**Math 222A** **W03 E.**

# The subdirect representation theorem

## 1. Direct products

Here is an attempt at a decomposition theorem using direct products:

Define an algebra $\mathcal{A}$ to be *directly indecomposable* if $|A| > 1$ and there are no $\mathcal{B}, \mathcal{C}$ with $\mathcal{A} \equiv \mathcal{B} \times \mathcal{C}$ except with $|B| = 1$ or $|C| = 1$.

Here is the statement you might hope for: "Every algebra is the direct product of directly indecomposable algebras (possibly infinitely many)." This is certainly true for finite algebras, but is false in general. In fact, let $\mathcal{A}$ be a vector space of countable dimension over the two-element field; observe that any directly indecomposable vector space has dimension 1 by a basis argument, but $\mathcal{A}$ has the wrong cardinality to be a direct product of either finitely many or infinitely many two-element vector spaces[1].

A modified concept, that of "subdirect products of subdirectly irreducible algebras", works much better.

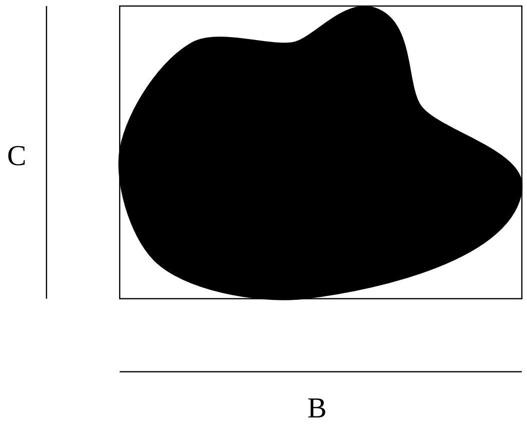


Figure 1: A subdirect product, heuristically

## 2. Subdirect products

2.1 *Definition.* A *subdirect product* of $\mathcal{B}$ and $\mathcal{C}$ is a subalgebra $\mathcal{A}_0$ of $\mathcal{B}\times\mathcal{C}$ such that the two coordinate projection maps carry $\mathcal{A}_0$ *onto* $\mathcal{B}$ and $\mathcal{C}$ respectively. In other words, every element of $B$ is used as a coordinate in $A_0$ and so is every element of $C$. A heuristic picture is given in Figure 1.

[1] Such a basis argument requires the Axiom of Choice, but there are similar examples that do not. See Problem E-8 and Problem E-9.

More generally, the same definition applies for a subalgebra of a direct product over any index set: $\mathcal{A} \subseteq \Pi_{\gamma\in\Gamma}\mathcal{B}_\gamma$, projection onto each factor.

You can see one virtue of subdirect products: $\mathcal{A}$ is obtained from $\mathcal{B}$ and $\mathcal{C}$, but also you can get from $\mathcal{A}$ back to $\mathcal{B}$ and $\mathcal{C}$ by taking homomorphic images.

Often we say that $\mathcal{A}$ "is" a subdirect product of some other algebras when we really mean that $\mathcal{A}$ is isomorphic to such a subdirect product.

## 3. Subdirect representations

Usually we want to use subdirect products "up to isomorphism".

3.1 *Definition.* A *subdirect representation* of an algebra $\mathcal{A}$ is an embedding $\mathcal{A} \hookrightarrow \prod_{\gamma\in\Gamma} \mathcal{B}_\gamma$ whose image is a subdirect product.

For example, a three-element chain (as a distributive lattice) has a subdirect representation as a subdirect product of two two-element chains, as in Figure 2.

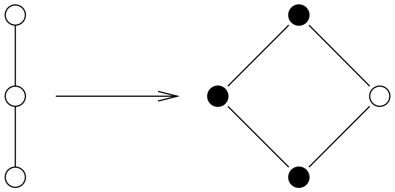


Figure 2: Subdirect representation of a 3-element chain

## 4. Subdirectly irreducible algebras

A subdirect product is said to be *trivial* if one of the coordinate projections is one-to-one, so that it is an isomorphism from $\mathcal{A}_0$ onto one of the factors.

Similarly, a subdirect representation of $\mathcal{A}$ is said to be *trivial* if the image is a trivial subdirect product of the factors. In that case, the factor is isomorphic to $\mathcal{A}$.

4.1 *Definition.* An algebra $\mathcal{A}$ is *subdirectly irreducible* (SI) if $|A| > 1$ and all subdirect representations of $\mathcal{A}$ are trivial.

4.2 *Theorem* (*Subdirect Representation Theorem*) Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.

For example, every distributive lattice is a subdirect product of two-element chains. (See Application 7.1 below.)

## 5. The internal point of view

5.1 *Observation.* If $\mathcal{A}$ has two congruence relations $\theta_1$ and $\theta_2$ with $\theta_1 \cap \theta_2 = 0$, then $\mathcal{A}$ has a subdirect representation $\mathcal{A} \hookrightarrow \mathcal{A}/\theta_1 \times \mathcal{A}/\theta_2$.

The reason is that the two natural homomorphisms of $\mathcal{A}$ onto $\mathcal{A}/\theta_i$ $(i = 1, 2)$ give a homomorphism of $\mathcal{A}$ into the direct product with kernel $\theta_1 \cap \theta_2 = 0$, so the homomorphism is an embedding. Composing with the projections gives back the natural homomorphisms, so this is a subdirect product.

More generally, if $\mathcal{A}$ has congruence relations $\theta_\gamma, \gamma \in \Gamma$ with $\cap_\gamma \theta_\gamma = 0$, then $\mathcal{A}/\cap_{\gamma\in\Gamma} \theta_\gamma \hookrightarrow \prod_{\gamma\in\Gamma} A/\theta_\gamma$.

5.2 *Observation.* Up to isomorphism, *any* subdirect representation of $\mathcal{A}$ is the same as an appropriate subdirect representation of the form given in Observation 5.1.

The reason: Given a subdirect representation $\phi : \mathcal{A} \hookrightarrow \prod_{\gamma\in\Gamma} \mathcal{B}_\gamma$, let $\mathcal{A}' = \phi(\mathcal{A})$, the image of $\phi$. Then for each $\gamma \in \Gamma$, the coordinate projection $\pi_\gamma$ takes $\mathcal{A}'$ onto $\mathcal{B}_\gamma$ with some kernel $\theta_\gamma$. The intersection of these kernels is the 0 congruence relation, since in any product two elements are equal when their projections on all factors are the same. Moreover, by the first isomorphism theorem, $\mathcal{B}_\gamma \cong A'/\theta_\gamma$. The mappings

$\mathcal{A} \hookrightarrow \prod_{\gamma\in\Gamma} \mathcal{B}_\gamma \overset{\pi_\gamma}{\twoheadrightarrow} \mathcal{B}_\gamma$

become

$\mathcal{A}' \hookrightarrow \prod_{\gamma\in\Gamma} \mathcal{A}'/\theta_\gamma \overset{\pi_\gamma}{\twoheadrightarrow} \mathcal{A}'/\theta_\gamma$, up to isomorphism.

5.3 *Proposition.* The following conditions are equivalent:

(1) $\mathcal{A}$ is subdirectly irreducible;

(2) $\cap_{\gamma\in\Gamma}\theta_\gamma = 0$ implies $\theta_\gamma = 0$ for some $\gamma \in \Gamma$;

(3) $0 \in \mathrm{Con}(\mathcal{A})$ is completely meet irreducible;

(4) $\mathrm{Con}(\mathcal{A})$ has a least element $> 0$ (the *monolith* of $\mathcal{A}$).

This gives an internal description of subdirect irreducibility.

## 6. The proof of the subdirect representation theorem

6.1 *Lemma.* Given $a \neq b$ in $\mathcal{A}$, there exists a congruence relation $\theta$ maximal with respect to the property $a \not\equiv b$ $(\theta)$.

*Proof.* Let $\mathcal{S} = \{\theta \in \mathrm{Con}(\mathcal{A}) : \langle a, b\rangle \notin \theta\}$. Then $\mathcal{S}$ is not empty, since $0 \in \mathcal{S}$. Suppose $\mathcal{C}$ is a chain of members of $\mathcal{S}$, where each relation is regarded as a subset of $\mathcal{A}\times\mathcal{A}$. Then $\bigcup_{\theta\in\mathcal{C}} \theta \in \mathcal{S}$, since all aspects of being in $\mathcal{S}$ (specifically, being an equivalence relation, being compatible with the operations of $\mathcal{A}$, and not containing $\langle a, b\rangle$) can be checked using finitely many elements at a time and so can be checked inside just one member of $\mathcal{C}$ at a time. Then by Zorn's Lemma, $\mathcal{S}$ has a maximal member. □

Let $\theta_{ab}$ be one such congruence relation maximal with respect to not identifying $a$ and $b$. Here $\theta_{ab}$ is in contrast to $\mathrm{con}(a, b)$, the smallest congruence relation that identifies $a$ and $b$. In fact, $\theta_{ab}$ can be described as a $\theta$ maximal with respect to the property $\theta \not\geq \mathrm{con}(a, b)$.

6.2 *Observation.* For $a \neq b$ in $\mathcal{A}$, in $\mathrm{Con}(\mathcal{A})$ there is a least element $> \theta_{ab}$, namely $\theta_{ab} \vee \mathrm{con}(a, b)$.

6.3 *Observation.* $\mathcal{A}/\theta_{ab}$ is subdirectly irreducible. Indeed, by Observation 1 and the Correspondence Theorem, $\mathrm{Con}(\mathcal{A}/\theta_{ab})$ has a least element $> 0$ and so is subdirectly irreducible.

6.4 *Observation.* $\bigcap_{a\neq b} \theta_{ab} = 0$ in $\mathrm{Con}(\mathcal{A})$, where $a, b$ range over $\mathcal{A}$.

*Proof* of the Representation Theorem. By Observation 6.4 we have $\mathcal{A} \hookrightarrow \prod_{a\neq b} \mathcal{A}/\theta_{ab}$, and by Observation 6.3 each $\mathcal{A}/\theta_{ab}$ is subdirectly irreducible.

## 7. An application

7.1 **Application.** It is easy to show that the only subdirectly irreducible distributive lattice is **2**. Consequences:

(i) Every distributive lattice is a subdirect product of copies of **2**.

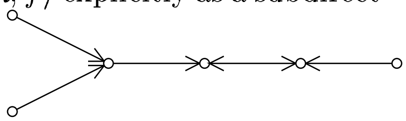
(ii) The variety of distributive lattices is the same as Var(**2**).

(iii) Every distributive lattice $L$ can be represented as a lattice of subsets of some set (perhaps not all subsets), with operations $\cup, \cap$.

## 8. Problems

**Problem E-1.** Prove Proposition 5.3.

**Problem E-2.** Represent the 1-unary algebra $\langle \mathcal{A}; f\rangle$ explicitly as a subdirect product of SI algebras, where $\mathcal{A}$ has diagram



**Problem E-3.** Let $L$ be a distributive lattice and let $a \in L$. Define $\phi_{\wedge \mathrm{a}} : L \to L$ by $\phi_{\wedge \mathrm{a}}(x) = x \wedge a$ and likewise $\phi_{\vee \mathrm{a}}$ by $\phi_{\vee \mathrm{a}}(x) = x \vee a$. As you know, these are lattice homomorphisms.

(a) Show that $\ker \phi_{\wedge a} \cap \ker \phi_{\vee a} = 0$. (Make a one-line proof based on the absorption law for lattices.)

(b) What embedding does (a) give?

(c) Show that the only SI distributive lattice is **2**. (Thus this fact is very elementary. The subdirection representation theorem then says that every distributive lattice is a subdirect product of copies of **2**, a deeper fact that depends on the Axiom of Choice.)

**Problem E-4.** Say how to represent the group $F_{Q_8}(2)$ as a subdirect product of subdirectly irreducible groups, using as few factors as possible, by referring to the diagram of its normal subgroups.

**Problem E-5.** (a) Which finite abelian groups are SI? (Use any facts you know about finite abelian groups and their subgroup diagrams. An SI abelian group has a smallest proper subgroup.)

(b) Find all SI abelian groups, finite and infinite. (They can be described as subgroups of the circle group—the multiplicative group of all complex numbers of absolute value 1.)

**Problem E-6.** (a) Show that an SI 1-unary algebra has no "fork", i.e., distinct elements $a, b, c$ with $c = f(a) = f(b)$.

(Method: Let $\langle a \rangle$ denote the subalgebra generated by $a$, and similarly for $b$. For a subalgebra $S$ of $\mathcal{A}$ let $\theta_S$ mean the congruence relation obtained by collapsing $S$ to a point. Show that $\theta_{\langle a \rangle} \cap \theta_{\langle b \rangle} \cap \mathrm{con}(a, b) = 0$ if $a, b$ give a fork. You may use the fact that $\mathrm{con}(a, b)$ is obtained by first identifying $f^i(a)$ with $f^i(b)$ for each $i$ and then seeing what equivalence relation that generates.)

(b) Using (a), try to find all finite SI 1-unary algebras whose diagram is connected.

(A useful observation: In an $n$-cycle, you get exactly the same congruences as for the abelian group $\mathbf{Z}_n$, so the congruence lattice of an $n$-cycle is isomorphic to Subgroup($\mathbf{Z}_n$).)

**Problem E-7.** Show that the finite SI 1-unary algebras are

(i) The algebra consisting of two fixed points,

(ii) the "cyclic" 1-unary algebras $\mathcal{C}_{p^k}$ of prime power order (with $k \geq 1$),

(iii) the algebras $\mathcal{D}_k, f$ where $\mathcal{D}_k = \{0, \ldots, k\}$ and $f(0) = 0$, $f(i) = i - 1$ for $i > 0$.

(iv) the two-component algebras where one component is a fixed point and the other is of kind (ii).

(In (ii), it is handy to make this observation, which you may justify very briefly: The congruence relations on an $n$-cycle regarded as a 1-unary algebra are exactly the same as those on the cycle regarded as the group or ring $\mathbf{Z}_n$. In all parts, you may justify briefly why these *are* SI; it is most important to explain why any finite SI must be of one of these forms.)

**Problem E-8.** Consider the ring $\mathcal{A} = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \ldots$, the "direct sum" of countably many copies of the ring $\mathbf{Z}_2$, or in other words, the subring of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \ldots$ consisting of the sequences that have only finitely many nonzero entries.

(a) Index the direct sum using $\omega = \{0, 1, 2 \ldots\}$. Show that the ideals of $\mathcal{A}$ correspond to subsets of $\omega$.

(b) Show that if $\mathcal{A} \equiv \mathcal{B} \times \mathcal{C}$, then at least one of $\mathcal{B}$ and $\mathcal{C}$ is isomorphic to $\mathcal{A}$. (Method: $\mathcal{A}$ would be the internal direct sum of corresponding ideals $I, J$, so that $I \cap J = (0)$ and $I + J = A$.)

(c) Show that $\mathcal{A}$ is not the direct product of directly indecomposable algebras. (Use a cardinality argument.)

**Problem E-9.** (a) Show that direct-product decompositions of a commutative ring with 1 into two factors correspond to idempotents (elements $e$ with $e^2 = e$).

(b) Let $R$ be the ring of all $\omega$-indexed sequences of zeros and ones that are "eventually constant", with sequences added and multiplied using the operations of $Z_2$ as a ring. Find all direct-product decompositions of $R$.

(c) In (b), does $R$ have a direct decomposition into directly indecomposable factors? (Why or why not?)

(d) What about the Boolean algebra $\text{Pow}_{fin}(X)$ for countably infinite $X$?

**Problem E-10.** Suppose that $\mathcal{A}$ is a finite algebra. An interesting question is whether $\text{Var}(\mathcal{A})$ contains finite SI algebras larger than $\mathcal{A}$, or even contains an infinite SI algebra. If $\mathcal{A}$ is a lattice, for example, there are no larger SI's; if $\mathcal{A}$ is a nonabelian $p$-group, the answer is that there are arbitrarily large finite SI's and also infinite ones. An easy case:

(a) Show that Shallon's algebra is SI, and in fact is simple. (Method: Think about $\text{con}(r, s)$ for different possible distinct elements $r, s$.)

More generally, Let $\mathcal{A}_n$ be the graph algebra based on a graph like Shallon's but with $n$ nodes, so that $\mathcal{A}_n$ has $n+1$ elements and Shallon's algebra is $\mathcal{A}_3$. Show that $\mathcal{A}_n$ is SI.

(b) Show that $\mathcal{A}_n \in \mathrm{Var}(\mathcal{A}_3)$. (Suggestion: Write $\mathcal{A}_3 = \{a_1, a_2, a_3, 0\}$. Inside $\mathcal{A}_3^n$, let $B$ be the subalgebra generated by elements whose entries are $a_1$'s (zero or more), then one $a_2$, and then the rest $a_3$'s. Let $\theta$ on $B$ be the equivalence relation obtained by identifying all elements of $B$ that have an entry of 0 and letting other blocks be singletons. Show that $\theta$ is a congruence relation on $B$. Then $B/\theta \cong \ldots$.)

(c) Can you find an infinite SI in $\mathrm{Var}(\mathcal{A}_3)$?