Boolean Lattices, Algebras, and Rings

1. Definitions

Definition. In a lattice L with 0 and 1, y is a *complement* of x if $x \wedge y = 0$, $x \vee y = 1$.

Proposition. In a *distributive* lattice, complements are unique.

Definition. A *Boolean* lattice is a distributive lattice with 0 and 1 in which every element has a complement and $0 \neq 1$.

The complement of x is denoted x'.

Definition. A *Boolean algebra* is a Boolean lattice in which complementation is regarded as an operation. It is best to regard 0 and 1 as constant operations. Thus the algebra has the form $\langle B; \vee, \wedge, 0, 1, ' \rangle$.

Thinking of complementation as an operation makes a difference when subalgebras or homomorphisms are considered. Thus a *Boolean subalgebra* is a sublattice that is also closed under complementation and contains 0 and 1. (In particular, a Boolean subalgebra cannot be empty, unlike a sublattice.)

Proposition. In a Boolean lattice B, the complementation map $x \mapsto x'$ is a "dual isomorphism", meaning an isomorphism of B with its dual (i.e., B upside-down). In other words, the complementation map is one-to-one and obeys de Morgan's laws $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

2. Examples

1. $\mathbf{2}^n$ $(n \geq 1)$, the most general finite example up to isomorphism. In particular, $\mathbf{2}$ is an example.

2. $\mathrm{Pow}(X)$ for any set X.

3. Any Boolean subalgebra of a Boolean algebra.

4. For any infinite set X, the lattice $\mathrm{Pow}_{fin}(X)$ of all finite and cofinite subsets of X.

5. Any interval $[a, b]$ of a Boolean algebra, with relativized operations.

6. Any direct product of Boolean lattices or Boolean algebras.

7. For any $n \geq 0$, the free Boolean algebra $\mathrm{FBA}(n)$, which is isomorphic to $\mathbf{2}^{2^n}$, with the exponent 2^n being just an integer.

   (In contrast, the free distributive lattice is $\mathbf{2}^{\mathbf{2}^n}$ (lattice exponent $\mathbf{2}^n$) with 0 and 1 deleted.)

8. For any Boolean algebra B and lattice ideal I, the lattice B/I of equivalence classes, where $x \equiv y$ means $x + y \in I$. (Recall that a nonempty subset I of a lattice L is an *ideal* if I is a downset closed under joins.)

9. For any infinite set X, the lattice $\mathrm{Pow}(X)/\mathbf{F}$ of all subsets of X modulo finite subsets. This means the lattice of all equivalence classes of subsets of X, where two subsets are considered equivalent if their symmetric difference [defined below] is finite.

10. The lattice of measurable subsets of the reals modulo sets of measure 0.

11. The lattice of equivalence classes of a first-order language, where equivalence means logical equivalence and the operations are "and", "or", and "not".

12. $\mathrm{Clopen}(X)$, where X is a topological space.

13. Any Boolean ring with 1, made into a Boolean algebra as below.

3. Boolean rings

Definition. A *Boolean ring* is a ring in which every element is idempotent.

Examples.

1. $\mathbf{Z}_2$ as a ring.

2. $\mathbf{Z}_2^n$ as a ring ($n \geq 1$).

3. Pow(X), made into a ring by letting multiplication be $\cap$, addition be the *symmetric difference* $A \Delta B = A \backslash B \cup B \backslash A$, and 0 be the empty set.

4. For an infinite set X, the subring of Pow(X) consisting of the finite subsets of X.

5. For any Boolean algebra $B = \langle B, \vee, \wedge, ', 0, 1\rangle$, the ring $\langle B, +, \cdot, 0\rangle$ obtained by defining xy to be $x \wedge y$ and $x + y$ to be $(x \wedge y') \vee (y \wedge x')$, the Boolean-algebra analogue of the symmetric difference.

All these examples except 4. are Boolean rings with 1.

Proposition 1. Any Boolean ring is of characteristic two (i.e., obeys $x + x = 0$ for all x).

Proposition 2. Any Boolean ring is commutative.

Proposition 3. Any Boolean ring with 1 can be made into a Boolean algebra by defining $x \wedge y = xy$, $x \vee y = x + y + xy$, and $x' = 1 - x$.

Proposition 4. For a Boolean algebra B, a subset is a lattice ideal if and only if it is a ring ideal with respect to the resulting ring structure.

4. Reduction of expressions to normal form.

A typical example:

$$
\begin{aligned}
&(x \vee (y' \vee z)')' = x' \wedge (y' \vee z)'' = x' \wedge (y' \vee z) && \text{(cmpl's inside)}\\
&= (x' \wedge y') \vee (x' \wedge z) && \text{(distribute)}\\
&= [(x' \wedge y') \wedge (z \vee z')] \vee [(x' \wedge z) \wedge (y \vee y')] && \text{(break into atoms)}\\
&= (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x' \wedge y' \wedge z)
\end{aligned}
$$

Any repeated meet-terms should be deleted. The final result is a join of distinct meets, with each meet involving all variables, possibly complemented. These meets correspond to the atoms in a free Boolean algebra, or equivalently, to the "puzzle pieces" in its Venn diagram.

Note. Determining whether an arbitrary Boolean expression reduces to 0 is the prototypical NP-complete problem. Many hard problems, such as the "traveling salesman problem", are equivalent to it in difficulty.

5. Complete Boolean lattices; atomic Boolean lattices

Definitions. In any lattice, the "sup" of a subset is its least upper bound, if it exists. Thus a sup is the same thing as a possibly infinite join. The sup of the empty subset is 0. Correspondingly, the "inf" of a subset is its greatest lower bound, if it exists, and the inf of the empty subset is 1. A lattice is *complete* if every subset has a sup and inf.

It is easy to show that if every subset in a lattice has a sup, then the lattice is already complete.

Definition. In a lattice with 0, an *atom* is an element that covers 0. A lattice is *atomic* if every element is the sup of some set of atoms.

6. A hard problem solved

A few decades ago, people were looking at alternate algebraic descriptions of Boolean algebras. H. Robbins looked at these axioms, which use join and complementation alone:

(1) $x \vee y = y \vee x$ (commutativity)

(2) $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity)

(3) $((x \vee y)' \vee (x \vee y')')' = x$ (a variant of $x = (x \wedge y') \vee (x \wedge y)$).

These conditions are obviously true in Boolean algebras. Robbins conjectured that they *define* Boolean algebras. This fact was finally proved in 1996 by a computer theorem-proving program, the first long-standing conjecture proved that way.

See `http://www.mcs.anl.gov/home/mccune/ar/robbins/index.html` .

7. Free Boolean algebras

FBA(3) $\cong \mathbf{2}^{2^3}$: 3 generators; 8 atoms; 256 elements.

FBA(4) $\cong \mathbf{2}^{2^4}$: 4 generators; 16 atoms; 65,536 elements.

FBA(5) $\cong \mathbf{2}^{2^5}$: 5 generators, 32 atoms, 4,294,967,296 elements.

FBA(6) $\cong \mathbf{2}^{2^6}$: 6 generators; 64 atoms; 18,446,744,073,709,551,616 elements.

FBA(7) $\cong \mathbf{2}^{2^7}$: 7 generators; 128 atoms; 340,282,366,920,938,463,463,374, 607,431,768,211,456 elements.

$\text{FBA}(8) \cong \mathbf{2}^{2^8}$: 8 generators; 256 atoms; 115,792,089,237,316,195,423,570, 985,008,687,907,853,269,984,665,640,564,039,457,584,007,913,129,639,936 elements.

8. Problems

Problem N-1. (a) On a sketch of $\mathbf{2}^4$, indicate two elements that generate $\mathbf{2}^4$ as a Boolean algebra. (b) Choose a third element at the same level as your two generators, and express it in Boolean normal form in terms of the generators.

Problem N-2. In the examples of §2, which are necessarily complete? Which are necessarily atomic?

Problem N-3. Prove Propositions 1 through 4 of §3 regarding Boolean rings.

Problem N-4. Decide which of the examples in §2 are atomic, which are atomless, and which (if any) are neither.

Problem N-5. Let X be a countably infinite set.

(a) Show that $\text{Pow}(X)$ contains a chain isomorphic to the chain $\mathbf{R}$ of reals.

(b) Show that $\text{Pow}(X)$ contains an uncountable antichain of elements whose pairwise meets are finite subsets of X.

Problem N-6. (a) Show that if p is an atom of a Boolean lattice B and $x \in B$, then either $p \leq x$ or $x \leq p'$, but not both.

(b) Show that in a complete Boolean lattice B, any atom p is *completely join-prime* (or sup-prime); in other words, $p \leq \sup S$ implies $p \leq s$ for some $s \in S$. (Suggestion: Somehow use p'.)

Problem N-7. Prove this representation theorem: Any atomic complete Boolean lattice is isomorphic to $\text{Pow}(X)$ for some set X. (A lattice is said to be *atomic* if every element is the sup of a set of atoms.)

(Notice that most of our representation theorems have used *some* subsets of a set; this representation theorem uses *all* subsets. Alternatively, this theorem can be regarded as a characterization of the lattices $\text{Pow}(X)$—they are the atomic complete Boolean lattices.)

Problem N-8. Let L be a Boolean lattice with prime ideal space $\Pi(L)$. Each lattice property of L should be reflected in some topological property of $\Pi(L)$. Here is one example: Show that L is atomic if and only if the isolated points of $\Pi(L)$ form a topologically dense subset.

(An *isolated point* in a topological space is a point that is open, as a singleton. A subset is *dense* if its closure is the whole space.)

Problem N-9. Let B be a Boolean lattice. Show that $\text{Open}(\Pi(B)) \cong \text{Ideals}(B)$, where Open() denotes the lattice of open sets of a topological space.

Problem N-10. As discussed in class, if $f : B \to C$ is a homomorphism of Boolean algebras, then there is a corresponding continuous map $\hat{f} : \Pi(C) \to \Pi(B)$, and if $h : X \to Y$ is a continuous map between Boolean spaces, then there is a corresponding homomorphism $\overline{h} : \text{Clopen}(Y) \to \text{Clopen}(X)$ of Boolean algebras.

Invent and state definitions for $\hat{f}$ and $\overline{h}$ (without writing the proof that they make sense) and then prove one of the two assertions in the following Proposition:

Proposition. $\overline{\hat{f}} = f$ and $\hat{\overline{h}} = h$, up to the identifications of Boolean algebras or Boolean spaces with their "double duals[1]."

Problem N-11. Show that any two countable, atomless Boolean algebras are isomorphic.

(A Boolean algebra is *atomless* if (surprise!) it has no atoms. An example of a countable, atomless Boolean algebra is $\text{FBA}(\aleph_0)$, the free Boolean algebra on countably many generators, which can be constructed by first making $\text{FBA}(1) \subseteq \text{FBA}(2) \subseteq \text{FBA}(3) \subseteq \dots$ using Venn diagrams and then taking their union—all the subsets you get at all stages. Another example is $\text{Clopen}(2 \times 2 \times 2 \times \dots)$, where 2 means $\{0, 1\}$ as a discrete topological space; this is the same as the lattice of all subsets of $2 \times 2 \times \dots$ that are describable by referring only to finitely many coordinates, for example, "the subset consisting of all sequences whose second and fourth entries are either 1 and 0 or 0 and 1".)

[1] "Dual" here is in the sense of categories, not order.