(801, #39) If z = f(x,y) where $x = r\cos(\theta)$, $y = r\sin(\theta)$ (a) find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$

and (b) show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Solution: To start

$$\frac{\partial z}{\partial r} = f_x(\cos(\theta)) + f_y(\sin(\theta)) \Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 = (f_x)^2\cos^2(\theta) + 2f_x f_y\cos(\theta)\sin(\theta) + (f_y)^2\sin^2(\theta)$$

Next,

$$\frac{\partial z}{\partial \theta} = f_x(-r\sin(\theta)) + f_y(r\cos(\theta)) \Rightarrow \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\left((f_x)^2\sin^2(\theta) - 2f_x f_y\sin(\theta)\cos(\theta) + (f_y)^2\cos^2(\theta)\right)$$

So, adding,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial r}\right)^2 = (f_x)^2 + (f_y)^2 = \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)$$