Introduction to lattices

1. Some definitions

In any partially ordered set P,

1. u is an *upper bound* of x and y if $x \leq u$ and $y \leq u$.
2. z is a *least upper bound* of x and y if
   (a) z is an upper bound of x and y, and
   (b) $z \leq u$ for all upper bounds u of x and y.

   We also can say that z is the *join* of x and y. We write $z = \mathrm{lub}(x, y)$ or $z = x \vee y$.
3. z is a *greatest lower bound* of x and y if ... We also say z is the *meet* of x and y and write $z = \mathrm{glb}(x, y)$ or $z = x \wedge y$.

2. Lattices

Definition. $\langle L, \leq \rangle$ is a *lattice* if it is a partially ordered set and *every* two elements have a least upper bound and greatest lower bound.

In other words, in a lattice $x \vee y$ and $x \wedge y$ are always defined. As with partially ordered sets, we usually just say "the lattice L".

3. Laws true in all lattices

Because $\vee$ and $\wedge$ are binary operations on a lattice, laws they satisfy can be considered.

(L1)	$x \vee x = x$ and $x \wedge x = x$	(idempotence)
(L2)	$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$	(commutativity)
(L3)	$x \vee (y \vee z) = (x \vee y) \vee z$	(associativity)
	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	
(L4)	$x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$	(absorption)

Note: By associativity, it is not ambiguous just to write $x \vee y \vee z$ and $x \wedge y \wedge z$.

Of course, many more laws follow from (L1)-(L4), but these four are critical in the following sense:

3.1 **Theorem.** If $\langle L, \vee, \wedge \rangle$ is an algebraic system satisfying the laws (L1)-(L4) and if $x \leq y$ is defined to mean $x \wedge y = x$, then $\langle L, \leq \rangle$ is a partially ordered set that is a lattice with least upper bound $\vee$ and greatest lower bound $\wedge$.

In other words, to define lattices using partial order is equivalent to defining them using (L1)-(L4).

4. Questions to ask, given a lattice L

1. Is L *distributive*?

This means that L obeys the distributive law

(D) $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

or the dual law, which is equivalent.

Examples. Chains, Pow(S), Div(n), not M_3, not N_5.

2. Is L *modular*?

This means that L obeys the modular law

(M) $\quad x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$, or equivalently,

(M′) $\quad (x \vee y) \wedge (x \vee z) = x \vee (y \wedge (z \vee x))$.

Examples. Normal(G), Subsp(V), any distributive lattice, M_3, not N_5.

3. Does L have a *top* element (usually denoted 1 or I) and/or a *bottom* element (usually denoted 0 or O)?

Examples. Any finite lattice has both, $\mathbf{R}$ has neither, ω has a bottom element but not a top element.

4. Is L *complemented*? (This requires top and bottom elements.)

This means that for each x there is at least one y with $x \wedge y = 0$, $x \vee y = 1$.

Examples. Pow(S), measurable subsets of $\mathbf{R}$.

Package: L is a *Boolean* lattice if L is distributive, has top and bottom elements, and is complemented.

5. Is L *complete*?

This means that *every* subset S of L has a least upper bound and greatest lower bound, not just the two-element subsets. We usually call these sup S and inf S, respectively.

Examples. Pow(S), $[a, b]$ in $\mathbf{R}$, any finite lattice.

5. Problems

Problem E-1. Which of the partially ordered sets listed as examples in the handout on that topic are lattices?

In cases where the lattice operations have more familiar names, give those names.

Problem E-2. A partially ordered set S is a *join-semilattice* if every two elements have a least upper bound.

(a) Show that a finite join-semilattice with bottom element is a lattice.

(b) Give an example of an infinite join-semilattice with bottom element that is not a lattice.

(c) Invent and prove a theorem for join-semilattices similar to Theorem 3.1.

Problem E-3. In a vector space V, the set of subspaces, ordered by inclusion, is a lattice. What are the meet and join operations, in more familiar terms?

Problem E-4. In a group G, the set of subgroups, ordered by inclusion, is a lattice. What is the meet, in more familiar terms? How can the join of two subgroups be described, in terms of elements?