ have the same weak equivalences and the same fibrations. Hence the result (see A.12 if needed). □

**Proposition 3.14.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories.*

(i) *Let $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}$ be a subset bigger than $\mathcal{D}$. In $\mathcal{S}^{\mathcal{C}}$, every $\mathcal{D}$-cofibration is an $\mathcal{E}$-cofibration and every trivial $\mathcal{D}$-cofibration is a trivial $\mathcal{E}$-cofibration. In particular, $\mathcal{D}$-cofibrant objects are $\mathcal{E}$-cofibrant.*

(ii) *If a morphism $\eta$ in $\mathcal{S}^{\mathcal{C}}$ is a (trivial) $\mathcal{D}$-cofibration, then $\eta(c)$ is a (trivial) cofibration in $\mathcal{S}$ for all $c \in \mathcal{C}$. In particular, a $\mathcal{D}$-cofibrant diagram $X \in \mathcal{S}^{\mathcal{C}}$ is $\mathcal{C}$-objectwise cofibrant, i.e. $X(c)$ is cofibrant in $\mathcal{S}$, for all $c \in \mathcal{C}$.*

*Proof.* Clearly, being a (trivial) $\mathcal{E}$-fibration is more than being a (trivial) $\mathcal{D}$-fibration. Therefore, the morphisms having the left lifting property with respect to (trivial) $\mathcal{D}$-fibrations, will have that property with respect to (trivial) $\mathcal{E}$-fibrations. This gives (i) (see A.12 if necessary). Now, by (i), for $\mathcal{E} = \mathcal{C}$, every (trivial) $\mathcal{D}$-cofibration is a (trivial) $\mathcal{C}$-cofibration. Then, to prove (ii), it suffices to know that a $\mathcal{C}$-cofibration is objectwise a cofibration. This is proven in [12, Prop. 11.6.3]. We give an alternative proof in Remark 8.8 below. □

**Examples 3.15.** We give a couple of "limit" examples for pairs $(\mathcal{C}, \mathcal{D})$.

(1) Assume that $\mathcal{D} = \varnothing$ is empty. Then, there is no condition to satisfy to be a $\mathcal{D}$-fibration or a $\mathcal{D}$-weak equivalence, and consequently, every morphism is a trivial $\mathcal{D}$-fibration. In this case, the $\mathcal{D}$-cofibrations are exactly the isomorphisms, as is easily checked.
(2) Let us assume that $\mathcal{C}$ is discrete (see B.5). In this situation, $\mathcal{S}^{\mathcal{C}}$ is the legitimate notion for the product $\prod \mathcal{S}$ of $|\operatorname{obj}(\mathcal{C})|$ copies of the model category $\mathcal{S}$. It is easy to check that $\mathcal{D}$-cofibrations are exactly those morphisms $\eta$ such that $\eta(c)$ is a cofibration when $c \in \mathcal{D}$, and an isomorphism when $c \notin \mathcal{D}$.

## 4. The notion of $\mathcal{D}$-codescent in $\mathcal{S}^{\mathcal{C}}$

For this section, we fix $\mathcal{S}$ a cofibrantly generated model category (see A.24), and we drop it from the notations. We define here the $\mathcal{D}$-codescent property for a functor $X \in \mathcal{S}^{\mathcal{C}}$, where $\mathcal{D}$ is a subcategory of $\mathcal{C}$. We also discuss some examples.

We start with the following observation.

*Remark* 4.1. Let $\mathcal{M}$ be a model category. One can distinguish different notions of "cofibrant substitutions". Namely, concerning the choice of an assignment

$$(\mathcal{Q}, \xi) \colon \mathcal{M} \longrightarrow \operatorname{arr}(\mathcal{M}), \quad X \longmapsto (\xi_X : \mathcal{Q}X \to X),$$

with $\mathcal{Q}X$ cofibrant and $\xi_X$ a weak equivalence, one can require or not $\mathcal{Q}$ to be functorial; one can only require that $\xi_X$ is a weak equivalence or one can further require that it is a fibration; finally, in the strictest sense, $\mathcal{Q}$ could be *the* functorial factorization **(MC 5)** (a) in $\mathcal{M}$ applied to the (unique) morphism $\varnothing \longrightarrow X$, in which case $\xi_X$ is a trivial fibration. We will not distinguish all these notions here for sake of readability, but will focus on the most rigid and the most flexible ones. So, following [12], we will say that $(\mathcal{Q}X, \xi_X)$ – or, abusively, $\mathcal{Q}X$ – is:

- *the cofibrant replacement* (and we write $Q$ in place of $\mathcal{Q}$) if it is obtained by the factorization axiom applied to $\varnothing \longrightarrow X$;
- *a cofibrant approximation* if $\mathcal{Q}X$ is cofibrant and $\xi_X$ is a weak equivalence.

We will see in the very useful Propositions 6.5 and 6.6 how these differences can be dealt with, and how flexible codescent is with this respect.

**Notation 4.2.** We denote the cofibrant replacement in $\mathcal{U}(\mathcal{C}, \mathcal{D})$ by

$$Q^{\mathcal{C}}_{\mathcal{D}} \colon \mathcal{U}(\mathcal{C}, \mathcal{D}) \longrightarrow \operatorname{arr}\big(\mathcal{U}(\mathcal{C}, \mathcal{D})\big), \quad X \longmapsto \big(\xi^{\mathcal{C},\mathcal{D}}_X \colon Q^{\mathcal{C}}_{\mathcal{D}}X \to X\big).$$

When $\mathcal{D} = \mathcal{C}$, we also write $\xi^{\mathcal{C}}_X$ and $Q_{\mathcal{C}}X$.

**Definition 4.3.** Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories, and let $X \in \mathcal{S}^{\mathcal{C}}$. We say that $X$ satisfies $\mathcal{D}$-*codescent* (or *codescent with respect to* $\mathcal{D}$) if the morphism

$$\xi_X^{\mathcal{C},\mathcal{D}} : Q_{\mathcal{D}}^{\mathcal{C}} X \longrightarrow X$$

in $\mathcal{U}(\mathcal{C},\mathcal{D})$ is a $\mathcal{C}$-weak equivalence; we sometimes say that $X$ is a $\mathcal{D}$-*codescending* object. For a given object $c \in \mathcal{C}$, we say that $X$ satisfies $\mathcal{D}$-*codescent at* $c$, if the morphism

$$\xi_X^{\mathcal{C},\mathcal{D}}(c) \colon Q_{\mathcal{D}}^{\mathcal{C}} X(c) \longrightarrow X(c)$$

is a weak equivalence in $\mathcal{S}$. Given a subset $\mathcal{A}$ of $\mathcal{C}$, we say that $X$ satisfies $\mathcal{D}$-*codescent on* $\mathcal{A}$, if it satisfies $\mathcal{D}$-codescent at every object $c \in \mathcal{A}$.

So, $X$ satisfies $\mathcal{D}$-codescent if and only if it satisfies $\mathcal{D}$-codescent on $\mathcal{C} \smallsetminus \mathcal{D}$.

* * *

Before starting the general theory (cf. Section 6 and following), we present a few basic, but hopefully instructive, examples.

**Example 4.4.** We first give two examples sitting at two opposite ends.

(1) Assume that $\mathcal{D} = \varnothing$. Then, by Example 3.15 (1), the initial object $\varnothing$ of $\mathcal{S}^{\mathcal{C}}$ is, up to isomorphism, the unique cofibrant object in $\mathcal{U}(\mathcal{C},\mathcal{D})$. Therefore, an $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at $c \in \mathcal{C}$ if and only if the unique morphism $\varnothing \longrightarrow X(c)$ is a weak equivalence in $\mathcal{S}$. In short, $X$ satisfies codescent exactly where $\varnothing \longrightarrow X(c)$ is a weak equivalence.

(2) Assume that $\mathcal{D} = \mathcal{C}$. Then, every $X$ satisfies $\mathcal{D}$-codescent everywhere. This is tautological: $\mathcal{D}$-codescent involves deciding whether a certain $\mathcal{D}$-weak equivalence is a $\mathcal{C}$-weak equivalence. Note however that not every $X$ is $\mathcal{D}$-cofibrant, for $X$ being $\mathcal{D}$-cofibrant requires $X(c)$ to be cofibrant in $\mathcal{S}$, for each $c \in \mathcal{C}$ (see Proposition 3.14 (ii)).

The next example illustrates the flavour of codescent quite well.

**Example 4.5.** Consider the category

$$\mathcal{C} \quad := \quad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "\mathrm{id}_d"] \arrow[r, "\alpha"] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

with only two objects $d$ and $c$ and one non-identity morphism $\alpha \colon d \longrightarrow c$. Let $\mathcal{D}$ be the full subcategory with $d$ as unique object. Giving an object $X \in \mathcal{S}^{\mathcal{C}}$ consists in giving two elements of $\mathcal{S}$, say $X_1$ and $X_2$, related by a morphism, say $x \colon X_1 \longrightarrow X_2$, which is $X(\alpha)$. To give a morphism $\eta \colon X \longrightarrow X'$ amounts to give two morphisms $\eta_1 : X_1 \longrightarrow X_1'$ and $\eta_2 \colon X_2 \longrightarrow X_2'$ such that $x'\eta_1 = \eta_2 x$ (with the obvious notations). Let us determine when an object

$$X \quad \overset{\text{def.}}{=} \quad X_1 \xrightarrow{x} X_2$$

is $\mathcal{D}$-cofibrant in $\mathcal{U}(\mathcal{C},\mathcal{D})$. By Proposition 3.14 (ii), we know that $X_1$ and $X_2$ must be cofibrant in $\mathcal{S}$. Now, consider the commutative square in $\mathcal{S}^{\mathcal{C}}$

$$
\begin{tikzcd}
\varnothing \arrow[r, bend left] \arrow[d] & Y \arrow[d, "p"] \\
X \arrow[r, bend right, "\mathrm{id}"'] \arrow[ur, dashed, "h"] & X
\end{tikzcd}
\quad \overset{\text{def.}}{=} \quad
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[rr, bend left] & \varnothing \arrow[d] \arrow[rr, bend left] & X_1 \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & X_1 \arrow[d, "x"] \\
X_1 \arrow[r, "x"] \arrow[rr, bend right, "\mathrm{id}"'] & X_2 \arrow[rr, bend right, "\mathrm{id}"'] & X_1 \arrow[r, "x"] & X_2
\end{tikzcd}
$$

where $Y$ and $p\colon Y \longrightarrow X$ are defined by the right-hand diagram. It is clear that $p$ is a trivial $\mathcal{D}$-fibration since it is a $\mathcal{D}$-isomorphism. If $X$ is $\mathcal{D}$-cofibrant, there must exist a lift $h\colon X \longrightarrow Y$ and it is easy to see that $h_1 = \mathrm{id}_{X_1}$, and that $h_2\colon X_2 \longrightarrow X_1$ is a two-sided inverse of $x$. So, for $X$ to be cofibrant, we need $x$ to be an isomorphism. Conversely, assume that $X_1$ and $X_2$ are cofibrant and that $x$ is an isomorphism. Consider a square

$$
\begin{tikzcd}
\varnothing \arrow[r, bend left] \arrow[d] & Y \arrow[d, "q"] \\
X \arrow[r, bend right, "v"'] \arrow[ur, dashed, "k"] & Z
\end{tikzcd}
\quad \overset{\text{def.}}{=} \quad
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[rr, bend left] & \varnothing \arrow[d] \arrow[rr, bend left] & Y_1 \arrow[r, "y"'] \arrow[d, "q_1"] & Y_2 \arrow[d, "q_2"] \\
X_1 \arrow[r, "x"] \arrow[rr, bend right, "v_1"'] & X_2 \arrow[rr, bend right, "v_2"'] & Z_1 \arrow[r, "z"] & Z_2
\end{tikzcd}
$$

where $q$ is a trivial $\mathcal{D}$-fibration. Since $X_1$ is cofibrant, there is a lift $k_1 : X_1 \longrightarrow Y_1$ such that $q_1 k_1 = v_1$. It is then easy to see that $k_1$ and $k_2 := y\, k_1\, x^{-1}$ define a lift $k\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$. In short,

$$X \in \mathcal{S}^{\mathcal{C}} \textit{ is } \mathcal{D}\textit{-cofibrant} \quad \textit{iff} \quad X(\alpha) \textit{ is an iso between cofibrant objects in } \mathcal{S}.$$

Using this, it is immediate to see that

$$X \in \mathcal{S}^{\mathcal{C}} \textit{ satisfies } \mathcal{D}\textit{-codescent} \quad \textit{iff} \quad X(\alpha) \textit{ is a weak equivalence.}$$

(This again illustrates the fact that there are many more objects satisfying $\mathcal{D}$-codescent than $\mathcal{D}$-cofibrant objects.) We leave it as an exercise for the interested reader to check that the same two statements hold if $\mathcal{C}$ is replaced by the category

$$
\begin{tikzcd}
\overset{d}{\bullet} \arrow[loop left, "\mathrm{id}_d"] \arrow[r, "\alpha"] & \overset{c}{\bullet} \arrow[loop right, "M"]
\end{tikzcd}
$$

with $M$ denoting any monoid of endomorphisms of $c$.

*Remark* 4.6. In Section 12, we will further illustrate the situation for $\mathcal{C}$ "extremely small", namely with 2 objects, and for $\mathcal{D}$ reduced to a one-object category. Although this sounds very limited and restrictive, these types of examples already contain the basic non-trivial general properties of codescent. We also point out that for a *torsion-free* discrete group $G$, the Baum-Connes Conjecture will be reformulated in [2] as a codescent statement with $\mathcal{C}$ a two-object category of the form

$$\begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "G"] \arrow[r, "\alpha"] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

and with $\mathcal{D}$ having $d$ as unique object.

## 5. Codescent versus descent

The present section is a heuristical discussion, that aims at putting codescent in some perspective, by comparison with the standard notion of descent in algebraic geometry and $K$-theory. The ideas discussed here only reflect the authors' current opinion and will not be used in the sequel.

Given a Grothendieck topology on $\mathcal{C}$, there is a model structure on simplicial presheaves $\mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{C}^{\mathrm{op}}}$ – which is due to Joyal and Jardine, see for instance [14] – in which the weak equivalences are tested stalkwise when the site has enough points (and we assume this for simplicity here). The cofibrations are openwise cofibrations, that is, cofibrations at each $c \in \mathcal{C}$. In this situation, dually to what happens with codescent, the cofibrations are clear and the fibrations are mysterious: they are defined by the right lifting property with respect to trivial cofibrations. Given a presheaf $Y \in \mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{C}^{\mathrm{op}}}$, it is then a legitimate question to look at the *fibrant* replacement

$$\zeta\colon Y \longrightarrow R(Y)\,,$$

which is, by definition, a stalkwise weak equivalence, and to wonder when this morphism $\zeta$ is indeed an openwise weak equivalence. This is exactly the *descent problem* for $Y$ with respect to the given Grothendieck topology. See for instance Mitchell [17] for a first introduction to these ideas. Similarly, one can – and should – consider presheaves of spectra, or with other values $\mathcal{S}$, as we also do here.

Thomason has proven that the algebraic $K$-theory spectrum he defines in [19] satisfies descent for both the Zariski and the Nisnevich topology.

It is legitimate to wonder if codescent is not merely a form of descent, up to some opposite-category-yoga. We explain now why we consider this as misleading. Of course, there is an isomorphism of categories between the category of functors from $\mathcal{C}$ to $\mathcal{S}$ and presheaves on $\mathcal{C}^{\mathrm{op}}$ with values in $\mathcal{S}^{\mathrm{op}}$, say

$$\alpha\colon \mathcal{S}^{\mathcal{C}} \overset{\cong}{\longleftrightarrow} (\mathcal{S}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$$

Therefore, there is a model structure on the right-hand side transported from $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, for an arbitrary choice of the subcategory $\mathcal{D}$. Note that this isomorphism of categories $\alpha$ is indeed *contravariant* and consequently, on the right, it is the fibrant replacement $R(-)$ which is now mysterious and hence interesting. Our codescent property for an $X \in \mathcal{S}^{\mathcal{C}}$ translates into a descent-like property: when is the morphism $\alpha X \longrightarrow R(\alpha X)$ from $\alpha X$ to its fibrant replacement an objectwise, *i.e.* openwise, weak equivalence?

This sounds very coherent but faces the following drawbacks, in our opinion:

(1) In principle, no one wants to work with the opposite category of simplicial sets $\mathcal{S} = \mathrm{s}\mathcal{S}\mathrm{ets}^{\mathrm{op}}$, or similarly with $\mathcal{T}\mathrm{op}^{\mathrm{op}}$, having the good old morphisms of "spaces" going backwards. In terms of marketing, it seems reasonable to stick to the usual maps of "spaces", in their usual direction. This commercial policy forces the category of values $\mathcal{S}$, and hence prevents us from doing the above $\alpha$-switching to $\mathcal{S}^{\mathrm{op}}$.

(2) More seriously, for a functor like algebraic $K$-theory of group rings, say $K(R[G])$ with $R$ varying among commutative unital rings and $G$ among discrete groups, there really are *two* different functorial dependencies of $K(R[G])$ involved. First, there is the dependence on the ring $R$, with morphisms induced by ring homomorphisms out of $R$, say $R \longrightarrow R'$, in the Zariski or Nisnevich site to fix the ideas; this is responsible for descent questions. Secondly, there is the dependence on the group $G$, with morphisms induced by group homomorphisms to $G$, say $\varphi\colon H \longrightarrow G$, where, typically, $H$ is a subgroup and $\varphi$ is a conjugation-inclusion; this is responsible for *co*descent. In symbols, we have:

$$K\big(R[H]\big) \xrightarrow{\text{codesc.}} K\big(R[G]\big) \xrightarrow{\text{desc.}} K\big(R'[G]\big)\,.$$

So, even if we perform the above $\alpha$-switch, we still have *two different "descents" involved.*

(3) Moreover, not only the two morphisms described above can occur simultaneously, but they are indeed going in *two opposite directions*. The two morphisms appearing in (2) could both go "from local to global" for instance or both "from global to local" but this is *not* the case. Namely, in the codescent situation, we know things about $X(d)$ and want to extend it to $X(c)$ but morally $X$ moves the information from $X(d)$ to $X(c)$, that is, from the "local object" to the "global object". In the descent problem, the restriction goes from $X(U)$ to $X(V)$ for $V \subset U$ and hence tends to go from the "global object" towards the "local objects". This "direction" of codescent is more formally explained by the Pruning Lemmas, see Remark 11.8 below.

Nevertheless, the analogy might be more important than the difference, at least conceptually speaking, and might also be a source of inspiration for attacking codescent questions. It would also be interesting to have some kind of unified treatment of both codescent and descent, not only in *one type* of conjectures as we achieve here and in [2], but really in *one common conjecture.*

## 6. Flexibility of codescent

The present section is the beginning of codescent theory itself. We establish the first properties related to the notion of codescent. We fix a cofibrantly generated model category $\mathcal{S}$ (see A.24) for the rest of the section.

Recall that Ken Brown's Lemma states, in particular, that if a functor between model categories takes trivial cofibrations between cofibrant objects to weak equivalences, then it takes all weak equivalences between cofibrant objects to weak equivalences (see [13, Lem. 1.1.12]).

**Proposition 6.1** (Rigidity of cofibrant objects).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. If a morphism $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$ is a $\mathcal{D}$-weak equivalence and if $X$ and $Y$ are $\mathcal{D}$-cofibrant, then $\eta$ is a $\mathcal{C}$-weak equivalence. Therefore, the cofibrant replacement*

$$Q^{\mathcal{C}}_{\mathcal{D}}\colon \mathcal{U}(\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{U}(\mathcal{C},\mathcal{D})$$

*takes $\mathcal{D}$-weak equivalences to $\mathcal{C}$-weak equivalences.*

*Proof.* Consider the identity functor $\mathcal{U}(\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{U}(\mathcal{C})$. We claim that it preserves all trivial cofibrations, which will be enough by Ken Brown's Lemma. This holds by the case $\mathcal{E} = \mathcal{C}$ in Proposition 3.14 (i), proving the first part.

For the second part, note that $Q^{\mathcal{C}}_{\mathcal{D}}$ preserves $\mathcal{D}$-weak equivalences, like any cofibrant replacement functor (see A.14 if necessary). Hence $Q^{\mathcal{C}}_{\mathcal{D}}$ turns $\mathcal{D}$-weak equivalences into $\mathcal{D}$-weak equivalences between cofibrant objects, which are $\mathcal{C}$-weak equivalences by the first part of the proof. □

**Corollary 6.2** (Codescent for cofibrant objects).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Then $\mathcal{D}$-cofibrant objects in $\mathcal{S}^{\mathcal{C}}$ satisfy $\mathcal{D}$-codescent.* □

For example, the constant functor $X = \varnothing$ in $\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent, whatever the subset $\mathcal{D}$ looks like. As Example 4.4 (2) shows, there are fortunately many more objects satisfying $\mathcal{D}$-codescent, than $\mathcal{D}$-cofibrant objects (see Example 4.5 as well).

As another application of Proposition 6.1, we get the result mentioned as a motivation in the Introduction, where $\mathcal{S}$ was merely chosen to be the category of topological spaces in order to fix the ideas.

**Corollary 6.3** (Rigidity of codescending objects).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Consider a $\mathcal{D}$-weak equivalence $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$. If $X$ and $Y$ satisfy $\mathcal{D}$-codescent, then $\eta$ is a $\mathcal{C}$-weak equivalence.*

*Proof.* By assumption, we have a commutative diagram

$$\begin{tikzcd}
Q^{\mathcal{C}}_{\mathcal{D}}X \arrow[r, "Q^{\mathcal{C}}_{\mathcal{D}}\eta", "\mathcal{D}\text{-weq}"'] \arrow[d, "\mathcal{C}\text{-weq}", "\xi^{\mathcal{C},\mathcal{D}}_X"'] & Q^{\mathcal{C}}_{\mathcal{D}}Y \arrow[d, "\mathcal{C}\text{-weq}"', "\xi^{\mathcal{C},\mathcal{D}}_Y"] \\
X \arrow[r, "\eta", "\mathcal{D}\text{-weq}"'] & Y
\end{tikzcd}$$

By Proposition 6.1, $Q^{\mathcal{C}}_{\mathcal{D}}\eta$ is a $\mathcal{C}$-weak equivalence, and the result follows by 2-out-of-3 for $\mathcal{C}$-weak equivalences (that is, in $\mathcal{U}(\mathcal{C})$). □

*Remark* 6.4. The class of $\mathcal{D}$-codescending objects in $\mathcal{S}^{\mathcal{C}}$ is *maximal* among the subclasses $\mathcal{K}$ of $\operatorname{obj}\left(\mathcal{S}^{\mathcal{C}}\right)$ such that every $\mathcal{D}$-weak equivalence between objects of $\mathcal{K}$ is a $\mathcal{C}$-weak equivalence. Indeed, let $\mathcal{K}$ be a bigger class, *i.e.* such a class containing all $\mathcal{D}$-codescending objects. If $X \in \mathcal{K}$, then $\xi^{\mathcal{C},\mathcal{D}}_X\colon Q^{\mathcal{C}}_{\mathcal{D}}X \longrightarrow X$ is a $\mathcal{D}$-weak equivalence and $Q^{\mathcal{C}}_{\mathcal{D}}X \in \mathcal{K}$ by assumption on $\mathcal{K}$ and by Corollary 6.2. It follows from Corollary 6.3 that $\xi^{\mathcal{C},\mathcal{D}}_X$ is a $\mathcal{C}$-weak equivalence. This proves that $X$ satisfies $\mathcal{D}$-codescent, as was to be shown.

* * *

**Proposition 6.5** (Local flexibility of codescent)**.**
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Then, for $X \in \mathcal{S}^{\mathcal{C}}$ and $c \in \mathcal{C}$, the following properties are equivalent:*

(i) *$X$ satisfies $\mathcal{D}$-codescent at $c$;*
(ii) *there exists a trivial $\mathcal{D}$-fibration $\eta\colon X' \longrightarrow X$ for some $X'$ which is $\mathcal{D}$-cofibrant and such that $\eta(c)$ is a weak equivalence;*
(iii) *for every trivial $\mathcal{D}$-fibration $\eta\colon X' \longrightarrow X$, where $X'$ is $\mathcal{D}$-cofibrant, $\eta(c)$ is a weak equivalence;*
(iv) *there exists a $\mathcal{D}\cup\{c\}$-weak equivalence $\eta\colon X' \longrightarrow X$ for some $X'$ which is $\mathcal{D}$-cofibrant;*
(v) *for every $\mathcal{D}$-weak equivalence $\eta\colon X' \longrightarrow X$, where $X'$ is $\mathcal{D}$-cofibrant, $\eta(c)$ is a weak equivalence.*

*Proof.* Since $\xi_X^{\mathcal{C},\mathcal{D}} : Q_{\mathcal{D}}^{\mathcal{C}}X \longrightarrow X$ is a trivial $\mathcal{D}$-fibration, one clearly has

$$(\mathrm{v}) \Longrightarrow (\mathrm{iii}) \Longrightarrow (\mathrm{i}) \Longrightarrow (\mathrm{ii}) \Longrightarrow (\mathrm{iv})\,.$$

(iv)$\Longrightarrow$(v) : Let $\eta\colon X' \longrightarrow X$ be a $\mathcal{D}\cup\{c\}$-weak equivalence where $X'$ is some $\mathcal{D}$-cofibrant object. Now, for a $\mathcal{D}$-weak equivalence $\zeta\colon Y \longrightarrow X$, where $Y$ is $\mathcal{D}$-cofibrant, consider the following commutative diagram obtained by applying the functorial cofibrant replacement $Q_{\mathcal{D}}^{\mathcal{C}}$ to everything in sight:

$$\begin{tikzcd}
Q_{\mathcal{D}}^{\mathcal{C}}X' \arrow[r, "Q_{\mathcal{D}}^{\mathcal{C}}\eta", "\mathcal{C}\text{-weq}"'] \arrow[d, "\xi_{X'}^{\mathcal{C},\mathcal{D}}"', "\mathcal{C}\text{-weq}"] & Q_{\mathcal{D}}^{\mathcal{C}}X \arrow[d, "\xi_{X}^{\mathcal{C},\mathcal{D}}"', "\mathcal{D}\text{-weq}"] & Q_{\mathcal{D}}^{\mathcal{C}}Y \arrow[l, "Q_{\mathcal{D}}^{\mathcal{C}}\zeta"', "\mathcal{C}\text{-weq}"] \arrow[d, "\xi_{Y}^{\mathcal{C},\mathcal{D}}"', "\mathcal{C}\text{-weq}"] \\
X' \arrow[r, "\eta", "\mathcal{D}\cup\{c\}\text{-weq}"'] & X & Y \arrow[l, "\zeta"', "\mathcal{D}\text{-weq}"]
\end{tikzcd}$$

The $\mathcal{C}$-weak equivalences are in fact $\mathcal{D}$-weak equivalences upgraded via rigidity of cofibrant objects 6.1. Now, $\eta(c)$ being a weak equivalence forces the same for $\xi_X^{\mathcal{C},\mathcal{D}}(c)$ by the left square and, in turn, that $\zeta(c)$ is a weak equivalence by the right square. □

**Proposition 6.6** (Global flexibility of codescent)**.**
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Then, for $X \in \mathcal{S}^{\mathcal{C}}$, the following properties are equivalent:*

(i) *$X$ satisfies $\mathcal{D}$-codescent;*
(ii) *there exists a trivial $\mathcal{D}$-fibration $\eta\colon X' \longrightarrow X$ for some $X'$ which is $\mathcal{D}$-cofibrant and such that $\eta$ is a $\mathcal{C}$-weak equivalence;*
(iii) *for every trivial $\mathcal{D}$-fibration $\eta\colon X' \longrightarrow X$, where $X'$ is $\mathcal{D}$-cofibrant, $\eta$ is a $\mathcal{C}$-weak equivalence;*
(iv) *there exists a $\mathcal{C}$-weak equivalence $\eta\colon X' \longrightarrow X$ for some $X'$ which is $\mathcal{D}$-cofibrant;*
(v) *for every $\mathcal{D}$-weak equivalence $\eta\colon X' \longrightarrow X$, where $X'$ is $\mathcal{D}$-cofibrant, $\eta$ is a $\mathcal{C}$-weak equivalence.*

*Proof.* As before, the only non-immediate implication is (iv)$\Longrightarrow$(v), which follows from a $\mathcal{C}$-objectwise application of (iv)$\Longrightarrow$(v) in Proposition 6.5. □

*Remark* 6.7. The bottom line of the global (resp. local) flexibility of codescent 6.6 (resp. 6.5) is that one can define the $\mathcal{D}$-codescent property (resp. at $c$) using *any* cofibrant approximation (4.1) in place of the cofibrant replacement that we used in Definition 4.3.

**Example 6.8.** Assume that $\mathcal{C}$ is a *discrete* category (see B.5) and that $\mathcal{D} \subset \mathcal{C}$. As seen in Example 3.15 (2), a diagram $X' \in \mathcal{S}^{\mathcal{C}}$ is $\mathcal{D}$-cofibrant if and only if it takes cofibrant values on $\mathcal{D}$ and the value $\varnothing$ (up to isomorphism) outside $\mathcal{D}$. Therefore, using local flexibility of codescent 6.5, one readily checks that $X$ satisfies $\mathcal{D}$-codescent if and only if $\varnothing \longrightarrow X(c)$ is a weak equivalence for every $c \in \mathcal{C} \smallsetminus \mathcal{D}$, without condition over $\mathcal{D}$.

*Remark* 6.9. The global (resp. local) flexibility of codescent 6.6 (resp. 6.5) also shows that if $\mathcal{D}$ and $\mathcal{E}$ are subcategories of a small category $\mathcal{C}$ and if the model categories $\mathcal{U}(\mathcal{C},\mathcal{D})$ and $\mathcal{U}(\mathcal{C},\mathcal{E})$ share the same weak equivalences and cofibrant objects, then $\mathcal{D}$-codescent (resp. at $c$) is equivalent to $\mathcal{E}$-codescent (resp. at $c$); see for instance Proposition 3.13.

* * *

**Proposition 6.10** (Weak invariance of codescent).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Let $\eta\colon X \longrightarrow Y$ be a morphism in $\mathcal{S}^{\mathcal{C}}$.*

(i) *Let $c \in \mathcal{C}$ and assume that $\eta$ is a $\mathcal{D}\cup\{c\}$-weak equivalence. Then $X$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $Y$ satisfies $\mathcal{D}$-codescent at $c$.*
(ii) *Assume that $\eta$ is a $\mathcal{C}$-weak equivalence. Then $X$ satisfies $\mathcal{D}$-codescent if and only if $Y$ satisfies $\mathcal{D}$-codescent.*

*Proof.* Choose $X'$ which is $\mathcal{D}$-cofibrant with a $\mathcal{D}$-weak equivalence $\xi\colon X' \longrightarrow X$. Consider the $\mathcal{D}$-weak equivalence $\zeta := \eta\circ\xi\colon X' \longrightarrow Y$. If $\eta(c)$ is a weak equivalence for some $c \in \mathcal{C}$, we have that $\xi(c)$ is a weak equivalence if and only if so is $\zeta(c)$. Now, (i) is a consequence of local flexibility of codescent 6.5, and (ii) follows. □

**Corollary 6.11.** *Let $F\colon \mathcal{S} \longrightarrow \mathcal{S}$ be an endofunctor of the model category $\mathcal{S}$ of values, and consider a natural transformation $\alpha\colon \mathrm{id}_{\mathcal{S}} \longrightarrow F$ or $\alpha\colon F \longrightarrow \mathrm{id}_{\mathcal{S}}$ such that $\alpha(s)$ is a weak equivalence in $\mathcal{S}$ for every $s$ in $\mathcal{S}$ – for instance, $F$ could be the fibrant or the cofibrant replacement in $\mathcal{S}$.*

*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Let $X \in \mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ and consider the composition $F\circ X \in \mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$. Then $X$ satisfies $\mathcal{D}$-codescent exactly where $F\circ X$ does. In particular, when deciding whether $X$ satisfies $\mathcal{D}$-codescent, one can always assume that $X$ is $\mathcal{C}$-objectwise cofibrant, fibrant or both.*

*Proof.* By assumption, $\alpha$ induces, objectwise, a natural transformation between $X$ and $F\circ X$, which is a $\mathcal{C}$-weak equivalence. The first result follows from weak invariance of codescent 6.10. The second is a direct consequence, noting that the fibrant replacement of a cofibrant object is fibrant *and* cofibrant. □

*Remark* 6.12. This Corollary stresses the fact that $X$ satisfying $\mathcal{D}$-codescent has essentially nothing to do with the fact that $X$ takes cofibrant or fibrant values in $\mathcal{S}$. It is more a question of knowing to what extent the interrelation of $\mathcal{D}$ and $\mathcal{C}$ is revealed by $X$ (see however Proposition 9.1 (ii) below; compare with Example 6.8).

## 7. Some Quillen adjunctions "forwards" for $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$

In the present section, we discuss various functors at the level of $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, related to a functorial change of the variable-categories $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{D}$. The title of the section will be justified at its end (see Remark 7.7 below).

Recall from A.16 the notion of Quillen adjunction, which should be thought of as a morphism in the "category" of model categories.

**Proposition 7.1.** *Let $F\colon \mathcal{S} \rightleftarrows \mathcal{T} :U$ be a Quillen adjunction between cofibrantly generated model categories. Then, the induced pair of functors*

$$F^{\mathcal{C}}\colon \mathcal{S}^{\mathcal{C}} \rightleftarrows \mathcal{T}^{\mathcal{C}} :U^{\mathcal{C}}\,,$$

*defined by $F^{\mathcal{C}}(X) := F\circ X$ and $U^{\mathcal{C}}(Y) := U\circ Y$, form a Quillen adjunction between $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ and $\mathcal{U}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ for any choice of $\mathcal{D}\subset\mathcal{C}$; in particular, $F^{\mathcal{C}}$ preserves cofibrant objects and weak equivalences between them.*

*Proof.* The functors $(F^{\mathcal{C}},U^{\mathcal{C}})$ are adjoint, see [12, Lem. 11.6.4]. Clearly, $U^{\mathcal{C}}$ preserves $\mathcal{D}$-fibrations and trivial $\mathcal{D}$-fibrations, since $U$ does preserve fibrations and trivial fibrations (see Remark A.17) and since, by the very definition, $\mathcal{D}$-weak equivalences and $\mathcal{D}$-fibrations are tested $\mathcal{D}$-objectwise. Therefore, $F^{\mathcal{C}}$ is a left Quillen functor (by A.17 again). The latter also yields the stated properties of $F^{\mathcal{C}}$. □

* * *

From now on, in this section, we shall not move the category of values $\mathcal{S}$, and we fix this notation below, *i.e.* $\mathcal{S}$ is a cofibrantly generated model category.

**Lemma 7.2.** *Let $\Phi\colon \mathcal{A}\longrightarrow\mathcal{C}$ be a functor between small categories, and consider the induced functor*

$$\Phi^*\colon \mathcal{S}^{\mathcal{C}}\longrightarrow\mathcal{S}^{\mathcal{A}},\quad X\longmapsto X\circ\Phi\,.$$

*Let $\mathcal{D}\subset\mathcal{C}$ and $\mathcal{B}\subset\mathcal{A}$ be subsets. Consider $\Phi^*$ as a functor between model categories $\mathcal{U}(\mathcal{C},\mathcal{D})\longrightarrow\mathcal{U}(\mathcal{A},\mathcal{B})$ and recall the terminology of A.15.*

(i) *If $\Phi(\mathcal{B})\subset\mathcal{D}$, then $\Phi^*$ preserves weak equivalences and fibrations.*
(ii) *If $\Phi(\mathcal{B})\supset\mathcal{D}$, then $\Phi^*$ detects weak equivalences and fibrations.*
(iii) *If $\Phi(\mathcal{B})=\mathcal{D}$, then $\Phi^*$ reflects weak equivalences and fibrations.*

*Proof.* Follows from Definition A.15, using that $\Phi^*\eta\,(b)=\eta\big(\Phi(b)\big)$ for $b\in\mathcal{B}$. □

**Definition 7.3.** Recall from 3.2 that a *pair of small categories* means a pair $(\mathcal{C},\mathcal{D})$, where $\mathcal{C}$ is a small category and $\mathcal{D}$ is a chosen subset of objects of $\mathcal{C}$. A *morphism* of such pairs, $\Phi\colon(\mathcal{A},\mathcal{B})\longrightarrow(\mathcal{C},\mathcal{D})$, is a functor $\Phi\colon\mathcal{A}\longrightarrow\mathcal{C}$ such that $\Phi(\mathcal{B})\subset\mathcal{D}$ (inclusion of sets of objects); when we write "$\Phi(\mathcal{B})=\mathcal{D}$", we really mean an equality of sets of objects.

**Definition 7.4.** By a *full inclusion of pairs*, $(\mathcal{A},\mathcal{B})\hookrightarrow(\mathcal{C},\mathcal{D})$, we mean a full inclusion $\mathcal{A}\hookrightarrow\mathcal{C}$ such that $\mathcal{B}$ is contained in $\mathcal{D}$. This is of course a morphism of pairs as defined above.

**Proposition 7.5.** *Let $\Phi\colon(\mathcal{A},\mathcal{B})\longrightarrow(\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Then, the functor $\Phi^*$ and its left adjoint $\Phi_*\colon\mathcal{S}^{\mathcal{A}}\longrightarrow\mathcal{S}^{\mathcal{C}}$ form a Quillen adjunction:*

$$\Phi_*\colon\mathcal{U}(\mathcal{A},\mathcal{B})\rightleftarrows\mathcal{U}(\mathcal{C},\mathcal{D}):\Phi^*\,.$$

*In particular, $\Phi_*$ preserves cofibrant objects and weak equivalences between them.*

*Proof.* The existence of the left adjoint $\Phi_*$ (also called the left Kan extension) is classical and is recalled in Appendix B. By Lemma 7.2 (i), $\Phi^*$ is a right Quillen functor, see Remark A.17. $\square$

**Corollary 7.6.** *Let $(\mathcal{A},\mathcal{B}) \hookrightarrow (\mathcal{C},\mathcal{D})$ be a full inclusion of pairs of small categories and consider the functors $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}}$ and $\mathrm{ind}^{\mathcal{C}}_{\mathcal{A}}$ as in Appendix B. Then*

$$\mathrm{ind}^{\mathcal{C}}_{\mathcal{A}} \colon \mathcal{U}(\mathcal{A},\mathcal{B}) \rightleftarrows \mathcal{U}(\mathcal{C},\mathcal{D}) : \mathrm{res}^{\mathcal{C}}_{\mathcal{A}}$$

*form a Quillen adjunction. In particular, the induction of a $\mathcal{B}$-cofibrant object is $\mathcal{D}$-cofibrant.*

*Proof.* Immediate from Proposition 7.5 and the definition of $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}}$ and $\mathrm{ind}^{\mathcal{C}}_{\mathcal{A}}$. $\square$

* * *

*Remark* 7.7. For a morphism of pairs $\Phi \colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$, the functor $\Phi^*$ and its *left* adjoint form a Quillen adjunction $\Phi_* \colon \mathcal{U}(\mathcal{A},\mathcal{B}) \rightleftarrows \mathcal{U}(\mathcal{C},\mathcal{D}) : \Phi^*$, as described in Proposition 7.5. This Quillen adjunction should be seen as "going from $\mathcal{U}(\mathcal{A},\mathcal{B})$ to $\mathcal{U}(\mathcal{C},\mathcal{D})$". From our point of view, this is the "forward" functorial direction of the construction $\mathcal{U}(-,-)$. This exists for *any* morphism of pairs $\Phi$.

However, there are *some* morphisms of pairs $\Phi \colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ where $\Phi^*$ and its *right* adjoint $\Phi_!$ also form a Quillen adjunction $\Phi^* \colon \mathcal{U}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{A},\mathcal{B}) : \Phi_!$, seen as a morphism of model categories $(\Phi^*, \Phi_!)$ going from $\mathcal{U}(\mathcal{C},\mathcal{D})$ to $\mathcal{U}(\mathcal{A},\mathcal{B})$, *i.e.* going "backwards". This is what we discuss in the next section.

## 8. Some Quillen adjunctions "backwards" for $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$

The reader opening the article at random is invited to read Remark 7.7 at the end of the previous section, before proceeding through this one.

Consider a morphism $\Phi$ of pairs (see 7.3). Here, we determine conditions guaranteeing that the functor $\Phi^*$, induced by $\Phi$, is a *left* Quillen functor (compare 7.5). Again, we fix a cofibrantly generated model category $\mathcal{S}$ (see A.24).

**Definition 8.1.** Let $\Phi \colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs. We shall say that $\Phi$ is *left glossy* if the following condition is satisfied: for every object $b \in \mathcal{B}$, there is a set of morphisms in $\mathcal{C}$

$$\big\{\beta_i \colon \Phi(b) \longrightarrow \Phi(b_i)\big\}_{i \in E_b}$$

all having source $\Phi(b)$ and with various targets $\Phi(b_i)$, such that

(i) the objects $b_i$ also belong to $\mathcal{B}$;
(ii) for every morphism $\alpha \colon \Phi(b) \longrightarrow \Phi(a)$ in $\mathcal{C}$ with $a \in \mathcal{A}$, there exists a *unique* pair $(i,\gamma)$, with $i$ an "index" in $E_b$ and $\gamma$ a morphism $b_i \longrightarrow a$ in $\mathcal{A}$, such that $\alpha = \Phi(\gamma) \circ \beta_i$, that is,

$$\begin{tikzcd}
\Phi(b) \arrow[rr, "\forall\alpha"] \arrow[dr, dashed, "(\exists! i \in E_b)\ \beta_i"'] & & \Phi(a) \\
& \Phi(b_i) \arrow[ur, dashed, "\Phi(\gamma)\ (\exists! \gamma \colon b_i \to a)"'] &
\end{tikzcd}$$

The terminology "glossy" is introduced to avoid using the mathematically overused expressions "good", "special" and the like.

Observe that condition (ii) has to be verified for all $a$ in $\mathcal{A}$, including those contained in $\mathcal{B}$ (see for instance the *two* conditions required in Example 12.5 below).

**Example 8.2.** Let $(\mathcal{A},\mathcal{B}) \hookrightarrow (\mathcal{C},\mathcal{D})$ be a full inclusion of pairs of small categories (see 7.4). Then, this inclusion $(\mathcal{A},\mathcal{B}) \hookrightarrow (\mathcal{C},\mathcal{D})$ is left glossy. It suffices to take for each $b \in \mathcal{B}$ the set $E_b := \{1\}$, with $b_1 := b$ and $\beta_1 := \mathrm{id}_b$.

**Example 8.3.** Here is an "extreme" example, which shows that left glossiness can be very far from fullness. Let $\mathcal{C}$ be a small category and let $\mathcal{C}'$ be the corresponding discrete subcategory (B.5), that is, with the same objects and only with the identities as morphisms. Then, the inclusion $(\mathcal{C}',\mathcal{C}') \hookrightarrow (\mathcal{C},\mathcal{C})$ is left glossy. It suffices to take for each $b \in \mathcal{C}'$ the set $E_b := \coprod_{c\in\mathcal{C}} \mathrm{mor}_{\mathcal{C}}(b,c)$, with, for every "index" $i\colon b \longrightarrow c$ in $E_b$, $b_i := c$ and $\beta_i := i$.

*Remark* 8.4. Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. For any $b \in \mathcal{B}$, consider the inclusion of comma categories (see B.1)

$$\big(\Phi(b) \searrow \Phi_{|\mathcal{B}}\big) \hookrightarrow \big(\Phi(b) \searrow \Phi\big)\,,$$

where $\Phi_{|\mathcal{B}}$ is the restriction of $\Phi$ to a functor $\mathcal{B} \longrightarrow \mathcal{D}$ (recall Convention 3.1 (ii)). Saying that $\Phi$ is left glossy is indeed tautologically equivalent to assuming that for every $b \in \mathcal{B}$, there is a *discrete* subcategory $\mathcal{E}_b \subset \big(\Phi(b) \searrow \Phi_{|\mathcal{B}}\big)$ such that the composite inclusion

$$\mathcal{E}_b \hookrightarrow \big(\Phi(b) \searrow \Phi_{|\mathcal{B}}\big) \hookrightarrow \big(\Phi(b) \searrow \Phi\big)$$

is an *initial* functor, as defined in [15, § IX.3, pp. 217–218] (this is also called *left cofinal* by some authors, like in [12, 14.2.1]). This $\mathcal{E}_b$ has nothing but the set $\{(b_i,\beta_i)\}_{i\in E_b}$ of Definition 8.1 as objects. The main consequence of initiality is that a limit over an initial subcategory 'coincides' with the limit over the whole category, see [15, § IX.3] or [12, Thm. 14.2.5 (2)]. Since a limit over a discrete category is merely the corresponding product, we have in particular that for any functor $Y\colon \mathcal{A} \longrightarrow \mathcal{S}$, the obvious morphism

$$\lim_{\left(a\,,\,\Phi(b) \overset{\alpha}{\to} \Phi(a)\right)\,\in\,\Phi(b)\searrow\Phi} Y(a) \quad\longrightarrow\quad \prod_{i\in E_b} Y(b_i)$$

is an isomorphism, natural in $Y$.

**Lemma 8.5.** *Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Assume that $\Phi$ is left glossy. Then, for $Y \in \mathcal{S}^{\mathcal{A}}$ and $b \in \mathcal{B}$, there is an isomorphism*

$$\Phi^*\Phi_!Y(b) \cong \prod_{i\in E_b} Y(b_i)\,,$$

*that is natural in $Y$ (where notations are kept as in Definition 8.1).*

*Proof.* By Definition B.3, we have the formula

$$\Phi_!Y(c) \quad=\quad \lim_{\left(a\,,\,c \overset{\alpha}{\to} \Phi(a)\right)\,\in\,c\searrow\Phi} Y(a)\,,$$

for $Y \in \mathcal{S}^{\mathcal{A}}$ and $c \in \mathcal{C}$. Applying it to $c := \Phi(b)$ with $b \in \mathcal{B}$, we get

$$\Phi^*\Phi_!Y(b) \quad=\quad \Phi_!Y(\Phi(b)) \quad=\quad \lim_{\left(a\,,\,\Phi(b) \overset{\alpha}{\to} \Phi(a)\right)\,\in\,\Phi(b)\searrow\Phi} Y(a) \quad\cong\quad \prod_{i\in E_b} Y(b_i)\,,$$

where the isomorphism on the right holds by Remark 8.4. $\square$

**Theorem 8.6.** *Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Assume that the following properties hold :*

(a) $\mathcal{D} = \Phi(\mathcal{B})$;

(b) $\Phi$ *is left glossy (see 8.1).*

*Then, the functor $\Phi^*$ and its right adjoint $\Phi_!\colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{C}}$ form a Quillen adjunction*

$$\Phi^*\colon \mathcal{U}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{A},\mathcal{B}) :\Phi_!\,.$$

*In particular, the functor $\Phi^*$ preserves cofibrations and fibrations, and reflects weak equivalences.*

*Proof.* We want to prove that $\Phi_!$ preserves fibrations and trivial fibrations (see A.17). By assumption (a) and by Lemma 7.2 (iii), it suffices to see that $\Phi^*\Phi_!$ preserves fibrations and trivial fibrations. Let $\eta\colon Y_1 \longrightarrow Y_2$ be a (trivial) $\mathcal{B}$-fibration in $\mathcal{U}(\mathcal{A},\mathcal{B})$. This means that $\eta(b)\colon Y_1(b) \longrightarrow Y_2(b)$ is a (trivial) fibration in $\mathcal{S}$ for every $b\in\mathcal{B}$. Fix an object $b\in\mathcal{B}$ and choose a set $\{\beta_i\colon \Phi(b)\longrightarrow\Phi(b_i)\}_{i\in E_b}$ like in Definition 8.1. By Lemma 8.5, we have $\Phi^*\Phi_!\eta(b)\cong\prod_{i\in E_b}\eta(b_i)$. Since $b_i\in\mathcal{B}$ for all $i\in E_b$, we deduce that $\Phi^*\Phi_!\eta(b)$ is a product of (trivial) fibrations in $\mathcal{S}$ and hence is again a (trivial) fibration (see A.12). Since this is true for an arbitrary $b\in\mathcal{B}$, the first result follows. For the "In particular" part, invoke Remark A.17, Proposition 7.5 and Lemma 7.2 (iii). □

**Corollary 8.7.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories and let $\mathcal{A}\subset\mathcal{C}$ be a full subcategory containing $\mathcal{D}$. Then, the functor $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}}$ and its right adjoint $\mathrm{ext}^{\mathcal{C}}_{\mathcal{A}}$ form a Quillen adjunction :*

$$\mathrm{res}^{\mathcal{C}}_{\mathcal{A}}\colon \mathcal{U}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{A},\mathcal{D}) :\mathrm{ext}^{\mathcal{C}}_{\mathcal{A}}\ .$$

*In particular, the restriction to $\mathcal{A}$ of a $\mathcal{D}$-cofibrant object is $\mathcal{D}$-cofibrant, and the functor $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}}$ preserves cofibrations and fibrations, and reflects weak equivalences.*

*Proof.* For the first part, apply Theorem 8.6 to the full inclusion $(\mathcal{A},\mathcal{D})\hookrightarrow(\mathcal{C},\mathcal{D})$ as in Example 8.2 with $\mathcal{B}:=\mathcal{D}$. The rest is clear. □

*Remark* 8.8. Let $\mathcal{C}$ be a small category. Let us prove directly that every $\mathcal{C}$-cofibration is objectwise a cofibration (see the proof of 3.14 (ii), where we referred to [12]). By Example 8.3 and Theorem 8.6, the restriction of our $\mathcal{C}$-cofibration to the corresponding discrete subcategory $\mathcal{C}'$ is a $\mathcal{C}'$-cofibration. On a discrete category, this is equivalent to being a cofibration objectwise as seen in Example 3.15 (2). Stress that Corollary 8.7 was not applied to the non-full subcategory $\mathcal{C}'$.

*Remark* 8.9. The assumption $\Phi(\mathcal{B})=\mathcal{D}$ which appears in Theorem 8.6, instead of our usual $\Phi(\mathcal{B})\subset\mathcal{D}$, is indeed not so restrictive. In fact, any morphism of pairs $\Phi\colon(\mathcal{A},\mathcal{B})\longrightarrow(\mathcal{C},\mathcal{D})$ can be written as a composition

$$(\mathcal{A},\mathcal{B})\longrightarrow(\mathcal{C},\Phi(\mathcal{B}))\hookrightarrow(\mathcal{C},\mathcal{D})\,,$$

where the first morphism is clearly surjective on the "$\mathcal{D}$-part" and where the second morphism is a full inclusion. Some of those full inclusions can be treated independently as we now explain.

* * *

We single out some particular full inclusions which still produce Quillen adjunction "backwards" (compare Remark 7.7).

**Definition 8.10.** Let $\mathcal{A}$ be a subset of a (small) category $\mathcal{C}$. We say that $\mathcal{A}$ is *left absorbant* in $\mathcal{C}$, if for every morphism $c \longrightarrow a$ in $\mathcal{C}$ with $a \in \mathcal{A}$, the object $c$ belongs to $\mathcal{A}$ as well.

**Lemma 8.11.** *Let $\mathcal{A} \hookrightarrow \mathcal{C}$ be a full subcategory of a small category $\mathcal{C}$, that is left absorbant in $\mathcal{C}$. Then, the right adjoint $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{C}}$ of the restriction functor $\mathrm{res}_{\mathcal{A}}^{\mathcal{C}}$ admits the following explicit description. For any $X \in \mathcal{S}^{\mathcal{A}}$, the functor $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X$ is equal to the functor $X$ on $\mathcal{A}$ and takes the value $*$ on objects of $\mathcal{C} \smallsetminus \mathcal{A}$, where $*$ is the terminal object in $\mathcal{S}$; this uniquely determines the functor $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X \colon \mathcal{C} \longrightarrow \mathcal{S}$ on morphisms.*

*Moreover, a natural transformation $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{A}}$ induces a natural transformation $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X \longrightarrow \mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} Y$ in the obvious way, namely as $\eta$ on $\mathcal{A}$ and as the identity of $*$ outside $\mathcal{A}$.*

*Proof.* Note that $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X$, as defined in the statement, is a well-defined functor on $\mathcal{C}$ because there are no morphisms $\gamma\colon c \longrightarrow a$ in $\mathcal{C}$, with $c \in \mathcal{C} \smallsetminus \mathcal{A}$ and $a \in \mathcal{A}$, by left absorbance of $\mathcal{A}$. So, the only morphisms in $\mathcal{C}$ for which $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X$ should be defined are those of $\mathcal{A}$, to which we apply $X$, and those with target outside $\mathcal{A}$, which we send to the only morphism in $\mathcal{S}$ with target $*$. The functoriality of $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X$ is an easy exercise. The functoriality of $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}}$ is an easy exercise as well.

The fact that this functor $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}}$ describes the right adjoint to $\mathrm{res}_{\mathcal{A}}^{\mathcal{C}}$ can be checked directly or using the description of $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}}$ which is given in B.3. Both ways use the left absorbance of $\mathcal{A}$ again. □

**Proposition 8.12.** *Let $(\mathcal{A}, \mathcal{B}) \hookrightarrow (\mathcal{C}, \mathcal{D})$ be a* full *inclusion of pairs of small categories (see 7.4). Assume that $\mathcal{A}$ is left absorbant in $\mathcal{C}$ as defined in 8.10. Assume further that $\mathcal{D} \cap \mathcal{A} = \mathcal{B}$. Then, the functor $\mathrm{res}_{\mathcal{A}}^{\mathcal{C}}$ and its right adjoint $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}}$ form a Quillen adjunction:*

$$\mathrm{res}_{\mathcal{A}}^{\mathcal{C}} \colon \mathcal{U}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{A}, \mathcal{B}) : \mathrm{ext}_{\mathcal{A}}^{\mathcal{C}}.$$

*In particular, the restriction to $\mathcal{A}$ of a $\mathcal{D}$-cofibrant object is $\mathcal{B}$-cofibrant, and the functor $\mathrm{res}_{\mathcal{A}}^{\mathcal{C}}$ preserves cofibrations, fibrations and weak equivalences.*

*Proof.* Using the description of $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} X$ given in Lemma 8.11, let us check that if a morphism $\eta$ is a (trivial) $\mathcal{B}$-fibration in $\mathcal{S}^{\mathcal{A}}$, then $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \eta$ is a (trivial) $\mathcal{D}$-fibration in $\mathcal{S}^{\mathcal{C}}$. The latter is tested $\mathcal{D}$-objectwise. For an object $d \in \mathcal{D}$, two cases can occur. Either $d$ does not belong to $\mathcal{A}$, in which case the source and target of $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \eta\,(d)$ are both equal to $*$, so that $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \eta\,(d)$ is an isomorphism; or $d$ does belong to $\mathcal{A}$, and hence to $\mathcal{B}$ by assumption, in which case $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \eta(d) = \eta(d)$ is a (trivial) fibration by choice of $\eta$. In both cases, $\mathrm{ext}_{\mathcal{A}}^{\mathcal{C}} \eta\,(d)$ is a (trivial) fibration. Hence the result.

The final sentence of the statement is an easy consequence; see Lemma 7.2 (i), Corollary 7.6 and Remark A.17. □

## 9. Functors reflecting codescent

In this section, we use the results of Sections 7 and 8 to move the codescent property from a triple $\mathcal{S}$, $\mathcal{C}$, $\mathcal{D}$ to another.

We first see how the change of the category of values $\mathcal{S}$ can reflect codescent. For the next statement, recall the terminology of A.15.

**Proposition 9.1.** *Let $F\colon \mathcal{S} \rightleftarrows \mathcal{T} : U$ be a Quillen adjunction between cofibrantly generated model categories. Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. Let $X \in \mathcal{S}^{\mathcal{C}}$ and $c \in \mathcal{C}$.*

(i) *If $F$ preserves weak equivalences and if $X$ satisfies $\mathcal{D}$-codescent at $c$, then $F \circ X$ also satisfies $\mathcal{D}$-codescent at $c$.*
(ii) *If $X$ is objectwise cofibrant and satisfies $\mathcal{D}$-codescent at $c$, then $F \circ X$ also satisfies $\mathcal{D}$-codescent at $c$.*
(iii) *If $F$ reflects weak equivalences, then $X$ satisfies $\mathcal{D}$-codescent exactly where $F \circ X$ does.*

*Proof.* Recall the notations introduced in Proposition 7.1, where it is proven that the functor $F^{\mathcal{C}}\colon \mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ preserves cofibrant objects. Consider a $\mathcal{D}$-cofibrant approximation (4.1) $\eta\colon X' \longrightarrow X$ of $X$ in $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$. Consider the morphism $F^{\mathcal{C}}\eta\colon F^{\mathcal{C}}X' \longrightarrow F^{\mathcal{C}}X$. Note that $F^{\mathcal{C}}X'$ is $\mathcal{D}$-cofibrant and let us check that $F^{\mathcal{C}}\eta$ is a $\mathcal{D}$-weak equivalence in $\mathcal{T}^{\mathcal{C}}$. In cases (i) and (iii), this is clear. The same is indeed true in case (ii), since $F$ preserves weak equivalences between cofibrant objects (see Remark A.17). So, $F^{\mathcal{C}}\eta\colon F^{\mathcal{C}}X' \longrightarrow F^{\mathcal{C}}X$ is a $\mathcal{D}$-cofibrant approximation of $F^{\mathcal{C}}X$ in $\mathcal{U}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$.

Let $c \in \mathcal{C}$. By local flexibility of codescent 6.5, we know that $X$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $\eta(c)$ is a weak equivalence, and that $F^{\mathcal{C}}X$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $F^{\mathcal{C}}\eta(c) = F(\eta(c))$ is a weak equivalence. The three stated results follow easily. □

Note that in (ii) above, it is enough for $X$ to be $\mathcal{D} \cup \{c\}$-objectwise cofibrant and to satisfy $\mathcal{D}$-codescent at $c$.

*Remark* 9.2. In real life, using weak invariance of codescent 6.10, we can always replace a given $X$ by a $\mathcal{C}$-objectwise cofibrant $Y$ which will satisfy $\mathcal{D}$-codescent exactly where $X$ does. For such a $Y$, we can apply part (ii) above, without requiring $F$ to preserve weak equivalences, to get that $F \circ Y$ satisfies $\mathcal{D}$-codescent where $X$ does.

**Example 9.3.** The typical situation where we want to apply Proposition 9.1, is when $F = |{-}|$ is the geometric realization, say, from simplicial sets to topological spaces. This reflects weak equivalences by the very definition of weak equivalences of simplicial sets. In other words, an $X \in s\mathcal{S}\mathrm{ets}^{\mathcal{C}}$ will satisfy codescent exactly where its realization $|X| \in \mathcal{T}\mathrm{op}^{\mathcal{C}}$ does (and similarly "in the pointed situation").

* * *

We now turn to the functor $\Phi_* : \mathcal{U}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{U}(\mathcal{C}, \mathcal{D})$ induced by a morphism $\Phi\colon (\mathcal{A}, \mathcal{B}) \longrightarrow (\mathcal{C}, \mathcal{D})$ of pairs of small categories (see 7.3). For the rest of this section, we fix a cofibrantly generated model category $\mathcal{S}$.

**Proposition 9.4.** *Let $\Phi\colon (\mathcal{A}, \mathcal{B}) \longrightarrow (\mathcal{C}, \mathcal{D})$ be a morphism of pairs of small categories, and fix an object $a \in \mathcal{A}$. Assume the following:*

(a) *$\mathcal{D} = \Phi(\mathcal{B})$;*
(b) *$\Phi^*\Phi_*$ preserves $\mathcal{B}$-weak equivalences (see A.15);*
(c) *$\Phi^*\Phi_*$ reflects $\{a\}$-weak equivalences (see A.15).*

*Consider a diagram $Y \in \mathcal{S}^{\mathcal{A}}$. Then $Y$ satisfies $\mathcal{B}$-codescent at $a$ if and only if $\Phi_*Y$ satisfies $\mathcal{D}$-codescent at $\Phi(a)$.*

*Proof.* By Proposition 7.5, the functor $\Phi_* : \mathcal{U}(\mathcal{A},\mathcal{B}) \longrightarrow \mathcal{U}(\mathcal{C},\mathcal{D})$ preserves cofibrant objects. In fact it also preserves weak equivalences, as follows readily from (a), (b) and Lemma 7.2. Let $\eta\colon Y' \longrightarrow Y$ be a $\mathcal{B}$-cofibrant approximation to $Y$ in $\mathcal{U}(\mathcal{A},\mathcal{B})$ (see 4.1). Then $\Phi_*\eta\colon \Phi_*Y' \longrightarrow \Phi_*Y$ is a $\mathcal{D}$-cofibrant approximation to $\Phi_*Y$. It is a weak equivalence at $\Phi(a)$ if and only if $\Phi^*\Phi_*\eta(a)$ is a weak equivalence which, in turn, amounts to $\eta(a)$ being a weak equivalence, by hypothesis (c). The result follows from local flexibility of codescent 6.5. $\square$

**Corollary 9.5** (Induction property for codescent)**.**
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories, and $\mathcal{A} \subset \mathcal{C}$ a full subcategory containing $\mathcal{D}$. Consider a diagram $Y \in \mathcal{S}^{\mathcal{A}}$ and $a \in \mathcal{A}$. Then $Y$ satisfies $\mathcal{D}$-codescent at $a$ if and only if $\operatorname{ind}_{\mathcal{A}}^{\mathcal{C}} Y$ does.*

*Proof.* The full inclusion $(\mathcal{A},\mathcal{D}) \longrightarrow (\mathcal{C},\mathcal{D})$ satisfies the hypotheses of Proposition 9.4, since $\operatorname{res}_{\mathcal{A}}^{\mathcal{C}} \circ \operatorname{ind}_{\mathcal{A}}^{\mathcal{C}} \cong \operatorname{id}$ (see B.4 (vii)). $\square$

* * *

Next, we present another application of Proposition 9.4. Compare the first part of Section 8, where we defined *left* glossiness to guarantee the existence of a Quillen adjunction "backwards", namely $(\Phi^*, \Phi_!)$, cf. 8.6. Later, in 9.14, we will see that this Quillen adjunction basically always preserves codescent. On the other hand, the *dual* notion of *right* glossiness will be used for the adjunction "forwards" $(\Phi_*, \Phi^*)$, which is essentially always a Quillen adjunction, but does not always preserve codescent. See the *tableau* in 9.17 below for a survey.

**Definition 9.6.** Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs. We shall say that $\Phi$ is *right glossy* if the following condition is satisfied: for every object $b \in \mathcal{B}$, there is a set of morphisms in $\mathcal{C}$

$$\big\{\beta_j : \Phi(b_j) \longrightarrow \Phi(b)\big\}_{j \in F_b}$$

all having target $\Phi(b)$ and with various sources $\Phi(b_j)$, such that

(i) the objects $b_j$ also belong to $\mathcal{B}$;
(ii) for every morphism $\alpha : \Phi(a) \longrightarrow \Phi(b)$ in $\mathcal{C}$ with $a \in \mathcal{A}$, there exists a *unique* pair $(j, \gamma)$, with $j$ an "index" in $F_b$ and $\gamma$ a morphism $a \longrightarrow b_j$ in $\mathcal{A}$, such that $\alpha = \beta_j \circ \Phi(\gamma)$, that is,

$$\begin{tikzcd} \Phi(a) \arrow[rr, "\forall\alpha"] \arrow[dr, dashed, "(\exists! \gamma:\, a \to b_j)\ \Phi(\gamma)"'] & & \Phi(b) \\ & \Phi(b_j) \arrow[ur, dashed, "\beta_j\ (\exists! j \in F_b)"'] & \end{tikzcd}$$

As for left glossiness, we point out that condition (ii) has to be verified for all $a$ in $\mathcal{A}$, including those belonging to $\mathcal{B}$.

**Example 9.7.** A full inclusion of pairs of small categories $(\mathcal{A},\mathcal{B}) \hookrightarrow (\mathcal{C},\mathcal{D})$ (see 7.4) is right glossy. It suffices to take for each $b \in \mathcal{B}$ the set $F_b := \{1\}$, with $b_1 := b$ and $\beta_1 := \operatorname{id}_b$.

**Example 9.8.** Here is an "extreme" example again, showing that right glossiness can be very far from fullness. Let $\mathcal{C}$ be a small category and let $\mathcal{C}'$ be the corresponding discrete subcategory (B.5). Then, the inclusion $(\mathcal{C}',\mathcal{C}') \hookrightarrow (\mathcal{C},\mathcal{C})$ is right glossy.

Indeed, it suffices to take for each object $b \in \mathcal{C}'$ the set $F_b := \coprod_{c \in \mathcal{C}} \mathrm{mor}_{\mathcal{C}}(c, b)$, with, for every "index" $j\colon c \longrightarrow b$ in $F_b$, $b_j := c$ and $\beta_j := j$.

*Remark* 9.9. Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Dually to Remark 8.4, one easily checks that for any $b \in \mathcal{B}$ and for any functor $Y \in \mathcal{S}^{\mathcal{A}}$, the obvious morphism

$$\coprod_{j \in F_b} Y(b_j) \quad \longrightarrow \quad \operatorname*{colim}_{\left(a\,,\,\Phi(a) \overset{\alpha}{\to} \Phi(b)\right) \,\in\, \Phi\searrow\Phi(b)} Y(a)$$

is an isomorphism, natural in $Y$.

**Lemma 9.10.** *Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Assume that $\Phi$ is right glossy. Then, for $Y \in \mathcal{S}^{\mathcal{A}}$ and $b \in \mathcal{B}$, there is an isomorphism*

$$\Phi^*\Phi_* Y(b) \cong \coprod_{j \in F_b} Y(b_j)\,,$$

*that is natural in $Y$ (where notations are kept as in Definition 9.6).*

*Proof.* The proof is dual to the one of Lemma 8.5, using Definition B.2 for $\Phi_*$ and the above Remark 9.9. □

**Definition 9.11.** We say that a model category $\mathcal{M}$ has the *coproduct property for weak equivalences* if for a set $\{f_k\}_{k\in K}$ of weak equivalences in $\mathcal{M}$, their coproduct $\coprod_{k\in K} f_k$ is a weak equivalence as well. If the converse is also true, we say that $\mathcal{M}$ has the *strong coproduct property for weak equivalences.*

*Remark* 9.12. For example, any of the model categories $\mathcal{T}$op, s$\mathcal{S}$ets, $\mathcal{S}$p or Ch($R$-mod) (with both model structures) introduced in Appendix A has the strong coproduct property for weak equivalences; for the category of spectra, see [16, Thm. 7.4 (ii)]; the other cases are easy.

**Proposition 9.13** (Right glossy invariance of codescent)**.**
*Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories, and let $a \in \mathcal{A}$ be an object. Assume the following:*

(a) *$\mathcal{D} = \Phi(\mathcal{B})$;*
(b) *$\Phi$ is right glossy (see 9.6);*
(c) *$\Phi^*\Phi_*$ reflects $\{a\}$-weak equivalences (see A.15);*
(d) *$\mathcal{S}$ has the coproduct property for weak equivalences (see 9.11).*

*Consider a diagram $Y \in \mathcal{S}^{\mathcal{A}}$. Then $Y$ satisfies $\mathcal{B}$-codescent at $a$ if and only if $\Phi_*Y$ satisfies $\mathcal{D}$-codescent at $\Phi(a)$. In particular, assuming condition (c) for every object $a \in \mathcal{A} \smallsetminus \mathcal{B}$, the functor $Y$ satisfies $\mathcal{B}$-codescent if and only if $\Phi_*Y$ satisfies $\mathcal{D}$-codescent on $\Phi(\mathcal{A})$.*

*Proof.* By (b), Lemma 9.10 applies. Combined with (d), this shows that $\Phi^*\Phi_*$ preserves $\mathcal{B}$-weak equivalences. So, with (a) & (c), all the hypotheses of Proposition 9.4 are satisfied and we get the result. □

* * *

Finally, we discuss the case of the backward functor $\Phi^*$ associated to a "reasonable" morphism of pairs $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$.

**Theorem 9.14** (Left glossy invariance of codescent).
*Let $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ be a morphism of pairs of small categories. Assume that the following holds:*

(a) $\mathcal{D} = \Phi(\mathcal{B})$;
(b) $\Phi$ *is left glossy (see 8.1).*

*Let $X \in \mathcal{S}^{\mathcal{C}}$ and $a \in \mathcal{A}$. Then $X$ satisfies $\mathcal{D}$-codescent at $\Phi(a)$ if and only if $\Phi^*X$ satisfies $\mathcal{B}$-codescent at $a$. In particular, $X$ satisfies $\mathcal{D}$-codescent on $\Phi(\mathcal{A})$ if and only if $\Phi^*X$ satisfies $\mathcal{B}$-codescent.*

*Proof.* From Theorem 8.6, we know that the functor $\Phi^*\colon \mathcal{U}(\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{U}(\mathcal{A},\mathcal{B})$ preserves cofibrant objects. It also reflects weak equivalences (see 7.2 (iii) if necessary). The result follows as above from local flexibility of codescent 6.5 by choosing a $\mathcal{D}$-cofibrant approximation to $X$ in $\mathcal{U}(\mathcal{C},\mathcal{D})$, moving it via $\Phi^*$ to a $\mathcal{B}$-cofibrant approximation to $\Phi^*X$ in $\mathcal{U}(\mathcal{A},\mathcal{B})$ and checking whether it is a weak equivalence at $a \in \mathcal{A}$. $\square$

*Remark* 9.15. If fact, assuming that $\mathcal{D} = \Phi(\mathcal{B})$ as in the theorem, a closer look at this proof shows that *as soon as $(\Phi^*, \Phi_!)$ is a Quillen pair, the functor $\Phi^*$ reflects codescent on $\mathcal{A}$.* Left glossiness is only used to guarantee that those functors do form a Quillen pair (cf. 8.6).

**Corollary 9.16** (Restriction property for codescent).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories and let $\mathcal{A} \subset \mathcal{C}$ be a* full *subcategory containing $\mathcal{D}$. Let $X \in \mathcal{S}^{\mathcal{C}}$ and $a \in \mathcal{A}$. Then $X$ satisfies $\mathcal{D}$-codescent at $a$ if and only if $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}} X$ does. In particular, $X$ satisfies $\mathcal{D}$-codescent* on $\mathcal{A}$ *if and only if $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}} X$ satisfies $\mathcal{D}$-codescent.*

*Proof.* Apply left glossy invariance 9.14 to the full inclusion $(\mathcal{A},\mathcal{D}) \hookrightarrow (\mathcal{C},\mathcal{D})$ which is left glossy as we have seen in Example 8.2. $\square$

*Remark* 9.17. It is worth making the following recapitulative observation on left and right glossiness. Suppose that $\Phi\colon (\mathcal{A},\mathcal{B}) \longrightarrow (\mathcal{C},\mathcal{D})$ is a morphism of pairs of small categories such that $\mathcal{D} = \Phi(\mathcal{B})$. Then, one has the following *tableau*:

| $(F,U)$ | Is $(F,U)$ a Quillen pair? | Whenever $(F,U)$ is a Quillen pair, does $F$ reflect codescent on $\mathcal{A}$? |
|---|---|---|
| $(\Phi_*, \Phi^*)$ | always (7.5) | if $\Phi$ is right glossy, but only conditionally† (9.13) |
| $(\Phi^*, \Phi_!)$ | if $\Phi$ is left glossy (9.14) | always (9.15) |

† see the exact conditions on $\mathcal{S}$ and $\Phi^*\Phi_*$ in Proposition 9.13.

* * *

Now, we illustrate left absorbance, defined in 8.10, giving an analogue of Corollary 9.16 without the assumption that $\mathcal{D} \subset \mathcal{A}$; this will turn to be extremely useful later on (and will be strongly generalized in Theorem 11.7).

**Proposition 9.18.** *Let $(\mathcal{A},\mathcal{B}) \hookrightarrow (\mathcal{C},\mathcal{D})$ be a full inclusion of pairs of small categories. Assume that $\mathcal{A}$ is left absorbant in $\mathcal{C}$. Assume further that $\mathcal{D}\cap\mathcal{A}=\mathcal{B}$. Let $X\in\mathcal{S}^{\mathcal{C}}$ and $a\in\mathcal{A}$. Then $X$ satisfies $\mathcal{D}$-codescent at $a$ if and only if $\operatorname{res}^{\mathcal{C}}_{\mathcal{A}} X$ satisfies $\mathcal{B}$-codescent at $a$.*

*Proof.* We know from Proposition 8.12 that $\operatorname{res}^{\mathcal{C}}_{\mathcal{A}}$ preserves weak equivalences and cofibrant objects. As in the proof of Theorem 9.14, the result follows from local flexibility of codescent 6.5. □

## 10. Basic properties of codescent

We collect in this section a series of simple results about codescent. These will concern the cofibrant approximations (4.1) in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ and some compatibility properties of codescent related to the notions of retract (A.4) and of weak retract (A.21). Again, we fix a cofibrantly generated model category $\mathcal{S}$ of "values" (see A.24).

We start with retracts, first showing that one can alter the subcategory $\mathcal{D}$ up to essential equivalence or even up to retract equivalence (see 3.12 for both definitions).

**Proposition 10.1** (Retract equivalence property for codescent).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories and let $\mathcal{E}$ be another subset of $\mathcal{C}$, which is retract equivalent to $\mathcal{D}$. A functor $X\in\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent exactly where it satisfies $\mathcal{E}$-codescent.*

*Proof.* By Proposition 3.13, an object $X'\in\mathcal{S}^{\mathcal{C}}$ is $\mathcal{D}$-cofibrant if and only if it is $\mathcal{E}$-cofibrant and a morphism $\eta\colon X'\longrightarrow X$ is a $\mathcal{D}$-weak equivalence if and only if it is an $\mathcal{E}$-weak equivalence. The result follows from local flexibility of codescent 6.5. □

The next result is a direct consequence (or can be proven directly).

**Corollary 10.2.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Then, an object $X\in\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at every object in $\mathcal{C}$ that is a retract of an object of $\mathcal{D}$.* □

**Proposition 10.3** (Weak retract invariance of codescent).
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Let $X$ be a $\mathcal{C}$-weak retract of $Y$, that is, a weak retract of $Y$ in the model category $\mathcal{U}(\mathcal{C})$ (and not merely in $\mathcal{U}(\mathcal{C},\mathcal{D})$), in the sense of A.21. If $Y$ satisfies $\mathcal{D}$-codescent at some $c\in\mathcal{C}$, then so does $X$.*

*Proof.* If $\eta\colon X\longrightarrow Y$ and $\eta'\colon Y\longrightarrow X$ are such that $\eta'\circ\eta$ is a $\mathcal{C}$-weak equivalence, then so is $Q^{\mathcal{C}}_{\mathcal{D}}(\eta'\circ\eta)$, by rigidity of cofibrant objects 6.1. By A.21, $\xi^{\mathcal{C},\mathcal{D}}_X(c)$ is a weak equivalence, since it is a weak retract of the weak equivalence $\xi^{\mathcal{C},\mathcal{D}}_Y(c)$. □

* * *

The next property can prove very useful. It is reminiscent of standard results in the framework of the Isomorphism Conjectures.

**Proposition 10.4** (Zoom-out property for codescent).
*Let $\mathcal{C}$ be a small category, and let $\mathcal{D}\subset\mathcal{E}\subset\mathcal{C}$ be subcategories. If for some $c\in\mathcal{C}$, $X$ satisfies $\mathcal{D}$-codescent on $\mathcal{E}\cup\{c\}$, then $X$ satisfies $\mathcal{E}$-codescent at $c$. In particular, if $X\in\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent, then it satisfies $\mathcal{E}$-codescent as well.*

*Proof.* There exists by assumption an $\mathcal{E} \cup \{c\}$-weak equivalence $\xi\colon X' \longrightarrow X$ with $X'$ being $\mathcal{D}$-cofibrant. By Proposition 3.14 (i), we know that $X'$ is also $\mathcal{E}$-cofibrant, hence the result using local flexibility of codescent 6.5. The rest follows from this (or directly from global flexibility of codescent 6.6). $\square$

* * *

So far, we did not use an explicit description of the cofibrant replacement in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ and we will keep doing so, except in the forthcoming discussion and in some examples below. This is possible thanks to local and global flexibilities of codescent, 6.5 and 6.6, which allow us to move from one cofibrant approximation to another. Unfolding the proof of the model structure of $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, we see that the existence of the cofibrant replacement is given formally by applying the small object argument to $\varnothing \longrightarrow X$. In the special case where $\mathcal{D} = \mathcal{C}$ and $\mathcal{S} = \mathrm{s}\mathcal{S}\mathrm{ets}$, there are more explicit (functorial) cofibrant approximations, as explained for instance in [6, §§ 2.6–2.10]. More generally, the knowledge of a cofibrant approximation on $\mathcal{U}_{\mathcal{S}}(\mathcal{D})$ can be transported to one on $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, as we now explain.

**Proposition 10.5.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories; suppose that $\mathcal{D}$ is* full *in $\mathcal{C}$. Let $(\mathcal{Q}_{\mathcal{D}}\,,\,\zeta^{\mathcal{D}})$ be a cofibrant approximation (4.1) in the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{D})$. We define $(\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}}\,,\,\zeta^{\mathcal{C},\mathcal{D}})$ on $\mathcal{S}^{\mathcal{C}}$ as follows. For $X \in \mathcal{S}^{\mathcal{C}}$, we set*

$$\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X := \mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\, \mathcal{Q}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X$$

*and we let $\zeta^{\mathcal{C},\mathcal{D}}_X$ be given by the composition*

$$\begin{tikzcd} \mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X \arrow[r, "\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\, \zeta^{\mathcal{D}}_{\mathrm{res}\, X}"] \arrow[rr, bend right, "\zeta^{\mathcal{C},\mathcal{D}}_X"'] & \mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, X \arrow[r, "\epsilon_X"] & X \end{tikzcd}$$

*where $\epsilon_X$ denotes the counit, at $X$, of the adjunction $\big(\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\,,\,\mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\big)$; in other words, $\zeta^{\mathcal{C},\mathcal{D}}_X$ is the morphism adjoint to $\zeta^{\mathcal{D}}_{\mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X}$. Then, $(\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}}\,,\,\zeta^{\mathcal{C},\mathcal{D}})$ is a cofibrant approximation in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$; it is functorial if so is $(\mathcal{Q}_{\mathcal{D}}\,,\,\zeta^{\mathcal{D}})$ (see 4.1).*

*Proof.* This is immediate from Corollary 7.6 applied to $(\mathcal{D},\mathcal{D}) \hookrightarrow (\mathcal{C},\mathcal{D})$ ($\mathcal{D}$ is full) which guarantees that $\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X$ is $\mathcal{D}$-cofibrant. To see that $\zeta^{\mathcal{C},\mathcal{D}}_X$ is a $\mathcal{D}$-weak equivalence, simply use that the unit $\eta\colon \mathrm{id} \longrightarrow \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} \circ \mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}$ is an isomorphism (see B.4 (vii)) : $\mathrm{res}^{\mathcal{C}}_{\mathcal{D}}\, \zeta^{\mathcal{C},\mathcal{D}}_X \circ \eta_{\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X} = \zeta^{\mathcal{D}}_{\mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X}$ is a $\mathcal{D}$-weak equivalence. $\square$

*Remark* 10.6. Let $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}$ be full inclusions of small categories. For $d \in \mathcal{D}$, let us denote by $\iota^{\mathcal{E}}_d\colon \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{E}}$ the left adjoint of the evaluation functor $\varepsilon_d\colon \mathcal{S}^{\mathcal{E}} \longrightarrow \mathcal{S}$ (compare with the proof of Theorem 3.5). Suppose that $I$ and $J$ designate chosen sets of generating cofibrations for $\mathcal{S}$. Then, the corresponding sets of generating cofibrations for $\mathcal{U}_{\mathcal{S}}(\mathcal{E},\mathcal{D})$ are, by virtue of Theorem 2.1,

$$\underline{I}^{\mathcal{E}}_{\mathcal{D}} := \bigcup_{d \in \mathcal{D}} \iota^{\mathcal{E}}_d(I) \qquad \text{and} \qquad \underline{J}^{\mathcal{E}}_{\mathcal{D}} := \bigcup_{d \in \mathcal{D}} \iota^{\mathcal{E}}_d(J)\,.$$

If the reader really prefers the cofibrant *replacement* to mere approximations, he could consider the following observation expressed using these notations :

$$\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}(\underline{I}^{\mathcal{D}}_{\mathcal{D}}) \cong \underline{I}^{\mathcal{C}}_{\mathcal{D}}\,.$$

This follows immediately from the fact that for every $d \in \mathcal{D}$ we have $\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\, \iota_d^{\mathcal{D}} \cong \iota_d^{\mathcal{C}}$. Unfortunately, one has only natural isomorphisms instead of equalities. It sounds reasonable to think that the small object arguments for $\underline{I}_{\mathcal{D}}^{\mathcal{D}}$ and for $\underline{I}_{\mathcal{D}}^{\mathcal{C}}$ are therefore compatible via the induction. We will not go into the details, because even if it has a rigorous formulation this compatibility is not needed here, as already explained.

*Remark* 10.7. Given an arbitrary cofibrantly generated *simplicial* model category $\mathcal{S}$, we devote Part II of the series [1] to the construction of explicit cofibrant approximations in the model category $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, not merely those obtained by the small object argument (since, in general, the latter tend to be formidable).

## 11. Pruning

In this section, we explain how to prune away unnecessary data in $\mathcal{C}$ and $\mathcal{D}$ without altering the codescent property of a given $X \in \mathcal{S}^{\mathcal{C}}$ at a given object $c \in \mathcal{C}$. As before, $\mathcal{S}$ is a fixed cofibrantly generated model category (see A.24).

Since in this section we will often pass from a category to a subcategory, we remind the reader of Convention 3.1, that unless otherwise mentioned a subcategory merely given by its objects is meant as the full subcategory on those objects.

**Proposition 11.1** (Covering property for codescent).
*Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories and let $\{\mathcal{C}_a\}_{a \in A}$ be a collection of full subcategories of $\mathcal{C}$, each of them containing $\mathcal{D}$. Suppose that the $\mathcal{C}_a$'s form a covering of $\mathcal{C}$,* i.e. $\mathrm{obj}(\mathcal{C}) = \bigcup_{a \in A} \mathrm{obj}(\mathcal{C}_a)$*. Then, a diagram $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent if and only if* $\mathrm{res}_{\mathcal{C}_a}^{\mathcal{C}} X$ *satisfies $\mathcal{D}$-codescent for all $a \in A$.*

*Proof.* This is an immediate consequence of Corollary 9.16. □

* * *

We can reduce the ambient category to the minimum, giving it the "shape of a funnel" with $\mathcal{D}$ as base and one object $c \in \mathcal{C}$ as vertex.

**Proposition 11.2** (Funneling Lemma).
*Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories and let $c \in \mathcal{C}$. A functor $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at $c$ if and only if its restriction* $\mathrm{res}_{\mathcal{D} \cup \{c\}}^{\mathcal{C}}(X)$ *satisfies $\mathcal{D}$-codescent.*

*Proof.* This follows directly from Corollary 9.16 applied to $\mathcal{A} := \mathcal{D} \cup \{c\}$. □

* * *

We can also prune away in $\mathcal{D}$ all objects which do not map to $c$, as we now explain.

**Notation 11.3.** Fix a (small) category $\mathcal{C}$. Let $\mathcal{D}$ be a subset of $\mathcal{C}$, and let $c \in \mathcal{C}$. We denote by $\mathcal{D}_c$ the subset of $\mathcal{D}$ of those objects which have at least one morphism to $c$ in $\mathcal{C}$, *i.e.*

$$\mathcal{D}_c := \left\{ d \in \mathcal{D} \,\middle|\, \mathrm{mor}_{\mathcal{C}}(d, c) \neq \varnothing \right\}.$$

**Lemma 11.4.** *Let $\mathcal{D}$ be a full subset of a (small) category $\mathcal{C}$, and $c \in \mathcal{C}$. Then, $\mathcal{D}_c$ is left absorbant in $\mathcal{D}$ as defined in 8.10. Similarly, $\mathcal{D}_c \cup \{c\}$ is left absorbant in $\mathcal{D} \cup \{c\}$, both $\mathcal{D}$ and $\mathcal{D} \cup \{c\}$ viewed as full subcategories of $\mathcal{C}$.*

*Proof.* By composition, any object $d \in \mathcal{D}$ having a morphism to some object having a morphism to $c$, has itself a morphism to $c$. So much for $\mathcal{D}_c$ and $\mathcal{D}$. For the other case, an object in $\mathcal{D} \cup \{c\}$ having a morphism to an object in $\mathcal{D}_c \cup \{c\}$ is either $c$ itself or clearly belongs to $\mathcal{D}_c$ by definition of the latter, or by the first part of the proof. □

**Theorem 11.5** (Pruning Lemma for objects).
*Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories, and let $c \in \mathcal{C}$. Then, for $X \in \mathcal{S}^{\mathcal{C}}$, the following properties are equivalent:*

(i) *$X$ satisfies codescent at $c$ with respect to $\mathcal{D}$;*
(ii) *$X$ satisfies codescent at $c$ with respect to $\mathcal{D}_c$.*

*Proof.* Consider the full inclusion of pairs of small categories

$$\big(\mathcal{D}_c \cup \{c\}\,,\,\mathcal{D}_c\big) \hookrightarrow \big(\mathcal{D} \cup \{c\}\,,\,\mathcal{D}\big)\,.$$

By Lemma 11.4 and since clearly $\mathcal{D} \cap (\mathcal{D}_c \cup \{c\}) = \mathcal{D}_c$, this inclusion satisfies the assumptions of Proposition 9.18. So, for any $Y \in \mathcal{S}^{\mathcal{D}\cup\{c\}}$, we know that $Y$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $\mathrm{res}^{\mathcal{D}\cup\{c\}}_{\mathcal{D}_c\cup\{c\}} Y$ satisfies $\mathcal{D}_c$-codescent at $c$. Apply this result to $Y = \mathrm{res}^{\mathcal{C}}_{\mathcal{D}\cup\{c\}} X$. Since

$$\mathrm{res}^{\mathcal{D}\cup\{c\}}_{\mathcal{D}_c\cup\{c\}} \circ \mathrm{res}^{\mathcal{C}}_{\mathcal{D}\cup\{c\}} = \mathrm{res}^{\mathcal{C}}_{\mathcal{D}_c\cup\{c\}}\,,$$

we have proven that $\mathrm{res}^{\mathcal{C}}_{\mathcal{D}\cup\{c\}} X$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $\mathrm{res}^{\mathcal{C}}_{\mathcal{D}_c\cup\{c\}} X$ satisfies $\mathcal{D}_c$-codescent at $c$. These two statements are respectively equivalent to (i) and (ii) by the Funneling Lemma 11.2. □

**Corollary 11.6.** *Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories and let $c \in \mathcal{C}$. Assume that no object $d \in \mathcal{D}$ possesses a morphism $d \longrightarrow c$ in $\mathcal{C}$. Then, a functor $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at $c$ if and only if the morphism $\varnothing \longrightarrow X(c)$ in $\mathcal{S}$ is a weak equivalence.*

*Proof.* By the Pruning Lemma 11.5, $X$ satisfies $\mathcal{D}$-codescent at $c$ if and only if it satisfies codescent at $c$ with respect to the empty subcategory. We conclude by Example 4.4 (1). □

* * *

Next, we see that the only important morphisms are those having their source in $\mathcal{D}$ and that we can drop all other morphisms from $\mathcal{C}$.

**Theorem 11.7** (Pruning Lemma for morphisms).
*Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. Define as follows a category $\mathcal{A}$ with the same objects as $\mathcal{C}$, and with the sets of morphisms given by*

$$\mathrm{mor}_{\mathcal{A}}(a,b) := \begin{cases} \mathrm{mor}_{\mathcal{C}}(a,b) & \text{if } a \in \mathcal{D} \\ \{\mathrm{id}_a\} & \text{if } a \notin \mathcal{D} \text{ and } a = b \\ \varnothing & \text{if } a \notin \mathcal{D} \text{ and } a \neq b\,. \end{cases}$$

*Then, this indeed defines a subcategory of $\mathcal{C}$ containing $\mathcal{D}$ as a left absorbant subset. Moreover, for a functor $X \in \mathcal{S}^{\mathcal{C}}$ and an object $c \in \mathcal{C}$, the following are equivalent:*

(i) *$X$ satisfies $\mathcal{D}$-codescent at $c$;*
(ii) *$\mathrm{res}^{\mathcal{C}}_{\mathcal{A}} X$ satisfies $\mathcal{D}$-codescent at $c$.*

*In particular, $X$ satisfies $\mathcal{D}$-codescent if and only if $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}} X$ satisfies $\mathcal{D}$-codescent.*

*Proof.* To check that $\mathcal{A}$ is really a subcategory of $\mathcal{C}$ as stated is straightforward and left to the reader. Consider the functor $\Phi\colon (\mathcal{A},\mathcal{D}) \longrightarrow (\mathcal{C},\mathcal{D})$ given by the (possibly non-full) inclusion. We claim that it satisfies the hypotheses of Theorem 9.14 on the left glossy invariance. Condition (a) is clear and we are left to prove condition (b), *i.e.* that $\Phi$ is left glossy (see 8.1). This is done like in Example 8.2 : for each $d \in \mathcal{D}$, we take $E_d := \{1\}$, with $d_1 := d$ and $\beta_1 := \mathrm{id}_d$. □

For instance, for $c \in \mathcal{C} \smallsetminus \mathcal{D}$, this shows that one can remove arbitrarily non-identity endomorphisms of $c$ ; conversely, one can add endomorphisms of $c$ as long as "$X$ remains a functor".

Note that the Pruning Lemma for morphisms 11.7 provides a (complicated) solution to the exercise stated at the end of Example 4.5 (at least as far as the second statement is concerned).

*Remark* 11.8. The Pruning Lemmas 11.5 and 11.7 give a clear "direction" to codescent. Namely, codescent goes from $\mathcal{D}$ to $\mathcal{C}$ in the sense that only the morphisms *out of* $\mathcal{D}$ to some given object $c$ will contribute to $\mathcal{D}$-codescent at $c$ and, for instance, not any of the morphisms from $c$ to an object of $\mathcal{D}$, and in fact not any of the morphisms out of $c$ whenever $c \notin \mathcal{D}$.

This conclusion might sound strange when compared to our earlier comment (3.8) that the morphisms of $\mathcal{D}$ were not important but merely the underlying set of objects $\mathrm{obj}(\mathcal{D})$. This remains undoubtedly true. What we say here is that in the ambient category $\mathcal{C}$, we can ignore the morphisms not taking their source in $\mathcal{D}$.

* * *

To state an important and illustrating consequence of the Pruning Lemmas and of the Funneling Lemma, we introduce a notation.

**Notation 11.9.** Let $\mathcal{E}$ be a subcategory of a small category $\mathcal{C}$, and let $c \in \mathcal{C} \smallsetminus \mathcal{E}$. We denote by $\mathcal{E} \veebar \{c\}$ the *subcategory* of $\mathcal{C}$ with $\mathrm{obj}(\mathcal{E}) \amalg \{c\}$ as set of objects, and with the following sets of morphisms :

$$\mathrm{mor}_{\mathcal{E}\veebar\{c\}}(a,b) := \begin{cases} \mathrm{mor}_{\mathcal{C}}(a,b) & \text{if } a \neq c \\ \{\mathrm{id}_c\} & \text{if } a = b = c \\ \varnothing & \text{if } a = c \text{ and } b \neq c\,. \end{cases}$$

For example, when $\mathcal{D}$ is full and distinct from $\mathcal{C}$, the category $\mathcal{A}$ occurring in the statement of 11.7 is, in some obvious sense, a patching of the subcategories $\mathcal{D} \veebar \{c\}$ with $c$ running over the set $\mathrm{obj}(\mathcal{C} \smallsetminus \mathcal{D})$.

Recall also Notation 11.3.

**Theorem 11.10.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories and consider $c \in \mathcal{C} \smallsetminus \mathcal{D}$. Then a functor $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at $c$ if and only if $\mathrm{res}^{\mathcal{C}}_{\mathcal{D}_c \veebar \{c\}} X$ satisfies $\mathcal{D}_c$-codescent (at $c$).*

*Proof.* By the Pruning Lemma for objects 11.5, the "codescent question" at $c$ for the pair $(\mathcal{C},\mathcal{D})$ is equivalent to that for $(\mathcal{C},\mathcal{D}_c)$; by the Funneling Lemma 11.2, the latter condition is in turn equivalent to the "codescent question" at $c$ for the pair $(\mathcal{D}_c \cup \{c\}, \mathcal{D}_c)$; finally, by the Pruning Lemma for morphisms 11.7, this is equivalent to the "codescent question" (at $c$) for the pair $(\mathcal{D}_c \veebar \{c\}, \mathcal{D}_c)$. □

It is sometimes possible to further prune away some data, using the retract equivalence property for codescent 10.1, and the glossy invariances of codescent 9.13 and 9.14.

## 12. Examples

We give here a class of simple examples, most of which are variations on the theme of Example 4.5. We let $\mathcal{S}$ be a cofibrantly generated model category. Recall also Convention 3.1.

To start with, as an application of rigidity of codescending objects 6.3, we illustrate, by an example, the fact that one can *not* expect that all objects in $\mathcal{S}^{\mathcal{C}}$ satisfy $\mathcal{D}$-codescent (at least whenever $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{D}$ are not "too trivial").

**Example 12.1.** Assume that there is a morphism $f\colon s \longrightarrow s'$ in $\mathcal{S}$ with $s \neq s'$, which is *not* a weak equivalence. Suppose that $\mathcal{D}$ is left absorbant (8.10) in the small category $\mathcal{C}$ and that $\mathcal{D} \neq \mathcal{C}$. (By the Pruning Lemma for morphisms 11.7, left absorbance is no effective restriction.) Let $X \in \mathcal{S}^{\mathcal{C}}$ be the constant diagram with value $s$. Let $Y \in \mathcal{S}^{\mathcal{C}}$ take the value $s$ on $\mathcal{D}$ and $s'$ outside, with

$$Y(\alpha) := \begin{cases} \mathrm{id}_s & \text{if } \alpha \text{ has source and target in } \mathcal{D} \\ f & \text{if } \alpha \text{ has source in } \mathcal{D} \text{ and target in } \mathcal{C}\smallsetminus\mathcal{D} \\ \mathrm{id}_{s'} & \text{if } \alpha \text{ has source and target in } \mathcal{C}\smallsetminus\mathcal{D}\,, \end{cases}$$

for every morphism $\alpha$ in $\mathcal{C}$. Define a morphism $\eta\colon X \longrightarrow Y$ in $\mathcal{S}^{\mathcal{C}}$ decreeing that $\eta$ is $\mathrm{id}_s$ on $\mathcal{D}$ and $f$ outside. Then, $\eta$ is a $\mathcal{D}$-weak equivalence but *not* a $\mathcal{C}$-weak equivalence. By rigidity of codescending objects 6.3, at least one of $X$ and $Y$ does *not* satisfy $\mathcal{D}$-codescent. For example, if we choose $s := \varnothing$, then $X$ satisfies $\mathcal{D}$-codescent and $Y$ does not. For $\mathcal{S} := \mathcal{T}\mathrm{op}_\bullet$, one can take $s' := *$ and then, for $\mathcal{D}$ empty, $Y$ satisfies $\mathcal{D}$-codescent and $X$ does not (see Example 4.4 (1)).

* * *

**Example 12.2.** Consider the general situation of a small category with two objects

$$\mathcal{C} \qquad := \qquad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "M"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\vdots"] \arrow[r] & \overset{c}{\bullet} \arrow[loop right, "N"] \arrow[l, bend left, "B"] \arrow[l, bend left=10] \end{tikzcd}$$

with $\mathcal{D} := \{d\}$, where $A$ and $B$ are arbitrary sets of morphisms from $d$ to $c$ and from $c$ to $d$ respectively, and $M$ and $N$ are arbitrary monoids of endomorphisms of $d$ and $c$ respectively. Fix a diagram $X \in \mathcal{S}^{\mathcal{C}}$. Combining the Funneling Lemma 11.2 and the Pruning Lemma for morphisms 11.7 (that is, applying Theorem 11.10), we deduce that $X$ satisfies $\mathcal{D}$-codescent if and only if its restriction to the category

$$\begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "M"] \arrow[r, shift left, "A"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

does. Next, we discuss a special case in which the monoid $M$ is reduced to the minimum.

**Example 12.3.** Consider the category

$$\mathcal{C} \quad := \quad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "\mathrm{id}_d"] \arrow[r, shift left, "A"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

with $A$ denoting a *non-empty* set of morphisms from $d$ to $c$, and let $\mathcal{D} := \{d\}$. A diagram $X \in \mathcal{S}^{\mathcal{C}}$ is the same thing as a set $\big\{X(d) \overset{X(\alpha)}{\longrightarrow} X(c)\big\}_{\alpha \in A}$ of morphisms in $\mathcal{S}$ with the same source and the same target, but without any further connection. The model category $\mathcal{U}(\mathcal{D})$ identifies canonically with $\mathcal{S}$. So, letting $(Q_{\mathcal{S}}, \xi^{\mathcal{S}})$ be the cofibrant replacement in $\mathcal{S}$, by Proposition 10.5, we have for $X$ the cofibrant approximation

$$\zeta_X^{\mathcal{C},\mathcal{D}} \colon \mathcal{Q}_{\mathcal{D}}^{\mathcal{C}} X = \operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Q_{\mathcal{S}} \operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X = \operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Q_{\mathcal{S}} X(d) \xrightarrow{\epsilon_X \circ \operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} \xi_{X(d)}^{\mathcal{S}}} X\,.$$

Consider a diagram $Y = Y(d)$ in $\mathcal{S} = \mathcal{S}^{\mathcal{D}}$. The comma categories $\mathcal{D} \searrow d$ and $\mathcal{D} \searrow c$ (see B.1) are discrete with, respectively, one object, namely $(d, \mathrm{id}_d)$, and $|A|$ objects, namely $(d, \alpha)$ with $\alpha \in A$. By B.2, we get canonical isomorphisms

$$\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Y(d) = \operatorname*{colim}_{\mathcal{D}\searrow d} Y(d) \cong Y \qquad \text{and} \qquad \operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Y(c) = \operatorname*{colim}_{\mathcal{D}\searrow c} Y(d) \cong \coprod_{\alpha \in A} Y\,.$$

For $\alpha \in A$, $\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Y(\alpha)$ is the canonical morphism $\iota_\alpha \colon Y \longrightarrow \coprod_{\alpha \in A} Y$ corresponding to the $\alpha$-term, as easily verified. Unravelling the construction of the morphism $\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} \xi_{X(d)}^{\mathcal{S}}$, one sees that the situation is as follows:

$$\begin{tikzcd} \mathcal{Q}_{\mathcal{D}}^{\mathcal{C}} X \arrow[d, "\zeta_X^{\mathcal{C},\mathcal{D}}"', "\mathcal{D}\text{-weq}"] \\ X \end{tikzcd} \quad \overset{\text{def.}}{=} \quad \begin{tikzcd} Q_{\mathcal{S}} X(d) \arrow[r, shift left, "\{\iota_\alpha\}_{\alpha \in A}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] \arrow[d, "\xi_{X(d)}^{\mathcal{S}}"', "\sim"] & \coprod_{\alpha \in A} Q_{\mathcal{S}} X(d) \arrow[d, "\left(X(\alpha) \circ \xi_{X(d)}^{\mathcal{S}}\right)_\alpha"] \\ X(d) \arrow[r, shift left, "\{X(\alpha)\}_{\alpha \in A}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & X(c) \end{tikzcd}$$

where the vertical morphism on the right-hand side is the one induced by the universal property of the coproduct. It is equal to the composition

$$\big(X(\alpha) \circ \xi_{X(d)}^{\mathcal{S}}\big)_\alpha \colon \textstyle\coprod_{\alpha \in A} Q_{\mathcal{S}} X(d) \xrightarrow{\coprod_\alpha \xi_{X(d)}^{\mathcal{S}}} \coprod_{\alpha \in A} X(d) \xrightarrow{(X(\alpha))_\alpha} X(c)\,.$$

So, by global flexibility of codescent 6.6, the functor $X$ satisfies $\mathcal{D}$-codescent if and only if $\big(X(\alpha) \circ \xi_{X(d)}^{\mathcal{S}}\big)_\alpha$ is a weak equivalence. *Suppose that a coproduct of weak equivalences in $\mathcal{S}$ is a weak equivalence* (compare 9.11). Then, by 2-out-of-3, we deduce that

$$X \in \mathcal{S}^{\mathcal{C}} \textit{ satisfies } \mathcal{D}\textit{-codescent} \quad \textit{iff} \quad \coprod_{\alpha \in A} X(d) \xrightarrow{(X(\alpha))_\alpha} X(c) \textit{ is a weq.}$$

For instance, when $A$ has two elements and $\mathcal{S} = \mathcal{T}\mathrm{op}$, the $\mathcal{C}$-diagram

$$X \quad \overset{\text{def.}}{=} \quad \begin{tikzcd} * \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{id}"'] & * \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

does *not* satisfy $\mathcal{D}$-codescent. The same diagram, but viewed as $\mathcal{T}\mathrm{op}_\bullet$-valued, does satisfy $\mathcal{D}$-codescent, since then $X$ is the initial object and is therefore $\mathcal{D}$-cofibrant.

**Example 12.4.** Let $\mathcal{C}$ be a small category and suppose that the full subcategory $\mathcal{D} \subset \mathcal{C}$ is such that $\operatorname{obj}(\mathcal{C}) = \mathcal{D} \amalg \{c_\infty\}$ with $c_\infty$ a terminal object in $\mathcal{C}$. Now, we apply Proposition 10.5 with $(\mathcal{Q}_\mathcal{D}\,,\, \zeta^\mathcal{D})$ denoting a cofibrant approximation (4.1) in the model category $\mathcal{U}_\mathcal{S}(\mathcal{D})$. Using the description of the induction functor given in B.2 and noticing that the comma category $\mathcal{D}\searrow c_\infty$ is canonically isomorphic to $\mathcal{D}$ viewed as a full subcategory of $\mathcal{C}$, one obtains that

$$X \in \mathcal{S}^\mathcal{C} \textit{ satisfies } \mathcal{D}\textit{-codescent} \quad \textit{iff} \quad \operatorname*{colim}_{\mathcal{D}} \mathcal{Q}_\mathcal{D} \operatorname{res}^\mathcal{C}_\mathcal{D} X \xrightarrow{\ \mu\ } X(c_\infty) \ \textit{ is a weq}$$

where $\mu$ is the canonical morphism (independently of the choice of $(\mathcal{Q}_\mathcal{D}\,,\, \zeta^\mathcal{D})$). This applies to the category

$$\mathcal{C} \quad := \quad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "M"] \arrow[r, "\alpha"] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

with $\mathcal{D} := \{d\}$ (recall Remark 4.6), giving another special case of Example 12.2.

$* * *$

Next, we give an example of left glossiness (see 8.1) for categories with two objects. Again, this treats some particular cases of Example 12.2.

**Example 12.5.** Let $M$ be a monoid and $M' \leqslant M$ a submonoid. Let $A$ be a non-empty right $M$-set, and $A' \subset A$ an $M'$-subset. Consider the functor $\Phi\colon \mathcal{A} \longrightarrow \mathcal{C}$ given by

$$\begin{tikzcd} \overset{b}{\bullet} \arrow[loop left, "M'"] \arrow[r, shift left, "A'"] \arrow[r, shift right, "\vdots"'] & \overset{a}{\bullet} \arrow[loop right, "\mathrm{id}_a"] \end{tikzcd} \qquad \xrightarrow{\quad \Phi \quad} \qquad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "M"] \arrow[r, shift left, "A"] \arrow[r, shift right, "\vdots"'] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

where $\mathcal{A}$ is depicted on the left and $\mathcal{C}$ on the right, and let $\mathcal{B} = \{b\}$ and $\mathcal{D} = \{d\}$. Then, $\Phi$ is left-glossy if and only if there exists a subset $L \subset M$ such that the two maps

$$M' \times L \longrightarrow M, \quad (m, \ell) \longmapsto m\ell \qquad \text{and} \qquad A' \times L \longrightarrow A, \quad (\alpha, \ell) \longmapsto \alpha \cdot \ell$$

are bijections. For instance, suppose $M := G$ is a *group acting transitively* on the non-empty set $A$. Choose an element $\alpha \in A$, and take $A' := \{\alpha\}$, $M' := \operatorname{Stab}_G(\alpha)$ (the stabilizer of $\alpha$ in $G$) and choose for $L$ any set of representatives of the right $G$-orbits $A/G$. This fulfills the required conditions. Consequently, the inclusion

$$\begin{tikzcd} \overset{b}{\bullet} \arrow[loop left, "\operatorname{Stab}_G(\alpha)"] \arrow[r, "\alpha"] & \overset{a}{\bullet} \arrow[loop right, "\mathrm{id}_a"] \end{tikzcd} \qquad \xrightarrow{\quad \Phi \quad} \qquad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "G"] \arrow[r, shift left, "A"] \arrow[r, shift right, "\vdots"'] & \overset{c}{\bullet} \arrow[loop right, "\mathrm{id}_c"] \end{tikzcd}$$

is left glossy (and then, Example 12.4 can be applied). In all these cases, left glossy invariance of codescent 9.14 applies to reflect codescent via $\Phi^* = \operatorname{res}^\mathcal{C}_\mathcal{A}$.

$* * *$

We pass to another type of examples.

**Example 12.6.** Let $\mathcal{C}$ be the "commutative-square-category", that is, the category presented by generators and relations as follows:

$$\mathcal{C} \ : \qquad \begin{tikzcd} & \overset{d}{\bullet} \arrow[dr, "\beta"] & \\ \overset{e}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] \arrow[rr, phantom, "\circlearrowleft"] & & \overset{c}{\bullet} \\ & \underset{d'}{\bullet} \arrow[ur, "\beta'"'] & \end{tikzcd} \qquad \text{with} \qquad \beta \circ \alpha = \beta' \circ \alpha'\,.$$

First, we let $\mathcal{E} := \{e\}$. Applying the Funneling Lemma 11.2 and invoking Example 4.5, we infer that

$$X \in \mathcal{S}^{\mathcal{C}} \textit{ satisfies } \mathcal{E}\textit{-codescent} \quad \textit{iff} \quad X(\alpha),\ X(\alpha'),\ X(\beta) \textit{ and } X(\beta') \textit{ are weq's.}$$

By 2-out-of-3, if suffices that three of these four morphisms are weak equivalences.

Now, we let $\mathcal{D} := \{e, d, d'\}$ (as always, viewed as a full subcategory of $\mathcal{C}$) and set $\gamma := \beta\alpha$. In [9, § 10], the same model category structure $\mathcal{U}(\mathcal{D})$ on $\mathcal{S}^{\mathcal{D}}$ is considered for this particular $\mathcal{D}$ (see Proposition 10.6 therein; in particular, an explicit description of cofibrations is given). Let $(Q_{\mathcal{D}}\,,\, \xi^{\mathcal{D}})$ be the cofibrant replacement in $\mathcal{U}(\mathcal{D})$. Consider a diagram $X \in \mathcal{S}^{\mathcal{C}}$. By Propositions 10.5, one has

$$\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X(c) = \operatorname*{colim}_{\left(a\,,\, a \overset{\delta}{\to} c\right) \,\in\, \mathcal{D}\searrow c} Q_{\mathcal{D}} \operatorname{res}^{\mathcal{C}}_{\mathcal{D}} X(a)\,.$$

Let us denote by $[\delta]$ the object $(a, \delta)$ in $\mathcal{D} \searrow c$. It is readily checked that the category $\mathcal{D}\searrow c$ looks as follows:

$$\mathcal{D}\searrow c \qquad = \qquad \begin{tikzcd} & {[\beta]} \arrow[loop right, "\mathrm{id}_{[\beta]}"] \\ {[\gamma]} \arrow[loop left, "\mathrm{id}_{[\gamma]}"] \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & \\ & {[\beta']} \arrow[loop right, "\mathrm{id}_{[\beta']}"] \end{tikzcd}$$

Therefore, taking a colimit over it amounts to taking the obvious pushout. Following [9, Prop. 10.7], this means that $\mathcal{Q}^{\mathcal{C}}_{\mathcal{D}} X(c)$ is a homotopy push-out. Therefore, *$X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent if and only if $X(c)$ is (weakly equivalent to) the homotopy push-out of $X(d)$ and $X(d')$ over $X(e)$.*

**Example 12.7.** Let $\mathcal{C}$ be the "non-commutative-square-category" presented by

$$\mathcal{C}\ : \qquad \begin{tikzcd} & \overset{d}{\bullet} \arrow[dr, "\beta"] & \\ \overset{e}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & & \overset{c}{\bullet} \\ & \overset{d'}{\bullet} \arrow[ur, "\beta'"'] & \end{tikzcd} \qquad \text{(without relations)}\,.$$

Let $\mathcal{E} := \{e\}$ and *suppose that a coproduct of weak equivalences in $\mathcal{S}$ is a weak equivalence*. Applying the Funneling Lemma 11.2 and invoking Example 12.3, we see that *a diagram $X \in \mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{E}$-codescent if and only if $X(\alpha)$ and $X(\alpha')$ as well as the morphism* $X(e)\coprod X(e) \xrightarrow{\left(X(\beta\alpha),\,X(\beta'\alpha')\right)} X(c)$ *are weak equivalences.*

* * *

We end this series of examples by presenting an example of right glossiness, where we suppose that the category of values $\mathcal{S}$ has the strong coproduct property for weak equivalences (see 9.11).

**Example 12.8.** Consider a morphism of pairs $\Phi\colon (\mathcal{A}, \mathcal{B}) \longrightarrow (\mathcal{C}, \mathcal{D})$, with $\mathcal{B} = \{b\}$ and $\mathcal{D} = \{d\}$, depicted in the obvious way as

$$\begin{tikzcd} \overset{b}{\bullet} \arrow[loop left, "N'"] \arrow[r, shift left, "A'"] \arrow[r, shift right] & \overset{a}{\bullet} \arrow[loop right, "M'"] \end{tikzcd} \qquad \xrightarrow{\ \Phi\ } \qquad \begin{tikzcd} \overset{d}{\bullet} \arrow[loop left, "N"] \arrow[r, shift left, "A"] \arrow[r, shift right] & \overset{c}{\bullet} \arrow[loop right, "M"] \end{tikzcd}$$

We suppose that $\Phi$ induces inclusions of $N'$, $A'$ and $M'$ in $N$, $A$ and $M$ respectively. Assume that there exists a subset $L \subset N$ such that the map $L \times N' \longrightarrow N$, $(\ell, n') \longmapsto \ell \cdot n'$ is bijective, as for example if $N$ and $N'$ are groups. Then, the functor $\Phi$ is right glossy. Indeed, it suffices to take as $\beta_j$'s the elements of $L$ (with $b_j := b$ for each $j$) in Definition 9.6. Similarly, assume further that there exists a subset $K \subset M$ such that the map $K \times (A' \amalg M') \longrightarrow A \amalg M$, $(k, x') \longmapsto k \cdot x'$ is bijective. Then, the functor $\Phi$, viewed as a morphism of pairs

$$\Phi \colon (\mathcal{A}, \mathcal{A}) \longrightarrow (\mathcal{C}, \mathcal{C}),$$

is right glossy, as a similar argument shows. By Lemma 9.10, given a diagram $X \in \mathcal{S}^{\mathcal{A}}$, we have a natural decomposition

$$\Phi^* \Phi_* X(a) \cong \coprod_K X(a).$$

This implies that condition (c) of Proposition 9.13 is fulfilled; the other conditions are already checked. As a consequence, the diagram $X$ satisfies $\mathcal{B}$-codescent if and only if the induced diagram $\operatorname{ind}_{\mathcal{A}}^{\mathcal{C}} X$ satisfies $\mathcal{D}$-codescent. This provides an example of induction property for codescent, without the assumption that the subcategory $\mathcal{A}$ be full in the ambient category $\mathcal{C}$ (compare with the induction property for codescent 9.5).

## 13. The homotopy category of $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$

Fix a cofibrantly generated model category $\mathcal{S}$ (see A.24). In this section, we analyze the homotopy category of the model category $\mathcal{U}(\mathcal{C}, \mathcal{D})$. We also reformulate the codescent property in the language of homotopy categories. Recall also Convention 3.1.

Concerning the homotopy category of a model category and related topics, we refer to [13, §§ 1.2–1.3] and to [12, §§ 8.3–8.5] (see also A.19, the subsequent paragraph and A.20).

**Notation 13.1.** Let $(\mathcal{C}, \mathcal{D})$ be a pair of small categories. We denote by $\operatorname{Ho}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ the homotopy category of the model category $\mathcal{U}(\mathcal{C}, \mathcal{D})$ introduced in 3.6, that is, the localization of $\mathcal{S}^{\mathcal{C}}$ with respect to $\mathcal{D}$-weak equivalences. We shall denote by $[X]$ the image of an $X \in \mathcal{S}^{\mathcal{C}}$ in $\operatorname{Ho}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$. When $\mathcal{C} = \mathcal{D}$, we also abbreviate $\operatorname{Ho}_{\mathcal{S}}(\mathcal{C}, \mathcal{C})$ by $\operatorname{Ho}_{\mathcal{S}}(\mathcal{C})$.

**Proposition 13.2.** *Let $(\mathcal{A}, \mathcal{B}) \hookrightarrow (\mathcal{C}, \mathcal{D})$ be a full inclusion of pairs of small categories. Then, the restriction $\operatorname{res}_{\mathcal{A}}^{\mathcal{C}}$ localizes at the level of homotopy categories to yield a functor $\operatorname{Res}_{\mathcal{A}}^{\mathcal{C}} \colon \operatorname{Ho}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) \longrightarrow \operatorname{Ho}_{\mathcal{S}}(\mathcal{A}, \mathcal{B})$ given by the formula*

$$\operatorname{Res}_{\mathcal{A}}^{\mathcal{C}}[X] = \big[\operatorname{res}_{\mathcal{A}}^{\mathcal{C}} X\big]$$

*for $X \in \mathcal{S}^{\mathcal{C}}$, and which is part of an adjoint pair*

$$\operatorname{Lind}_{\mathcal{A}}^{\mathcal{C}} \colon \operatorname{Ho}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}) \rightleftarrows \operatorname{Ho}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) : \operatorname{Res}_{\mathcal{A}}^{\mathcal{C}},$$

*with the functor $\operatorname{Lind}_{\mathcal{A}}^{\mathcal{C}}$ being characterized by the formula*

$$\operatorname{Lind}_{\mathcal{A}}^{\mathcal{C}}[Y] = \big[\operatorname{ind}_{\mathcal{A}}^{\mathcal{C}}(Q_{\mathcal{B}}^{\mathcal{A}} Y)\big],$$

*for $Y \in \mathcal{S}^{\mathcal{A}}$, where $Q_{\mathcal{B}}^{\mathcal{A}} Y$ is the $\mathcal{B}$-cofibrant replacement of $Y$ in $\mathcal{U}(\mathcal{A},\mathcal{B})$. Moreover, the unit of the adjunction is an isomorphism, i.e.* $\eta\colon \mathrm{id} \xrightarrow{\cong} \mathrm{Res}_{\mathcal{A}}^{\mathcal{C}} \circ L\mathrm{ind}_{\mathcal{A}}^{\mathcal{C}}$.

(The functor $L\mathrm{ind}_{\mathcal{A}}^{\mathcal{C}}$ was denoted by $\mathcal{I}\mathrm{nd}_{\mathcal{A}}^{\mathcal{C}}$ in the Introduction.)

*Proof.* The restriction localizes since it preserves weak equivalences; it is characterized by the formula indicated in the statement (see [13, Lem. 1.2.2 (i)]). For the rest of the proof, we refer to A.20. The pair of adjoint functors of the statement is the derived pair of the Quillen pair of Corollary 7.6. The localization $\mathrm{Res}_{\mathcal{A}}^{\mathcal{C}}$ is then also naturally isomorphic to the total right derived functor $R\,\mathrm{res}_{\mathcal{A}}^{\mathcal{C}}$. On the other hand, the total left derived functor $L\mathrm{ind}_{\mathcal{A}}^{\mathcal{C}}$ is characterized by the given formula. Now, recall that the unit $\mathrm{id} \longrightarrow \mathrm{res}_{\mathcal{A}}^{\mathcal{C}} \circ \mathrm{ind}_{\mathcal{A}}^{\mathcal{C}}$ is an isomorphism, see B.4 (vii). Unravelling the construction of the derived adjunction (see for instance [13, Proof of Lemma 1.3.10]), one checks that the stated fact about the counit $\eta$ follows. $\square$

*Remark* 13.3. Some care is needed with these derived functors. It might happen that the Quillen adjunction $(F, U)$ is an equivalence of categories and that the derived adjunction is not. As an exercise, the reader could look at the Quillen adjunction given by the identity (!) itself, $\mathrm{id}\colon \mathcal{U}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{C}) :\mathrm{id}$, and unfold the definition of the derived adjunction (see A.20). See also Theorem 13.9 below.

* * *

**Lemma 13.4.** *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor admitting a right adjoint $U\colon \mathcal{B} \longrightarrow \mathcal{A}$. Assume that the unit of the adjunction is an isomorphism, i.e. $\eta\colon \mathrm{id} \xrightarrow{\cong} U \circ F$. Given an object $b \in \mathcal{B}$, there exists an object $a \in \mathcal{A}$ such that $F(a) \cong b$ in $\mathcal{B}$ if and only if the counit of the adjunction at $b$ is an isomorphism, i.e. $\epsilon_b\colon FU(b) \xrightarrow{\cong} b$.*

*Proof.* The condition is clearly sufficient, simply take $a := U(b)$. Conversely, assume that $\beta\colon F(a) \longrightarrow b$ is an isomorphism in $\mathcal{B}$ for some object $a \in \mathcal{A}$. Denote by $\alpha\colon a \longrightarrow U(b)$ the morphism that is adjoint to $\beta$. We have commutative diagrams

$$\begin{tikzcd} U(F(a)) \arrow[r, "U(\beta)"] & U(b) \\ a \arrow[u, "\eta_a"] \arrow[ur, "\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} F(a) \arrow[r, "F(\alpha)"] \arrow[dr, "\beta"'] & F(U(b)) \arrow[d, "\epsilon_b"] \\ & b \end{tikzcd}$$

giving the usual connection between the adjunction, the unit and the counit (see [15, Thm. IV.1.1, p. 82]). Now, by assumption, in the left-hand diagram, $\eta_a$ and $U(\beta)$ are isomorphisms, consequently, so is $\alpha\colon a \longrightarrow U(b)$. Using this in the right-hand diagram, $\beta$ and $F(\alpha)$ are isomorphisms and hence $\epsilon_b$ too. $\square$

**Theorem 13.5** (Codescent via homotopy categories)**.**
*Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories (consider $\mathcal{D}$ as full in $\mathcal{C}$). Let*

$$L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\colon\ \mathrm{Ho}_{\mathcal{S}}(\mathcal{D}) \rightleftarrows\ \mathrm{Ho}_{\mathcal{S}}(\mathcal{C}) : \mathrm{Res}_{\mathcal{D}}^{\mathcal{C}}$$

*be the adjunction of 13.2. Then, for a diagram $X \in \mathcal{S}^{\mathcal{C}}$, the following are equivalent:*

(i) *$X$ satisfies $\mathcal{D}$-codescent;*

(ii) *the image $[X]$ of $X$ in $\mathrm{Ho}_{\mathcal{S}}(\mathcal{C})$ belongs up to isomorphism to the image of the functor $L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}$.*

*When (i) and (ii) hold, the isomorphism of (ii) can be realized for instance by the counit of the above adjunction at $[X]$, i.e. one has* $\epsilon_{[X]}\colon L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}} \circ \mathrm{Res}_{\mathcal{D}}^{\mathcal{C}}\,[X] \xrightarrow{\cong} [X]$.

*Proof.* The adjunction is a special case of the one of Proposition 13.2 applied to the full inclusion $(\mathcal{D},\mathcal{D}) \hookrightarrow (\mathcal{C},\mathcal{C})$. Consider the $\mathcal{D}$-cofibrant replacement

$$\xi_X^{\mathcal{D}}\colon Q_{\mathcal{D}}\operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X \longrightarrow \operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X$$

of $\operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X$ in $\mathcal{U}(\mathcal{D})$. Applying $\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}}$ to it yields a $\mathcal{D}$-cofibrant approximation

$$\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}} Q_{\mathcal{D}}\operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X \xrightarrow{\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}}\xi_X^{\mathcal{D}}} \operatorname{ind}_{\mathcal{D}}^{\mathcal{C}}\operatorname{res}_{\mathcal{D}}^{\mathcal{C}} X \xrightarrow{\ \epsilon_X\ } X\,,$$

where $\epsilon$ is the counit of the adjunction $\big(\operatorname{ind}_{\mathcal{D}}^{\mathcal{C}}\,,\,\operatorname{res}_{\mathcal{D}}^{\mathcal{C}}\big)$, as we already saw in Proposition 10.5 ($\mathcal{D}$ is full). By global flexibility of codescent 6.6, $X$ satisfies $\mathcal{D}$-codescent if and only if the above morphism is a $\mathcal{C}$-weak equivalence. By the very construction of the derived adjunction (again, see [13, Proof of Lemma 1.3.10]), the latter is, in turn, equivalent to say that the counit $\epsilon_{[X]}\colon L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\circ\operatorname{Res}_{\mathcal{D}}^{\mathcal{C}}[X] \longrightarrow [X]$ is an isomorphism. One concludes via Lemma 13.4, since by Proposition 13.2, the unit of the adjunction $\big(L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\,,\,\operatorname{Res}_{\mathcal{D}}^{\mathcal{C}}\big)$ is an isomorphism. □

*Remark* 13.6. We deduce that *the notion of codescent does not depend on the choice of the model structure* $\mathcal{U}(\mathcal{C},\mathcal{D})$ *on* $\mathcal{S}^{\mathcal{C}}$. The above statement can be done in the language of Dwyer-Kan, Heller, Dugger and Hirschhorn. In this spirit, *statement (ii) in 13.5 can be taken as a definition of codescent.* We did not choose this definition because it makes the notion of codescent at an object $c$ more complicated and because condition (ii) is less concrete than our definition.

* * *

The following is a sort of converse to the zoom-out property 10.4.

**Proposition 13.7** (Iterating codescent).
*Let $\mathcal{C}$ be a small category and let $\mathcal{D}\subset\mathcal{E}\subset\mathcal{C}$ be subcategories. Let $X\in\mathcal{S}^{\mathcal{C}}$ and let $c\in\mathcal{C}$. Assume that the following hold:*

(a) *$X\in\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{E}$-codescent at $c$;*
(b) *$\operatorname{res}_{\mathcal{E}}^{\mathcal{C}} X$ satisfies $\mathcal{D}$-codescent at all objects of $\mathcal{E}_c$ (see 11.3).*

*Then $X\in\mathcal{S}^{\mathcal{C}}$ satisfies $\mathcal{D}$-codescent at $c$. In particular, if $X$ satisfies $\mathcal{E}$-codescent and if $\operatorname{res}_{\mathcal{E}}^{\mathcal{C}} X$ satisfies $\mathcal{D}$-codescent, then $X$ satisfies $\mathcal{D}$-codescent.*

*Proof.* By the Pruning Lemma for objects 11.5 and the Funneling Lemma 11.2, we know that we can reduce the question to the following full subcategories of $\mathcal{C}$:

$$\mathcal{D}_c\subset\mathcal{E}_c\subset\mathcal{E}_c\cup\{c\}\,.$$

In other words, it suffices to prove the second part of the statement, *i.e.* we can assume that $X$ satisfies $\mathcal{E}$-codescent and that $\operatorname{res}_{\mathcal{E}}^{\mathcal{C}} X$ satisfies $\mathcal{D}$-codescent. Consider the two successive adjunctions

$$\mathrm{Ho}_{\mathcal{S}}(\mathcal{D}) \overset{L\mathrm{ind}_{\mathcal{D}}^{\mathcal{E}}}{\rightleftarrows} \mathrm{Ho}_{\mathcal{S}}(\mathcal{E}) \overset{L\mathrm{ind}_{\mathcal{E}}^{\mathcal{C}}}{\rightleftarrows} \mathrm{Ho}_{\mathcal{S}}(\mathcal{C})\,.$$

The explicit formula for Res given in the statement of Proposition 13.2 shows that the composite of the right adjoints is $\operatorname{Res}_{\mathcal{D}}^{\mathcal{E}}\circ\operatorname{Res}_{\mathcal{E}}^{\mathcal{C}}=\operatorname{Res}_{\mathcal{D}}^{\mathcal{C}}$. Therefore, we also have a natural isomorphism of functors (cf. [15, Cor. IV.1.1, p. 85; Thm. IV.8.1, p. 103])

$$L\mathrm{ind}_{\mathcal{E}}^{\mathcal{C}}\circ L\mathrm{ind}_{\mathcal{D}}^{\mathcal{E}}\cong L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\ .$$

Now, the result follows readily from a triple application of Theorem 13.5; indeed,

$$[X]\cong L\mathrm{ind}_{\mathcal{E}}^{\mathcal{C}}\operatorname{Res}_{\mathcal{E}}^{\mathcal{C}}[X]\cong L\mathrm{ind}_{\mathcal{E}}^{\mathcal{C}}\big(L\mathrm{ind}_{\mathcal{D}}^{\mathcal{E}}\operatorname{Res}_{\mathcal{D}}^{\mathcal{E}}\operatorname{Res}_{\mathcal{E}}^{\mathcal{C}}[X]\big)\cong L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\operatorname{Res}_{\mathcal{D}}^{\mathcal{C}}[X]\,,$$

where the first two isomorphisms come, respectively, from the facts that $X$ satisfies $\mathcal{E}$-codescent and that $\mathrm{res}^{\mathcal{C}}_{\mathcal{E}} X$ satisfies $\mathcal{D}$-codescent. □

*Remark* 13.8. It is also possible to give a direct proof of this result without using the homotopy categories. We leave it to the motivated reader, as a good familiarizing exercise.

* * *

Now, we provide a description of the homotopy category of $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$.

**Theorem 13.9.** *Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. Then the adjunction*

$$L\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}\colon \mathrm{Ho}_{\mathcal{S}}(\mathcal{D}) \rightleftarrows \mathrm{Ho}_{\mathcal{S}}(\mathcal{C},\mathcal{D}) :\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}$$

*is an equivalence of categories.*

*Proof.* The adjunction is given by Proposition 13.2 applied to the full inclusion of pairs $(\mathcal{D},\mathcal{D}) \hookrightarrow (\mathcal{C},\mathcal{D})$. By the latter proposition, it only remains to prove that the counit of the adjunction, $\epsilon\colon L\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}} \circ \mathrm{Res}^{\mathcal{C}}_{\mathcal{D}} \longrightarrow \mathrm{id}$, is an isomorphism. Recall that a morphism in a model category becomes an isomorphism in the homotopy category if and only if it is a weak equivalence (see [13, Thm. 1.2.10 (iv)]). Since the weak equivalences on both $\mathcal{U}(\mathcal{C},\mathcal{D})$ and $\mathcal{U}(\mathcal{D})$ are the $\mathcal{D}$-weak equivalences, it follows easily that $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}$ detects isomorphisms. Applying this to the above counit and remembering that the unit $\eta$ of the adjunction is already known to be an isomorphism, the result follows (recall the equality $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\,\epsilon_{[X]} \circ \eta_{\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}[X]} = \mathrm{id}_{\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}[X]}$ for all $[X] \in \mathrm{Ho}_{\mathcal{S}}(\mathcal{D})$, by general properties of adjunctions: see [15, (8) on p. 82]). □

*Remark* 13.10. In other words, *we have constructed on $\mathcal{S}^{\mathcal{C}}$ a model structure which is Quillen equivalent to Hirschhorn's model structure on $\mathcal{S}^{\mathcal{D}}$.* If, at this point, the reader gets the impression that codescent is indeed easier than what it seemed in Definition 4.3, then we have reached our goal! This notion should not be underestimated though: we will see in [2] that this nice and simple property is in fact related to deep and central mathematical problems.

## 14. The codescent locus

In this section, we observe that many statements can be very conveniently reformulated, using the notion of codescent locus, that we next introduce. This part can be read completely independently of the rest of the paper, except for the Introduction; for a more detailed account, the reader may quickly refer to 3.2–3.7 (for the definition of the model category $\mathcal{U}(\mathcal{C},\mathcal{D})$) and to 4.1–4.3 (for the definition of $\mathcal{D}$-codescent and of $\mathcal{D}$-codescent at a given $c \in \mathcal{C}$). This can serve as an index for the whole paper.

We start by recalling Convention 3.1: by a subset of a small category, we mean a subset of its class of objects; by a subcategory given by a set of objects without further mention, we mean the corresponding full subcategory.

**Definition 14.1.** Let $(\mathcal{C},\mathcal{D})$ be a pair of small categories. The *$\mathcal{D}$-codescent locus* of a functor $X \in \mathcal{S}^{\mathcal{C}}$ is the subset of those objects of $\mathcal{C}$, where $X$ satisfies $\mathcal{D}$-codescent; we denote it by $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$.

For the terminology and notations used in the next statement, we indicate the following references to the rest of the paper:

- *closed under retracts* (A.4 (i) and (ii));
- *retract equivalent* (3.12); see also *essentially equivalent* (3.12);
- res and ind (beginning of Appendix B and B.2);
- $\mathcal{D}_c$ and $\mathcal{E}_c$ (11.3); $\mathcal{D}_c \veebar \{c\}$ (11.9);
- $\mathcal{C}$*-weak equivalence* (3.4 (ii));
- *weak retract* (A.21).

**Proposition 14.2.** *Let $(\mathcal{C},\mathcal{D})$ and $(\mathcal{C},\mathcal{E})$ be pairs of small categories (see 3.2), and consider an object $X \in \mathcal{S}^{\mathcal{C}}$. The following properties hold:*

(i) *The set $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$ contains $\mathcal{D}$ and is closed under retracts.*
(ii) *If $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$, then $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X) \subset \mathcal{C}\mathrm{od}_{\mathcal{E}}(X)$ holds.*
(iii) *If $\mathcal{E}$ is retract equivalent to $\mathcal{D}$, then $\mathcal{C}\mathrm{od}_{\mathcal{E}}(X) = \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$.*
(iv) *The restriction $\mathrm{res}^{\mathcal{C}}_{\mathcal{C}\mathrm{od}_{\mathcal{D}}(X)} X$ satisfies $\mathcal{D}$-codescent.*
(v) *The set $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$ is the union $\bigcup \mathrm{obj}(\mathcal{A})$ over all full subcategories $\mathcal{A}$ of $\mathcal{C}$ such that $\mathrm{res}^{\mathcal{C}}_{\mathcal{A}} X$ satisfies $\mathcal{D}$-codescent.*
(vi) *One has $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X) = \mathcal{D} \cup \bigcup_{c\in\mathcal{C}\smallsetminus\mathcal{D}} \mathcal{C}\mathrm{od}_{\mathcal{D}_c}(\mathrm{res}^{\mathcal{C}}_{\mathcal{D}_c\veebar\{c\}} X)$.*
(vii) *Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$ containing $\mathcal{D}$. Then, for an object $Y \in \mathcal{S}^{\mathcal{A}}$, one has $\mathcal{C}\mathrm{od}_{\mathcal{D}}(Y) = \mathcal{C}\mathrm{od}_{\mathcal{D}}(\mathrm{ind}^{\mathcal{C}}_{\mathcal{A}} Y) \cap \mathrm{obj}(\mathcal{A})$.*
(viii) *If $Y \in \mathcal{S}^{\mathcal{C}}$ is $\mathcal{C}$-weakly equivalent to $X$, then $\mathcal{C}\mathrm{od}_{\mathcal{D}}(X) = \mathcal{C}\mathrm{od}_{\mathcal{D}}(Y)$.*
(ix) *If $Y \in \mathcal{S}^{\mathcal{C}}$ is a weak retract of $X$ in $\mathcal{U}(\mathcal{C})$, then $\mathcal{C}\mathrm{od}_{\mathcal{D}}(Y) \subset \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$.*
(x) *If $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}$, then $\{c \in \mathcal{C}\mathrm{od}_{\mathcal{E}}(X) \,|\, \mathcal{E}_c \subset \mathcal{C}\mathrm{od}_{\mathcal{D}}(\mathrm{res}^{\mathcal{C}}_{\mathcal{E}} X)\} \subset \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$.*

*Proof.*
(i) is Corollary 10.2 (clearly, $\mathcal{D} \subset \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$).
(ii) is the *zoom-out property for codescent* 10.4.
(iii) is the *retract equivalence property for codescent* 10.1.
(iv) follows from the *restriction property for codescent* 9.16.
(v) follows from the *covering property for codescent* 11.1.
(vi) follows from *funneling and pruning*, see Theorem 11.10.
(vii) is the *induction property for codescent* 9.5.
(viii) is the *weak invariance of codescent* 6.10.
(ix) is the *weak retract invariance of codescent* 10.3.
(x) is *iterating codescent* 13.7. □

At this point, for the reader using this section as an index or as a survey, we also refer to the *Funneling Lemma* 11.2 and to the *Pruning Lemmas* 11.5 and 11.7 in connection with (vi) above.

*Remark* 14.3. We point out that statement (v) in Proposition 14.2 tells that there is a maximal full subcategory of $\mathcal{C}$, where $X$ satisfies $\mathcal{D}$-codescent. On the other hand, in general, there is no minimal (full, say) subcategory $\mathcal{D}_0$ of $\mathcal{C}$ such that $X$ satisfies $\mathcal{D}_0$-codescent. For example, if $\mathcal{D}$ and $\mathcal{E}$ are essentially equivalent (see 3.12), then $X$ satisfies $\mathcal{D}$-codescent exactly where it satisfies $\mathcal{E}$-codescent (by the retract equivalence property for codescent 10.1); however, as easy examples show, $\mathcal{D}$ and $\mathcal{E}$ may well be non-empty and have no common object (see also Example 4.4 (1) and (2)).

Proposition 9.1 can also be reformulated as follows, using the terminology of A.15 (the proof is clear).

**Proposition 14.4.** *Let $F\colon \mathcal{S} \longrightarrow \mathcal{T}$ be a left Quillen functor between cofibrantly generated model categories. Then, for $X \in \mathcal{S}^{\mathcal{C}}$, the following holds:*

(i) *If $F$ preserves weak equivalences or if $X$ is $\mathcal{C}$-objectwise cofibrant, then we have $\mathcal{C}\mathrm{od}_{\mathcal{D}}(F \circ X) \supset \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$.*

(ii) *If $F$ reflects weak equivalences, then $\mathcal{C}\mathrm{od}_{\mathcal{D}}(F \circ X) = \mathcal{C}\mathrm{od}_{\mathcal{D}}(X)$ holds.*

## Appendix A. Recollection on model categories

The following can be found in the original work of Quillen [18], whereas the modern terminology is to be found for instance in [10], [12] and [13].

Here and in the body of the text, we try to give the definitions in such a way that the non-specialist can get the feeling of those concepts; on the other hand, the proofs are written so that the specialist can easily check the details.

**Definition A.1.** Let $\mathcal{A}$ be a category and let $f\colon a \longrightarrow b$ and $g\colon x \longrightarrow y$ be two morphisms in $\mathcal{A}$. One says that $f$ has the *left lifting property with respect to $g$* if for every commutative (solid) diagram

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[d, "f"'] & x \arrow[d, "g"] \\ b \arrow[r, "v"'] \arrow[ur, dashed, "h"] & y \end{tikzcd}$$

in $\mathcal{A}$ (with $u$ and $v$ arbitrary), there exists a "lift" $h\colon b \longrightarrow x$ making the above diagram commute. In this case, $g$ is of course said to have the *right lifting property with respect to $f$*. Given a collection of morphisms $\mathcal{K}$ in $\mathcal{A}$, we denote by $\mathrm{LLP}(\mathcal{K})$ the collection of morphisms having the left lifting property with respect to all $k \in \mathcal{K}$. Dually, $\mathrm{RLP}(\mathcal{K})$ is the collection of morphisms having the right lifting property with respect to all $k \in \mathcal{K}$.

**Notation A.2.** Let $\mathcal{A}$ be a category. We denote by $\mathrm{arr}(\mathcal{A})$ the *category of arrows of $\mathcal{A}$*, whose objects are morphisms $a \longrightarrow a'$ in $\mathcal{A}$, whose morphisms are the corresponding commutative squares in $\mathcal{A}$, and with concatenation as composition.

**Definition A.3.** Given a category $\mathcal{A}$, a *functorial factorization* $(\alpha, \beta)$ consists of a factorization of an arbitrary morphism $f$ as $f = \beta(f) \circ \alpha(f)$, in a functorial way with respect to $f$, in the sense that $\alpha$ and $\beta$ must be functors $\mathrm{arr}(\mathcal{A}) \longrightarrow \mathrm{arr}(\mathcal{A})$, such that the source of $\beta$ equals the target of $\alpha$, as functors $\mathrm{arr}(\mathcal{A}) \longrightarrow \mathcal{A}$.

**Definition A.4.** (i) Let $\mathcal{A}$ be a category. An object $a$ of $\mathcal{A}$ is called a *retract* of the object $b \in \mathcal{A}$, if there exist morphisms $\alpha\colon a \longrightarrow b$ and $\beta\colon b \longrightarrow a$ such that $\beta \circ \alpha = \mathrm{id}_a$.

(ii) A subcategory $\mathcal{A}'$ of a category $\mathcal{A}$ is called *closed under retracts* (in $\mathcal{A}$), if whenever $a \in \mathcal{A}$ is a retract in $\mathcal{A}$ of some $a' \in \mathcal{A}'$, then $a$ belongs to $\mathcal{A}'$ too.

(iii) A morphism $f$ in a category $\mathcal{B}$ is a *retract* of the morphism $g$, if $f$ is a retract of $g$ in the category $\mathcal{A} := \mathrm{arr}(\mathcal{B})$, in the sense of (i).

Before the next definition, we recall a few useful notions. A category is called *small* if its underlying class of objects is a set. A *small* (co)limit is a (co)limit over a small category. A category is *complete* (resp. *cocomplete*) if it admits all small limits (resp. all small colimits).

**Definition A.5.** A *model category* is a quadruple $(\mathcal{M}, \mathcal{W}eq, \mathcal{C}of, \mathcal{F}ib)$, where $\mathcal{M}$ is a category, and $\mathcal{W}eq$, $\mathcal{C}of$ and $\mathcal{F}ib$ are classes of morphisms, called *weak equivalences*, *cofibrations* and *fibrations* respectively, and satisfying the following axioms:

**(MC 1)** The category $\mathcal{M}$ is complete and cocomplete.

**(MC 2)** The class of morphisms $\mathcal{W}eq$ satisfies the 2-out-of-3 property: given a composition $g \circ f$, if two out of $f$, $g$ and $g \circ f$ are weak equivalences, then so is the third.

**(MC 3)** The classes $\mathcal{W}eq$, $\mathcal{C}of$ and $\mathcal{F}ib$ are closed under retracts, that is, if $f$ is a retract of $g$, and if $g$ belongs to one of those classes, so does $f$.

**(MC 4)** (a) $\mathcal{C}of \subset \mathrm{LLP}(\mathcal{W}eq \cap \mathcal{F}ib)$;
(b) $\mathcal{F}ib \subset \mathrm{RLP}(\mathcal{W}eq \cap \mathcal{C}of)$.

**(MC 5)** (a) There exists a functorial factorization $(\alpha, \beta)$ such that, for every morphism $f$ in $\mathcal{M}$, $\alpha(f) \in \mathcal{C}of$ and $\beta(f) \in \mathcal{W}eq \cap \mathcal{F}ib$.
(b) There exists a functorial factorization $(\gamma, \delta)$ such that, for every morphism $f$ in $\mathcal{M}$, $\gamma(f) \in \mathcal{W}eq \cap \mathcal{C}of$ and $\delta(f) \in \mathcal{F}ib$.

For simplicity, we generally write $\mathcal{M}$ for $(\mathcal{M}, \mathcal{W}eq, \mathcal{C}of, \mathcal{F}ib)$.

**Definition A.6.** Let $\mathcal{M}$ be a model category. A morphism in $\mathcal{W}eq \cap \mathcal{C}of$ (resp. $\mathcal{W}eq \cap \mathcal{F}ib$) is called a *trivial cofibration* (resp. a *trivial fibration*).

We will denote an isomorphism in a category by "$\xrightarrow{\cong}$" and a weak equivalence in a model category by "$\xrightarrow{\sim}$".

Note that a model category $\mathcal{M}$ being complete and cocomplete, it has an initial object $\varnothing$ and a terminal object $*$ (in both cases, such an object is unique up to a unique isomorphism, and, for convenience, we can once and for all fix one and put the article "the" in front of it).

**Definition A.7.** An object $X$ in a model category $\mathcal{M}$ is called *cofibrant* if the morphism $\varnothing \longrightarrow X$ in $\mathcal{M}$ is a cofibration; it is called *fibrant* if the morphism $X \longrightarrow *$ in $\mathcal{M}$ is a fibration.

For the following three examples, we refer to [18] and to [13].

**Example A.8.** The category $\mathcal{T}op$ of (all) topological spaces is a model category with the classes $\mathcal{W}eq$ and $\mathcal{C}of$ having the usual meaning, and with the Serre fibrations forming the class $\mathcal{F}ib$. The initial object is the empty space $\varnothing$ and the terminal object is the point, $* = pt$. For this structure, every topological space is fibrant, and among the cofibrant spaces are the CW-complexes. Similar results hold for the category $\mathcal{T}op_\bullet$ of pointed topological spaces (with all *well-pointed* CW-complexes being cofibrant objects).

**Example A.9.** Let $\mathrm{s}\mathcal{S}ets := \mathcal{S}ets^{\mathbf{\Delta}^{\mathrm{op}}}$ be the category of simplicial sets. It has a model category structure with weak equivalences being those morphisms which induce a weak homotopy equivalence on the realization, cofibrations being monomorphisms (*i.e.* degreewise injections of sets), and fibrations being the Kan fibrations, *i.e.* the class $\mathrm{RLP}(J)$, where $J := \{\Lambda^n_k \hookrightarrow \Delta^n \mid n > 0 \text{ and } 0 \leq k \leq n\}$. In this case, all simplicial sets are cofibrant, and the fibrant ones are precisely the *Kan complexes*. Similar results hold for the category $\mathrm{s}\mathcal{S}ets_\bullet$ of pointed simplicial sets.

**Example A.10.** Let $R$ be a unital ring and let $\mathcal{M} := \mathrm{Ch}(R\text{-mod})$ be the category of chain complexes of left $R$-modules. Then, $\mathcal{M}$ has two standard model category structures, both with $\mathcal{W}eq$ being the class of quasi-isomorphisms (isomorphism on homology groups). For one of them, one takes for $\mathcal{F}ib$ the class of degreewise epimorphisms and defines $\mathcal{C}of := \mathrm{LLP}(\mathcal{W}eq \cap \mathcal{F}ib)$; in this case, every chain complex is fibrant. For the other structure, $\mathcal{C}of$ is the class of degreewise monomorphisms and $\mathcal{F}ib := \mathrm{RLP}(\mathcal{W}eq \cap \mathcal{C}of)$; here, every chain complex is cofibrant.

**Example A.11.** The category $\mathcal{S}\mathrm{p}$ of spectra (of pointed simplicial sets, say) has a model category structure with weak equivalences being the $\pi_*^s$-isomorphisms, where $\pi_*^s$ denotes the stable homotopy groups. We refer the reader to [4] for details on the model structure on $\mathcal{S}\mathrm{p}$.

**Proposition A.12.** *Let $\mathcal{M}$ be a model category. The following holds:*

(i) *We have $\mathcal{C}of = \mathrm{LLP}(\mathcal{W}eq \cap \mathcal{F}ib)$ and $\mathcal{W}eq \cap \mathcal{C}of = \mathrm{LLP}(\mathcal{F}ib)$.*
(ii) *We have $\mathcal{F}ib = \mathrm{RLP}(\mathcal{W}eq \cap \mathcal{C}of)$ and $\mathcal{W}eq \cap \mathcal{F}ib = \mathrm{RLP}(\mathcal{C}of)$.*
(iii) *Any two of the classes $\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}eq$ determine the third one.*
(iv) *The class of cofibrations is closed under transfinite compositions, pushouts and coproducts. The same is true for trivial cofibrations.*
(v) *The class of fibrations is closed under pullbacks and products. The same is true for trivial fibrations.*

*Proof.* See [12, Propositions 7.2.3, 7.2.4, 7.2.5, 7.2.7, 7.2.12 and 10.3.4]. □

**Definition-Notation A.13.** Let $\mathcal{M}$ be a model category. For a given object $X$ in $\mathcal{M}$, applying the functorial factorization **(MC 5)** (a) to the morphism $\varnothing \longrightarrow X$, one obtains a functor

$$\mathcal{M} \longrightarrow \mathrm{arr}(\mathcal{M}), \quad X \longmapsto (\xi_X : QX \to X),$$

with $QX \in \mathcal{M}$ cofibrant and $\xi_X$ a trivial fibration; $QX$ is called the *cofibrant replacement* of $X$. Similarly, applying the functorial factorization **(MC 5)** (b) to the morphism $X \longrightarrow *$, one gets a functor

$$\mathcal{M} \longrightarrow \mathrm{arr}(\mathcal{M}), \quad X \longmapsto (\phi_X : X \to RX),$$

with $RX \in \mathcal{M}$ fibrant and $\phi_X$ a trivial cofibration; $RX$ is called the *fibrant replacement* of $X$.

*Remark* A.14. The cofibrant replacement and the fibrant replacement functors $Q, R\colon \mathcal{M} \longrightarrow \mathcal{M}$ both preserve weak equivalences. This is an immediate consequence of the 2-out-of-3 property of weak equivalences **(MC 2)**.

* * *

**Definition A.15.** For a functor $\Psi\colon \mathcal{M} \longrightarrow \mathcal{N}$ between model categories, we say that

(i) $\Psi$ *preserves weak equivalences* if the following holds: if a morphism $\eta$ is a weak equivalence in $\mathcal{M}$, then $\Psi(\eta)$ is a weak equivalence in $\mathcal{N}$;

(ii) $\Psi$ *detects weak equivalences* if the following holds: if a morphism $\eta$ is such that $\Psi(\eta)$ is a weak equivalence in $\mathcal{N}$, then $\eta$ is a weak equivalence in $\mathcal{M}$;

(iii) $\Psi$ *reflects weak equivalences* if the following holds: a morphism $\eta$ is a weak equivalence in $\mathcal{M}$ if and only if $\Psi(\eta)$ is a weak equivalence in $\mathcal{N}$.

Similarly for the meaning of *preserving, detecting* or *reflecting fibrations*, and so on.

**Definition A.16.** Given two model categories $\mathcal{M}$ and $\mathcal{N}$ and a pair of adjoint functors

$$F\colon \mathcal{M} \rightleftarrows \mathcal{N} : U\,,$$

we say that it is a *Quillen adjunction* if the left adjoint $F$ preserves cofibrations and trivial cofibrations (compare Remark A.17 below). In this situation, $F$ is called a *left Quillen functor* and $U$ a *right Quillen functor*; one also says that $F$ and $U$ form a *Quillen pair.*

*Remark* A.17. A pair of adjoint functors $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ as above is a Quillen adjunction if and only if the right adjoint $U$ preserves fibrations and trivial fibrations. See [13, § 1.3.1] for details. A left Quillen functor always preserves cofibrant objects, since it preserves the initial object and cofibrations; it also preserves weak equivalences between cofibrant objects, by Ken Brown's Lemma (see for instance [13, Lem. 1.1.12]). Similarly, a right Quillen functor preserves fibrant objects and weak equivalences between them.

The above adjoint pair $(F, U)$ can be thought of as a morphism *from* the model category $\mathcal{M}$ *to* the model category $\mathcal{N}$. The basic example is the geometric realization $|{-}|\colon s\mathcal{S}\mathrm{ets} \longrightarrow \mathcal{T}\mathrm{op}$ which has the singular functor $\mathrm{Sing}\colon \mathcal{T}\mathrm{op} \longrightarrow s\mathcal{S}\mathrm{ets}$ as right adjoint.

* * *

**Definition A.18.** A *localization* of a category $\mathcal{M}$ with respect to a class of morphisms $\mathcal{W}$ in $\mathcal{M}$ is a functor $q\colon \mathcal{M} \longrightarrow \mathcal{H}$ to some other category $\mathcal{H}$ such that

(a) $q(w)$ is an isomorphism in $\mathcal{H}$ for all $w \in \mathcal{W}$;
(b) $q$ is universal for property (a), that is, for every functor $t\colon \mathcal{M} \longrightarrow \mathcal{T}$ to a category where $t(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a unique factorization

$$\begin{tikzcd} \mathcal{M} \arrow[r, "t"] \arrow[d, "q"'] & \mathcal{T} \\ \mathcal{H} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

As usual, when it exists, such a localization is unique, up to a unique isomorphism, and we write $\mathcal{M}[\mathcal{W}^{-1}] := \mathcal{H}$.

For the next result, we refer to [13, § 1.2, pp. 7–13] and to [12, § 8.3, pp. 147–151] for instance.

**Proposition-Definition A.19.** *If $\mathcal{M}$ is a model category, then the localization of $\mathcal{M}$ with respect to $\mathcal{W}eq$ exists; it is called the* homotopy category of $\mathcal{M}$*, and is denoted by*

$$\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}eq^{-1}]\,.$$

To construct $\mathrm{Ho}(\mathcal{M})$, consider the full subcategory $\mathcal{M}_{cf}$ of $\mathcal{M}$ on those objects which are both cofibrant and fibrant. There is an equivalence relation on each set of morphisms in $\mathcal{M}_{cf}$ such that $\mathrm{Ho}(\mathcal{M})$ can be realized as a quotient of $\mathcal{M}_{cf}$ by these relations. The functor $q\colon \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$ is induced by the composite of the fibrant and the cofibrant replacement functors. Again, see the details in [13, § 1.2] and in [12, § 8.3].

**Proposition-Definition A.20.** *A Quillen adjunction* $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ *induces a so-called* derived adjunction

$$LF\colon \operatorname{Ho}(\mathcal{M}) \rightleftarrows \operatorname{Ho}(\mathcal{N}) : RU$$

*where the so-called* total left derived functor $LF$ *and* total right derived functor $RU$ *are essentially defined to be* $F$ *pre-composed with the cofibrant replacement in* $\mathcal{M}$ *and* $U$ *pre-composed with the fibrant replacement in* $\mathcal{N}$*, respectively.*

See details for instance in [13, § 1.3, pp. 13–22] (see in particular Definition 1.3.6 and Lemma 1.3.10 therein); see also [12, §§ 8.4–8.5, pp. 151–158].

* * *

*Remark* A.21. Let $\mathcal{M}$ be a model category. We call a morphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ a *weak retract* of the morphism $g\colon A \longrightarrow B$, if there exist morphisms $\alpha\colon f \longrightarrow g$ and $\beta\colon g \longrightarrow f$ in $\operatorname{arr}(\mathcal{M})$ such that both the $X$- and the $Y$-component of $\beta \circ \alpha$ are weak equivalences, as follows:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\sim"] & A \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "\sim"'] & B \arrow[r] & Y \end{tikzcd}$$

The reader can prove as an exercise that if $g$ is a weak equivalence, then so is $f$. To do this, it suffices to consider the image diagram in the homotopy category. We shall sometimes say that an object $X$ is a *weak retract* of another object $A$, meaning that $\mathrm{id}_X$ is a weak retract of $\mathrm{id}_A$, or equivalently that there exist morphisms $\eta\colon X \longrightarrow A$ and $\zeta\colon A \longrightarrow X$ such that $\zeta \circ \eta$ is a weak equivalence.

* * *

The rest of this appendix, except for the definition of a cofibrantly generated model category (in A.24 below), will only be needed in Section 2, so, the reader tempted to rush through or even to skip that section may just have a rapid look at part (iii) and (iv) of Definition A.24 and at Example A.26, and then directly proceed to Appendix B. What we next recall is some terminology extracted directly from [13, § 2.1, pp. 28–29], without unfolding all set-theoretical details.

**Definition A.22.** Let $\mathcal{A}$ be a category and let $K$ be a *set* of morphisms. A morphism in $\mathcal{A}$ is called a *relative* $K$*-cell complex* if it is a transfinite composition of pushouts of elements of $K$. We denote by $K$-cell the class of relative $K$-cell complexes.

For the next definition, recall that an ordinal $\lambda$ is called $\kappa$*-filtered*, where $\kappa$ is some cardinal, if it is a limit ordinal and if $\lambda_0 \subset \lambda$ is such that $|\lambda_0| \leq \kappa$, then $\sup \lambda_0 < \lambda$.

**Definition A.23.** An object $a$ in a category $\mathcal{A}$ is called *small relative to a class of morphisms* $\mathcal{K}$ if there exists a cardinal $\kappa$ such that for every $\kappa$-filtered ordinal $\lambda$ and for every $\lambda$-sequence

$$a_0 \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_\beta \longrightarrow \ldots$$

in $\mathcal{A}$, with the morphism $a_\beta \longrightarrow a_{\beta+1}$ in $\mathcal{K}$ whenever $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \operatorname{mor}_{\mathcal{A}}(a, a_\beta) \longrightarrow \operatorname{mor}_{\mathcal{A}}(a, \operatorname*{colim}_{\beta<\lambda} a_\beta)$$

is a bijection. (More precisely, in this case, one says that $a$ is *$\kappa$-small* relative to $\mathcal{K}$.) In short, a morphism out of the object $a$ to a "linear" colimit, say $\operatorname{colim}_\beta a_\beta$, is already – and essentially in a unique way – a morphism out of $a$ to some $a_\beta$.

**Definition A.24.** A model category $(\mathcal{M}, \mathcal{W}eq, \mathcal{C}of, \mathcal{F}ib)$ is called *cofibrantly generated* if there exist two *sets* of morphisms $I$ and $J$ such that:

(i) the domains of the morphisms in $I$ are small relative to $I$-cell;
(ii) the domains of the morphisms in $J$ are small relative to $J$-cell;
(iii) $\mathcal{F}ib = \mathrm{RLP}(J)$;
(iv) $\mathcal{W}eq \cap \mathcal{F}ib = \mathrm{RLP}(I)$.

The (elements of the) sets $I$ and $J$ are called the *generating cofibrations* and the *generating trivial cofibrations* respectively.

*Remark* A.25. Of course, if the domain of every morphism in $I \cup J$ is merely *small*, that is, small relative to the whole of $\mathcal{M}$, then conditions (i) and (ii) trivially hold.

**Examples A.26.** The categories $\mathcal{T}$op, $\mathcal{T}$op$_\bullet$, s$\mathcal{S}$ets, s$\mathcal{S}$ets$_\bullet$, Ch($R$-mod) (with both indicated model structures) and $\mathcal{S}$p of Examples A.8, A.9, A.10 and A.11 are cofibrantly generated model categories. This can also be found in [13], except for the case of spectra, for which, as in A.11 above, we refer to Appendix A of [3] for a more detailed discussion. As an illustration, for $\mathcal{T}$op, one can take

$$I := \{S^{n-1} \hookrightarrow D^n \,|\, n \geq 0\} \quad \text{and} \quad J := \{D^n \hookrightarrow D^n \times [0,1] \,|\, n \geq 0\}$$

(inclusion of the $(n-1)$-sphere in the closed $n$-disk as its boundary, with $S^{-1} := \varnothing$, and, respectively, the inclusion at level 0).

* * *

**Definition A.27.** For a category $\mathcal{C}$ and a class $\mathcal{K}$ of morphisms in $\mathcal{C}$, we set

$$\mathrm{cof}(\mathcal{K}) := \mathrm{LLP}(\mathrm{RLP}(\mathcal{K})) \quad \text{and} \quad \mathrm{fib}(\mathcal{K}) := \mathrm{RLP}(\mathrm{LLP}(\mathcal{K})).$$

It is a general fact that $K$-cell $\subset \mathrm{cof}(K)$ as follows immediately from A.12.

**Theorem A.28** (Kan)**.** *Let $\mathcal{C}$ be a complete and cocomplete category. Suppose that $\mathcal{W}$ is a class of morphisms in $\mathcal{C}$, and that $I$ and $J$ are sets of morphisms in $\mathcal{C}$. Then, there is a cofibrantly generated model category structure on $\mathcal{C}$ with $I$ as generating cofibrations, $J$ as generating trivial cofibrations, and $\mathcal{W}$ as weak equivalences if and only if the following conditions are satisfied:*

(K1) *the class $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts;*
(K2) *the domains of $I$ are small relative to $I$-*cell*;*
(K3) *the domains of $J$ are small relative to $J$-*cell*;*
(K4) $J$-cell $\subset \mathcal{W} \cap \mathrm{cof}(I)$*;*
(K5) $\mathrm{RLP}(I) \subset \mathcal{W} \cap \mathrm{RLP}(J)$*;*
(K6) *either* $\mathcal{W} \cap \mathrm{cof}(I) \subset \mathrm{cof}(J)$ *or* $\mathcal{W} \cap \mathrm{RLP}(J) \subset \mathrm{RLP}(I)$.

*Proof.* This is [13, Thm. 2.1.19] and also [12, Thm. 11.3.1]. □

## Appendix B. Left and right Kan extensions

Fix a category $\mathcal{S}$ of "values" and denote by $\mathcal{S}^{\mathcal{A}}$ the category of functors from a small category $\mathcal{A}$ to $\mathcal{S}$. We generally assume that $\mathcal{S}$ is complete and cocomplete.

Let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between small categories. Consider the functor

$$\Phi^*\colon \mathcal{S}^{\mathcal{B}} \longrightarrow \mathcal{S}^{\mathcal{A}}, \quad X \longmapsto X \circ \Phi\,.$$

In the case of an inclusion $\mathrm{Incl}\colon \mathcal{A} \hookrightarrow \mathcal{B}$ of a (not necessarily full) subcategory, the functor $\mathrm{Incl}^*$ is just the usual *restriction*

$$\mathrm{res}^{\mathcal{B}}_{\mathcal{A}} := \mathrm{Incl}^*\colon \mathcal{S}^{\mathcal{B}} \longrightarrow \mathcal{S}^{\mathcal{A}}, \quad X \longmapsto X_{|\mathcal{A}}\,.$$

By general considerations, $\Phi^*$ has a left and a right adjoint. The *left* and *right Kan extensions* $\Phi_*$ and $\Phi_!$ are explicit descriptions of these adjoints. Their definition requires the use of so-called "comma categories".

**Definition B.1.** Let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between small categories and let $b \in \mathcal{B}$. One defines the *comma category* $\Phi \searrow b$ as follows. Its objects are the pairs $(a, \beta)$ consisting of an object $a \in \mathcal{A}$ and a morphism $\beta\colon \Phi(a) \longrightarrow b$. A morphism $\alpha\colon (a_1, \beta_1) \longrightarrow (a_2, \beta_2)$ is a morphism $\alpha\colon a_1 \longrightarrow a_2$ in $\mathcal{A}$ such that the following diagram commutes in $\mathcal{B}$:

$$\begin{tikzcd}
\Phi(a_1) \arrow[r, "\beta_1"] \arrow[d, "\Phi(\alpha)"'] & b \arrow[d, equal] \\
\Phi(a_2) \arrow[r, "\beta_2"] & b
\end{tikzcd}$$

Dually, the *comma category* $b \searrow \Phi$ consists of the pairs $\big(a\,,\, b \xrightarrow{\beta} \Phi(a)\big)$ and of the morphisms $\alpha\colon (a_1, \beta_1) \longrightarrow (a_2, \beta_2)$ with $\alpha\colon a_1 \longrightarrow a_2$, such that $\Phi(\alpha) \circ \beta_1 = \beta_2$.

When $\Phi = \mathrm{Incl}\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is an inclusion, we denote these two categories by $\mathcal{A} \searrow b$ and $b \searrow \mathcal{A}$ respectively.

**Definition B.2.** Let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between small categories and let $\mathcal{S}$ be a cocomplete category. For any $Y \in \mathcal{S}^{\mathcal{A}}$, the *left Kan extension* $\Phi_* Y \in \mathcal{S}^{\mathcal{B}}$ of $Y$ is defined to be, for every $b \in \mathcal{B}$,

$$\Phi_* Y(b) := \operatorname*{colim}_{\left(a\,,\,\Phi(a) \xrightarrow{\beta} b\right) \,\in\, \Phi \searrow b} Y(a)\,.$$

This construction is functorial in $b \in \mathcal{B}$ and in $Y \in \mathcal{S}^{\mathcal{A}}$. This gives a functor

$$\Phi_*\colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{B}}\,.$$

In the special case where $\Phi = \mathrm{Incl}\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is an inclusion, we shall denote by

$$\mathrm{ind}^{\mathcal{B}}_{\mathcal{A}} := \mathrm{Incl}_*\colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{B}}$$

the *induction* from $\mathcal{A}$ to $\mathcal{B}$ (the notation is motivated by the analogy with representation theory).

**Definition B.3.** Let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between small categories and let $\mathcal{S}$ be a complete category. For any $Y \in \mathcal{S}^{\mathcal{A}}$, the *right Kan extension* $\Phi_! Y \in \mathcal{S}^{\mathcal{B}}$ of $Y$ is defined to be

$$\Phi_! Y(b) := \lim_{\left(a\,,\, b \xrightarrow{\beta} \Phi(a)\right) \,\in\, b \searrow \Phi} Y(a)$$

for any $b \in \mathcal{B}$. As before, this yields a functor

$$\Phi_!\colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{B}}\,.$$

In the special case where $\Phi = \mathrm{Incl}\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is an inclusion, we shall denote by

$$\mathrm{ext}_{\mathcal{A}}^{\mathcal{B}} := \mathrm{Incl}_!\colon \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{B}}$$

the *extension* from $\mathcal{A}$ to $\mathcal{B}$ (this non-standard notation is motivated by the interpretation of this functor as something extending the given data from $\mathcal{A}$ to $\mathcal{B}$).

**Lemma B.4.** *Let $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between small categories and let $\mathcal{S}$ be a category which is complete and cocomplete.*

(i) *The functor $\Phi_*$ is left adjoint to $\Phi^*$.*
(ii) *The functor $\Phi_!$ is right adjoint to $\Phi^*$.*
(iii) *Denote by $\varnothing_{\mathcal{S}}$ and $*_{\mathcal{S}}$ the initial and terminal objects of $\mathcal{S}$ respectively. Let $\varnothing$ be the initial object of $\mathcal{S}^{\mathcal{A}}$ or $\mathcal{S}^{\mathcal{B}}$, which is $\varnothing_{\mathcal{S}}$ objectwise; and similarly for the terminal object $*$ of $\mathcal{S}^{\mathcal{A}}$ or $\mathcal{S}^{\mathcal{B}}$. Then $\Phi^*(\varnothing) = \varnothing$, $\Phi^*(*) = *$, $\Phi_*(\varnothing) = \varnothing$ and $\Phi_!(*) = *$ hold.*

*If $\Psi\colon \mathcal{B} \longrightarrow \mathcal{C}$ is a further functor into a small category $\mathcal{C}$, then, we have:*

(iv) $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$
(v) $(\Psi \circ \Phi)_* \cong \Psi_* \circ \Phi_*$
(vi) $(\Psi \circ \Phi)_! \cong \Psi_! \circ \Phi_!$

*Furthermore, in case $\Phi = \mathrm{Incl}\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is a* full *inclusion, the unit $\eta$ of the adjunction $\big(\mathrm{ind}_{\mathcal{A}}^{\mathcal{B}}, \mathrm{res}_{\mathcal{A}}^{\mathcal{B}}\big)$ is an isomorphism:*

(vii) $\eta\colon\ \mathrm{id} \xrightarrow{\cong} \mathrm{res}_{\mathcal{A}}^{\mathcal{B}} \circ \mathrm{ind}_{\mathcal{A}}^{\mathcal{B}}\,.$

*Proof.* See [15, Chapter 10]. Part (iii) follows from the fact that for any category $\mathcal{E}$, the objects $\varnothing$ and $*$, if they exist, are respectively the colimit and the limit of the empty diagram with values in $\mathcal{E}$. A left adjoint preserves colimits and a right adjoint preserves limits. The proof of (vii) is straightforward and uses the fact that $\mathcal{A}$ is full in $\mathcal{B}$ to see that the object $(a, \mathrm{id}_a)$ is final in the comma category $\mathrm{Incl}\searrow a$. Hence the colimit on $\mathrm{Incl}\searrow a$ is simply the evaluation at $a$. $\square$

For (not necessarily full) inclusions $\mathcal{A} \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{C}$ of categories, note that part (iv) of the lemma reads

$$\mathrm{res}_{\mathcal{A}}^{\mathcal{C}} = \mathrm{res}_{\mathcal{B}}^{\mathcal{C}} \circ \mathrm{res}_{\mathcal{A}}^{\mathcal{B}}\,,$$

a formula that will be used without further comment.

**Definition B.5.** We call a category $\mathcal{C}$ *discrete* if it is small and its only morphisms are the identities (in other words, if $\mathcal{C}$ is "essentially a set").

*Remark* B.6. Consider the special case where $\mathcal{A} = \{*\}$ is the discrete category with only one object. A functor $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ simply consists in the choice of an object $b := \Phi(*)$ in $\mathcal{B}$. Then, $\Phi^* = \mathrm{ev}_b$ is the evaluation at $b$, and, its left adjoint $\iota_b := \Phi_*$, which is a functor $\mathcal{S} = \mathcal{S}^{\mathcal{A}} \longrightarrow \mathcal{S}^{\mathcal{B}}$, boils down to

$$\iota_b(s)(c) = \coprod_{\mathrm{mor}_{\mathcal{B}}(b,c)} s\,,$$

for each $s \in \mathcal{S}$ and each $c \in \mathcal{B}$. This also shows that $\mathcal{A}$ has to be full in $\mathcal{B}$ in part (vii) of Lemma B.4.

## References


[1] P. Balmer and M. Matthey, Codescent theory II : Cofibrant approximations, *Preprint*, 2003.
[2] P. Balmer and M. Matthey, Model theoretic reformulation of the Baum-Connes and Farrell-Jones conjectures, *Adv. in Math.*, to appear.
[3] P. Balmer and M. Matthey, *In preparation.*
[4] A. K. Bousfield and E. M. Friedlander, Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets, In : Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), Vol. II, 80–130, *Lecture Notes in Math.* **658**, Springer, 1978.
[5] E. Dror Farjoun, Homotopy and homology of diagrams of spaces, In : Algebraic topology (Proc. Workshop, Seattle/Wash. 1985), 93–134, *Lecture Notes in Math.* **1286**, Springer, 1987.
[6] D. Dugger, Universal homotopy theories, *Adv. in Math.* **164** (2001), 144–176.
[7] W. G. Dwyer and D. M. Kan, Function complexes for diagrams of simplicial sets, *Nederl. Akad. Wetensch. Indag. Math.* **45** (1983), 139–147.
[8] W. G. Dwyer and D. M. Kan, Equivalences between homotopy theories of diagrams, In *Algebraic topology and algebraic $K$-theory*, pp. 180–205, Annals of Mathematics Studies 113, 1983.
[9] W. G. Dwyer and J. Spalinski, Homotopy theories and model categories, In *Handbook of algebraic topology*, 73–126, North-Holland, Amsterdam, 1995.
[10] P. Goerss and J. R. Jardine, Simplicial homotopy theory, Progress in Mathematics 174, Birkhäuser, 1999.
[11] A. Heller, Homotopy theories, *Mem. Amer. Math. Soc.* **71**, 1988.
[12] P. Hirschhorn, Model categories and their localizations, Math. Surveys of the AMS 99, 2003.
[13] M. Hovey, Model categories, Math. Surveys of the AMS 63, 1999.
[14] J. Jardine, Simplicial presheaves, *Journal of Pure and Applied Algebra* **47** (1987), 35–87.
[15] S. Mac Lane, Categories for the working mathematician, *Graduate Texts in Math.* 5, Springer.
[16] M. A. Mandell, J. P. May, S. Schwede and B. Shipley, Model categories of diagram spectra, *Proc. London Math. Soc.* **82** (2001), 441–512.
[17] S. Mitchell, Hypercohomology spectra and Thomason's descent theorem, In *Algebraic $K$-theory (Toronto, ON, 1996)*, pp. 221–277, Fields Inst. Commun., 16, AMS, Providence, RI, 1997.
[18] D. Quillen, Homotopical Algebra, *Lecture Notes in Math.* **43**, Springer, 1967.
[19] R. Thomason and T. Trobaugh, Higher algebraic $K$-theory of schemes and of derived categories, In *The Grothendieck Festschrift III*, pp. 247–435, Progr. Math., 88, Birkhäuser 1990.



Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland
*E-mail address*: `paul.balmer@math.ethz.ch`
*URL*: `http://www.math.ethz.ch/~balmer`

Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland
*E-mail address*: `michel.matthey@math.ethz.ch`
*URL*: `http://www.math.ethz.ch/~matthey`