Complete lattices

1. The concept

Definition. A lattice L is said to be *complete* if (i) every subset S of L has a least upper bound (denoted $\sup S$) and (ii) every subset of L has a greatest lower bound (denoted $\inf S$).

Observation 1. A complete lattice has top and bottom elements, namely $0 = \sup \emptyset$ and $1 = \inf \emptyset$.

Observation 2. If (i) holds, then L already satisfies (ii).

2. Examples

(1) $[0, 1] \subseteq \mathbf{R}$

(2) $\mathrm{Pow}(X)$

(3) Any finite lattice.

(4) $\mathrm{Subsp}(V)$, for any vector space V.

(5) $\mathrm{Subgp}(G)$, for any group G.

(6) $\mathcal{I}(L)$, the lattice of ideals of a lattice L, together with the empty set if L has no zero element.

(7) The family of closed subsets of a topological space X.

(7′) The family of open subsets of a topological space X.

(8) Any *closure system* $\mathcal{C}$ on a set X.

This means a family $\mathcal{C}$ of subsets of X such that

(i) $\mathcal{C}$ is closed under arbitrary intersections;

(ii) $X \in \mathcal{C}$.

(9) The family of closed sets under a *closure operator* on a set X.

This means a map $A \to \overline{A}$ of $\mathrm{Pow}(X) \to \mathrm{Pow}(X)$ such that

(a) $A \subseteq \overline{A}$

(b) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

(c) $\overline{\overline{A}} = \overline{A}$

A is *closed* when $\overline{A} = A$.

(10) The family of closed subspaces of a Hilbert space.

(11) A "conditionally complete" lattice with $0, 1$ adjoined (if needed).

A lattice L is said to be *conditionally complete* if any subset bounded above has a least upper bound and any subset bounded below has a greatest lower bound.

(12) A well ordered set with top element.

A chain C is said to be *well ordered* if every nonempty subset in C has a least element.

(13) Any complete sublattice of a complete lattice.

(14) CC(P), the "completion by cuts" of a partially ordered set P. (See §3)

(15) Part(X), the lattice of partitions of a set X, or equivalently, the lattice of equivalence relations on X.

Partitions are usually compared as equivalence relations: $\theta \subseteq \psi$ when every θ-block is a ψ-block. Under this ordering, the largest partition is the one that consists of one block containing all elements, and the smallest partition is the one where all blocks are singletons, or in other words, the equality relation.

Sometimes in analysis the ordering of partitions is by refinement, which is the reverse: the finest partition (with singleton blocks) is the largest.

3. Polarities

We start with a *context*, by which is meant a triple (X, Y, ρ), where X and Y are sets and ρ is a binary relation between X and Y, i.e., $\rho \subseteq X \times Y$.

For any subset A of X, let $A^{\uparrow} = \{y \in Y : a\rho y \text{ for all } a \in A\}$, and for any subset B of Y, let $B^{\downarrow} = \{x \in X : x\rho b \text{ for all } b \in B\}$. Then it can be checked that

1. $A_1 \subseteq A_2 \Rightarrow A_1^{\uparrow} \supseteq A_2^{\uparrow}$ and $B_1 \subseteq B_2 \Rightarrow B_1^{\downarrow} \supseteq B_2^{\downarrow}$;
2. $A \subseteq A^{\uparrow\downarrow}$ and $B \subseteq B^{\downarrow\uparrow}$ (where $A^{\uparrow\downarrow}$ means $(A^{\uparrow})^{\downarrow}$, etc.);

3. $A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$ and $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$;

4. $A \mapsto \bar{A}$ and $B \mapsto \hat{B}$ are closure operations, where $\bar{A} = A^{\uparrow\downarrow}$ and $\hat{B} = B^{\downarrow\uparrow}$;

5. the lattice of closed subsets of X is dually isomorphic to the lattice of closed subsets of Y.

The maps $A \mapsto A^{\uparrow}$ and $B \mapsto B^{\downarrow}$ are said to constitute a *polarity*, and the correspondence of closed subsets is called a *Galois correspondence*, after the example of Galois theory mentioned below.

Examples.

(1) The completion by cuts CC(P): $(P, P, \leq)$

(2) Orthogonality: $x \rho y$ is $x \perp y$ in one sense or another

(a) $X = Y = \mathbf{R}^n$

(b) Like (a) but for Hilbert space.

(c) $X = Y = R$, a ring; $x \rho y$ means $xy = 0$.

(3) Commuting: $x \rho y$ is $yx = xy$

(a) $X = Y = G$, a group

(b) $X = Y = R$, a ring

(4) Vanishing: Y consists of functions on X to some group; $x \rho f$ means $f(x) = 0$.

(a) X is any subset of the reals $\mathbf{R}$, Y is the set of continuous functions $f : X \to \mathbf{R}$;

(b) $X = V$, a vector space over a field F; $Y = V^*$, the "dual space" consisting of all linear functionals on V, i.e., all linear maps $f : V \to \mathbf{F}$;

(c) $X = B$, a Banach space over $\mathbf{C}$; $Y = B^*$, the "dual space" consisting of all bounded linear functionals $f : B \to \mathbf{C}$ (the complex numbers);

(d) $X = \mathbf{C}^n$, $Y = \mathbf{C}[X_1, \ldots, X_n]$, the ring of polynomials in n variables $X_1, \ldots, X_n$;

(e) $X = G$, a finite group, Y is the group of characters of X, i.e., homomorphisms χ of G into the unit-circle group; here $x \rho \chi$ means $\chi(x) = 1$.

(5) Fixing: X is some set, Y is a set of functions from X to X, and $x \rho f$ means $f(x) = x$.

(a) $X = K$, the smallest subfield of $\mathbf{C}$ containing all roots of some polynomial; $Y = \mathrm{Aut}(K)$, the group of automorphisms of K;

(b) X is a set, and $Y = G$, a group acting on X; $x \rho g$ means $gx = x$.

(c) The specific case of (b) where the group G acts on itself by conjugation: $\lambda_g(x) = g^{-1}xg$. (Is this equivalent to another example mentioned?)

(6) Convexity:

(a) $X = \mathbf{R}^n$, Y is the space of affine functionals on $\mathbf{R}^n$, and $\mathbf{x} \rho f$ means $f(\mathbf{x}) > 0$.

(An affine functional is a map $f : \mathbf{R}^n \to \mathbf{R}$ of the form $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a} + \mathbf{b}$, for constant vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$. In other words, $\mathbf{x} \mapsto f(\mathbf{x}) - f(\mathbf{0})$ is a linear transformation.)

(b) $X = \mathbf{R}^{n+1}$, Y is the dual space of linear functionals on $\mathbf{R}^{n+1}$, and $\mathbf{x} \rho f$ means $f(\mathbf{x}) > 0$.

(b') $X = Y = \mathbf{R}^{n+1}$ and $\mathbf{x} \rho \mathbf{y}$ means $\mathbf{x} \cdot \mathbf{y} > 0$.

(7) Hull-kernel constructions of closure systems; ρ is $\in$:

(a) $X = \mathbf{R}^n$ and Y is the set of closed half-spaces (= 6(a));

(b) $X = L$, a distributive lattice, and $Y = \Pi_L$, the set of prime ideals of L;

(c) $X = R$, a commutative ring with 1, and Y is the set of prime ideals of R;

(d) $X = R$, a commutative ring with 1, and Y is the set of maximal ideals of R.

(8) Model theory: ρ is satisfaction.

(a) First-order model theory, say for groups: X is the class of all groups, Y is the set of all first-order sentences in the language of groups, and $G \rho S$ means that G satisfies the sentence S, i.e., S is true about G. (Here X is a class instead of a set, but polarities work the same as usual.)

(b) Equational model theory, say for lattices: X is the class of all lattices, Y is the set of all possible laws involving $\wedge$ and $\vee$, and ρ is satisfaction.

(9) Concept lattices (R. Wille and his co-workers)

X is a finite set of objects, Y is a finite set of attributes, and $x \rho y$ means that the object x has the attribute y. A *concept* is a pair (A, B) with A a closed subset of X, B a closed subset of Y, and $A \leftrightarrow B$ under the Galois correspondence. We write $(A_1, B_1) \leq (A_2, B_2)$ if $A_1 \subseteq A_2$.

See the last page of this handout for some examples.

4. Problems

Problem S-1. State and prove a simple correspondence between closure systems and closure operators on the same set.

Problem S-2. Prove this fact due to Tarski:

Theorem. Let L be a complete lattice and let $f : L \to L$ be isotone (i.e., order-preserving). Then f has a fixed point. In other words, there is an element $p \in L$ with $f(p) = p$. (Suggestion: Let $A = \{x \in L : f(x) \geq x\}$ and let $p = \sup A$.)

Problem S-3. The *completion by cuts* $\mathrm{CC}(P)$ of a partially ordered set P is the lattice of $^{-}$closed subsets of P under the polarity $(P, P, \leq)$. It can be shown that P is isomorphically embedded in $\mathrm{CC}(P)$ by $p \mapsto (p] = \{q \in P : q \leq p\}$, with all existing meets and joins preserved. Draw diagrams of the completion by cuts of the two partially ordered sets shown in Figure 1. Notice that the two middle elements in the second diagram are *not* comparable. (Suggestion: Start by identifying the closures of singletons. Their meets become set intersections. Joins can be found dually.)

Figure 1: Two partially ordered sets

Problem S-4. Find a simple description (up to isomorphism) of the completion by cuts of the chain of rational numbers.

Problem S-5. For these contexts (X, Y, ρ), describe the closed subsets in these examples, stating whether your description is both a necessary and sufficient condition or just necessary, and giving proof where asked:

(a) $X = \mathbf{R}$ (the reals), Y is the set of all continuous functions $f : \mathbf{R} \to \mathbf{R}$, and $x \rho f$ means $f(x) = 0$. Prove your description of the X-closed subsets.

(b) $X = Y = \mathbf{R}^3$ and $x \rho y$ means $x \cdot y \geq 0$.

Problem S-6. In the context (X, Y, ρ), the statement that a particular subset A of X is closed actually has the force of an existence statement and so can be highly nontrivial. Specifically, if A is closed, then for each $x \notin A$ there exists $y \in Y$ such that $a \rho y$ for all $a \in A$ but not $x \rho y$. Write such existence statements in these more specific contexts from §3, using your knowledge of what the closed subsets of X are:

(a) Example 2(a).

(b) Example 4(a).

(c) Example 7(a).