

Convex sets and convex polyhedra

1. Convexity

Some subsets of \mathbf{R}^2 are *convex*, and others are not, as in Figure 1.

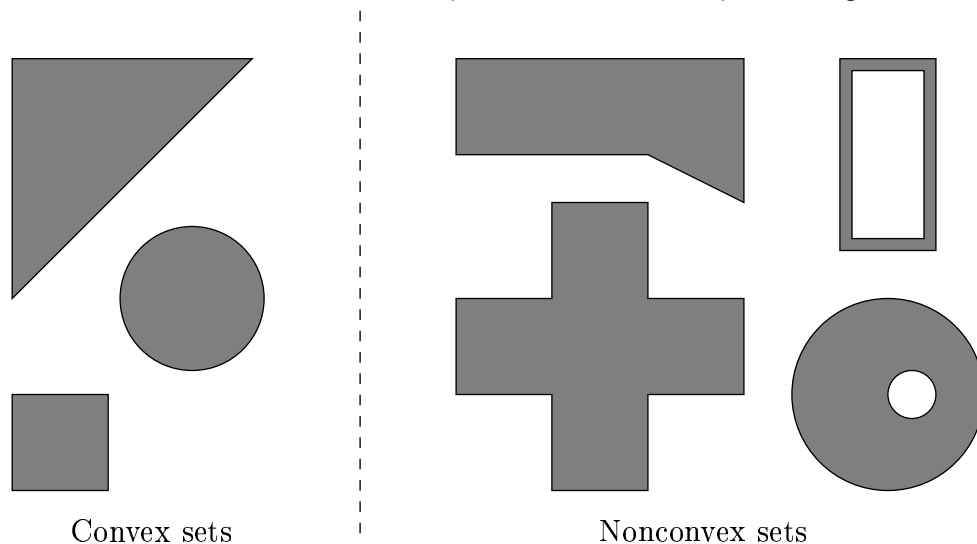


Figure 1: Convex and nonconvex sets

It is important to observe that convexity is a property of the whole set, and not just of its boundary. For example, the circle $x^2 + y^2 = 1$ in \mathbf{R}^2 is *not* convex, but the circular disk $x^2 + y^2 \leq 1$ *is* convex.

Convexity can be tested by looking at line segments with both end points in the set. In fact, this test makes a good official definition, in any number of dimensions:

Definition. A subset C of \mathbf{R}^n is **convex** if for every two points P, Q in C , the whole line segment \overline{PQ} is in C .

In \mathbf{R}^3 , for example, a solid cube is convex, a tetrahedron is convex, and a ball (solid sphere) is convex, but a torus is not and a (hollow) sphere is not.

2. The convex hull of a set

Let S_0 be any subset of \mathbf{R}^n , convex or not. For example, S_0 could be a subset of \mathbf{R}^2 with an indentation, or it could even consist of finitely many points. The *convex hull* of S_0 is the smallest convex set containing S_0 , which does exist; see Figure 2 for examples.

Proposition 2.1. For any subset S_0 of \mathbf{R}^n , there is a convex set C containing S_0 in \mathbf{R}^n that is smallest, in the sense that C is contained in all other convex sets that contain S_0 .

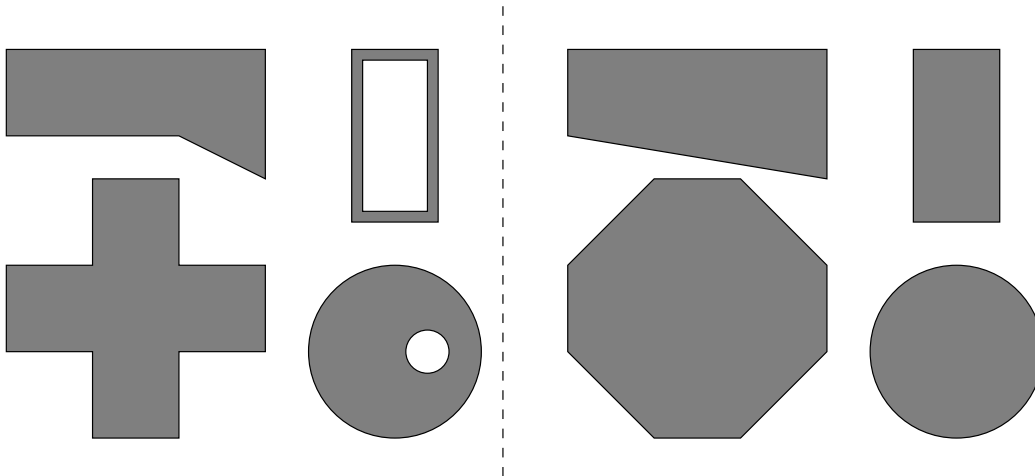


Figure 2: Constructing convex hulls

Proof. Let C be the intersection of all convex sets containing S_0 . Then C is clearly contained in all convex sets that contain S_0 , so the only question is whether C itself is convex. But it is, because if P and Q are two points in C , then P and Q are in each convex set containing S_0 , so all of the segment \overline{PQ} is, and so \overline{PQ} is all in the intersection of those convex sets, which is C .

Definition. For any subset S_0 of \mathbf{R}^n , the smallest convex set containing \mathbf{R}^n is the **convex hull** of S_0 .

3. Convex combinations

Definition. A *convex combination* of points (or equivalently, vectors) P_1, \dots, P_k is a linear combination $c_1P_1 + \dots + c_kP_k$ in which

- (i) the sum of the coefficients is 1 and
- (ii) the coefficients are nonnegative.

Equivalently, a convex combination is a weighted average in which the weights are nonnegative and add to 1. The term *convex* combination comes from the connection with convexity shown in Theorems 3.1 and 3.2 below.

Examples. (1) $\frac{1}{2}P_1 + \frac{1}{2}P_2$, the ordinary average of P_1 and P_2 , is a convex combination of P_1 and P_2 .

(2) More generally, the ordinary average of k points P_1, \dots, P_k is a convex combination of them.

(3) For two points P, Q , the points on the line segment PQ have the form $P + t(P - Q) = (1 - t)P + tQ$, where $0 \leq t \leq 1$, and so are convex

combinations of P and Q .

Theorem 3.1. If $S_0 = \{P_1, \dots, P_k\}$ (a finite subset of \mathbf{R}^n), then the convex hull of S_0 consists of every point that is a convex combination of P_1, \dots, P_k .

Theorem 3.2. If S_0 is any subset of \mathbf{R}^n , then the convex hull of S_0 consists of every point that is a convex combination of a finite subset of S_0 .

Note. A linear combination in which the coefficients have sum 1 is called a *barycentric* combination. Thus a convex combination is a barycentric combination in which the coefficients are also nonnegative.

4. Convex polyhedra

By a *polyhedron* let us mean a solid in \mathbf{R}^3 whose boundary consists of finitely many planar polygons. Examples are a cube and the “regular polyhedra” shown in Figures 5, 6, and 7.

This definition is too informal, however. We don’t want to allow a solid that is in two or more pieces; we don’t want to allow a solid that extends infinitely far (such as the part of \mathbf{R}^3 *outside* a tetrahedron); we don’t want to allow a solid that doesn’t include its boundary (such as the “open” cube described by $0 < x < 1, 0 < y < 1, 0 < z < 1$); and we don’t want to allow a solid with no thickness (for example, a single triangle).

A better definition is to say that a polyhedron is a solid that can be obtained by gluing together finitely many tetrahedra. A cube could be made in this way, for example.

Note that the plural of “polyhedron” is “polyhedra”¹.

The easiest kind of polyhedron to deal with is a *convex* polyhedron.

Proposition 4.1. A subset C of \mathbf{R}^3 is a convex polyhedron if and only if C is the convex hull of a finite set and is not contained in a plane.

5. Hidden-surface removal for convex polyhedra

Let us consider a convex polyhedron C and a projection of it on some viewplane.

If the projection is perspective, let V_0 be the viewpoint. In this case, we assume of course that V_0 is not a point of C .

¹This a Greek plural like “one criterion, two criteria.” Some people mistakenly say “a criteria” instead of “a criterion.”

If the projection is parallel, let \mathbf{v} be a vector giving the direction of the viewpoint. In the past, we have considered \mathbf{v} and $-\mathbf{v}$ to be equivalent; now, though, we should not, because the question of which faces are hidden depends on the direction from which we are looking. See Figure 3.

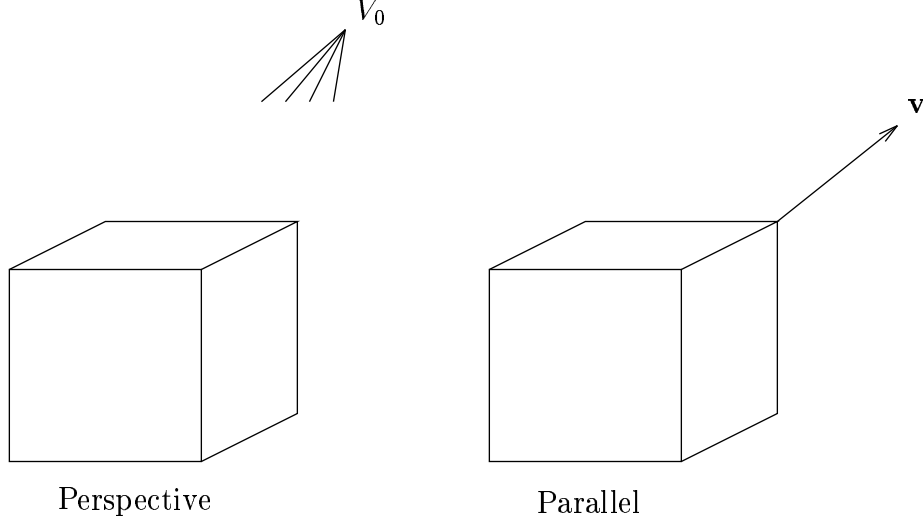


Figure 3: Viewing setups

Observation. Because C is convex, each face is (a) entirely visible, (b) entirely hidden, or (c) seen edgewise.

Here (c) means that the plane of the face contains V_0 or is parallel to \mathbf{v} .

For a given face $P_0P_1 \dots P_{k-1}$, here is a method for computing which among (a), (b), (c) holds. Let's assume that no three of these vertices are in a straight line and that no other vertices of the polyhedron are in the plane of this face.

Step 0. If the projection is perspective, let $\mathbf{v} = V_0 - P_0$. Thus whether the projection is perspective or parallel, we have a “line-of-sight” vector \mathbf{v} that, if drawn at P_0 , points away from the polyhedron.

Step 1. Find a normal \mathbf{N} to the face. One way is simply to let $\mathbf{N} = (P_0 - P_1) \times (P_2 - P_1)$.

Step 2. Choose another vertex Q of the polyhedron, not on the given face, and let $\mathbf{w} = Q - P_0$, a vector that definitely points inward. Check the sign of the dot product $\mathbf{w} \cdot \mathbf{N}$. If $\mathbf{w} \cdot \mathbf{N} > 0$, then \mathbf{N} is an inward normal; define a new normal $\mathbf{N}_{out} = -\mathbf{N}$. If $\mathbf{w} \cdot \mathbf{N} < 0$, then \mathbf{N} is an outward normal already; let $\mathbf{N}_{out} = \mathbf{N}$.

Step 3. Check the dot product $\mathbf{v} \cdot \mathbf{N}_{out}$.

(a) If $\mathbf{v} \cdot \mathbf{N}_{out} > 0$, then the face is *visible*.

- (b) If $\mathbf{v} \cdot \mathbf{N}_{out} < 0$, then the face is *hidden*.
(c) If $\mathbf{v} \cdot \mathbf{N}_{out} = 0$, then the face is *seen edgewise*.

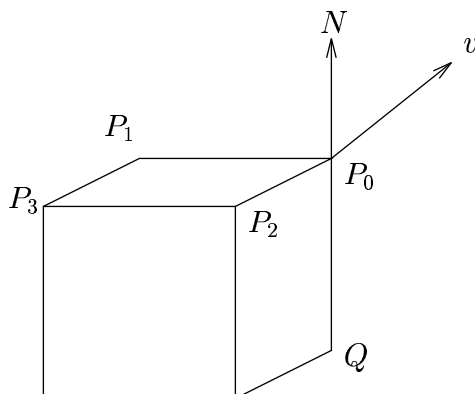


Figure 4: Testing a face for visibility

Observe that the viewplane itself is irrelevant for this method (Figure 4).

To make a picture of a convex polyhedron with hidden faces removed, simply plot the image of each visible face. If you have a pen plotter, for example, just compute the images of the vertices of visible faces and plot line segments between them corresponding to the edges of those faces.

Remark. If three consecutive vertices of a face are allowed to be collinear, then Step 1 may fail, because the cross product may be the zero vector. In this case, either try different choices of three vertices until three are found that are not collinear, or else let \bar{P} be the average of all the vertices of the face and let $\mathbf{N} = (P_0 - \bar{P}) \times (P_1 - \bar{P})$. (\bar{P} will automatically be in the interior of the face.) Similarly, if two different faces are allowed to lie in one plane, then in Step 2, Q may be in the plane of the given face and not tell you anything about \mathbf{N} ; in this case, instead let Q be the average of all the vertices of the polyhedron (a point definitely not on the given face).

6. Exercises

Problem S-1. Which of these subsets of \mathbf{R}^2 are convex? (a) a single point, (b) a line, (c) the upper half plane $y \geq 0$, (d) a filled-in semicircle, (e) the whole plane, (f) the empty set. (Note: Any statement you make about “all points of the empty set” is true, since there are no counterexamples. For example, all points of the empty set are green.)

Problem S-2. Which of these sets are convex? (a) in \mathbf{R} , an interval $[a, b]$; (b) in \mathbf{R}^3 , the sphere $x^2 + y^2 + z^2 = 1$; (c) in \mathbf{R}^3 , the solid ellipsoid $x^2 + 2y^2 + 3z^2 \leq 1$.

Problem S-3. In \mathbf{R}^2 , consider four points P, Q, R, S , one in each of the four quadrants. Must the origin be in their convex hull? If yes, explain why; if no, give an example.

Problem S-4. (a) In \mathbf{R}^2 , is a line segment convex? (b) Is the answer the same in \mathbf{R}^3 ?

Problem S-5. Find an explicit example to show that a linear combination of points $c_1P_1 + \cdots + c_kP_k$ might not be in the convex hull of $\{P_1, \dots, P_k\}$

(a) if $c_i \geq 0$ for all i but $c_1 + \cdots + c_k \neq 1$;

(b) if $c_1 + \cdots + c_k = 1$ but not all of c_1, \dots, c_k are ≥ 0 .

(You may need to choose coordinate axes.)

Problem S-6. As mentioned in Section 3, a *barycentric* combination is a linear combination in which the coefficients add to 1.

(a) Show that any translation preserves barycentric combinations. In other words, if $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ and $Q = c_1P_1 + \cdots + c_kP_k$ with $c_1 + \cdots + c_k = 1$, then $T(Q) = c_1T(P_1) + \cdots + c_kT(P_k)$.

(b) Show, conversely, that if a linear combination is preserved by all translations, or even by one nontrivial translation, then the linear combination is barycentric.

(c) Show that, in fact, if a linear combination is barycentric then it is preserved by all affine transformations. (Method: For \Rightarrow , use (a) and the observation that homogeneous linear combinations preserve *all* linear combinations, so for affine \dots ; for \Leftarrow , use (b) and the observation that translations are a particular kind of affine transformation.)

Problem S-7. Prove that the image of a convex set under an affine transformation is convex. In other words, if S is a convex subset of \mathbf{R}^n and $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is an affine transformation, then the image $T(S)$ is also convex. (You may assume the fact that the image under an affine transformation of the line segment joining two given points is the line segment joining the images of those points. In other words, $T(\overline{PQ}) = \overline{T(P)T(Q)}$. To prove the statement, just try to apply the definition of a convex set to $T(S)$. Remember, every point in $T(S)$ is of the form $T(P)$ for some point P in S .)

Problem S-8. Use Problem S-7 to show that the image of a convex set under an orthogonal or oblique projection is still convex.

Problem S-9. Invent an example to show that a point in \mathbf{R}^2 can be a convex combination of *four* given points in more than one way, i.e., with more than one list of coefficients.

Problem S-10. Invent an example of a non-convex polyhedron that, when viewed with an orthographic projection, partially hides one of its own edges. (Your answer will be a sketch of the image. Use dotted lines to represent hidden parts of edges.)

Problem S-11. Explain why the method of Section 5 is valid. (What does the sign of a dot product say about the angle between the two vectors?)

Problem S-12. For the tetrahedron $P = (2, 0, 0)$, $Q = (0, 3, 0)$, $R = (0, 0, 4)$, $S = (1, 1, 1)$, which faces are visible from above by an orthographic projection, i.e., from the direction $\mathbf{k} = (0, 0, 1)$? Use a computer method that would be valid for any tetrahedron.

Problem S-13. (a) In testing visibility for the case of a finite viewpoint V_0 , as in §4, we choose $\mathbf{v} = V_0 - P_0$ and then look at the value of $\mathbf{N}_{out} \cdot \mathbf{v}$. Show that this value is the same if we use any P_i in place of P_0 . (Method: To compare $\mathbf{N}_{out} \cdot (V_0 - P_0)$ with $\mathbf{N}_{out} \cdot (V_0 - P_i)$, expand and subtract the two and then try to isolate the factor $P_0 - P_i$.)

(b) There is a similar issue for $\mathbf{N} \cdot \mathbf{w}$, where $\mathbf{w} = Q - P_0$ with Q being a test point not on the face. State what the issue is and show something similar to part (a).

Problem S-14. Suppose P and Q are each a convex combination of P_1, \dots, P_n . Show that each point of the line segment joining P and Q is a convex combination of P_1, \dots, P_n . (Start from the definition of a convex combination, without quoting Theorem 3.1 or Theorem 3.2. Suggestion: To get an idea, try the case $n = 3$, first with some numerical coefficients and then with letters as coefficients.)

Problem S-15. Show that any point Q that is a convex combination of points P_1, \dots, P_n can be obtained by constructing line segments repeatedly. More specifically, as a first stage construct the segment $\overline{P_1 P_2}$; as a second stage construct a segment from a point on the first segment to P_3 ; as a third stage construct a segment from a point on the second segment to P_4 , and so on; the problem is to show that at stage $n - 1$ you can get the desired point Q , if the line segments at each stage are chosen carefully.

Equivalently, show that any convex combination of P_1, \dots, P_n is on a line segment from some convex combination of P_1, \dots, P_{n-1} to P_n .

(Method for the case $n = 3$: For the point $Q = c_1P_1 + c_2P_2 + c_3P_3$, with $c_i \geq 0$ and $c_1 + c_2 + c_3 = 1$, rewrite Q as

$$Q = (c_1 + c_2)\left(\frac{c_1}{c_1 + c_2}P_1 + \frac{c_2}{c_1 + c_2}P_2\right) + c_3P_3, \text{ or equivalently, as}$$

$$Q = (1 - t)(c'_1P_1 + c'_2P_2) + tP_3, \text{ for } t = c_3 \text{ and suitable } c'_i.$$

Why is this point on a line segment from a convex combination of P_1 and P_2 to P_3 ? It would be bad if the denominator were zero, but then $Q = P_3$, which is an easy case.)

Problem S-16. Use Problem S-14 and Problem S-15 to prove Theorem 3.1.

(Method: For points P_1, \dots, P_n , let H be the convex hull of the P_i and let C be the set of all points that are linear combinations of the P_i . You want to show that $H = C$. Do this by showing that $C \subseteq H$ (C is contained in H , i.e., every point of C is a point of H) and also that $H \subseteq C$. Explain how Problem S-14 shows that C is convex; since C contains all the P_i (why?), C contains the smallest convex set containing the P_i , which is what? Explain how Problem S-15 shows that any point of C is contained in any convex set containing the P_i , in particular, what?)

Problem S-17. Prove Theorem 3.2, using any preceding exercises.

(Method: Do like Problem S-16, with suitably modified definitions of H and C . The difference will be that in applying Problem S-14, the line segment will be between two convex combinations of different lists of points, but that's OK since the two lists can be merged into one longer list; in applying Problem S-15, even the list of P_i will depend on which Q is used.)

Problem S-18. A *regular octahedron* is a polyhedron with six vertices and eight faces, each an equilateral triangle. Consider the regular octahedron with vertices $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 2)$. Find a face that is visible from $(3, 3, 3)$ but not from $(1, 1, 1)$. (You may use intuition to decide which face, but for your final answer use a method that is suitable for computer. See Figure 5.)

Problem S-19. A *regular icosahedron* is a polyhedron with twenty sides, each an equilateral triangle. Five sides meet at each vertex. See Figure 6.

(a) How many vertices and how many edges does a regular icosahedron have? (Method: "Overcount" by counting each vertex [or edge] of each face separately, and then divide by the number of faces that each vertex [or edge] is on.)

A regular icosahedron can be made from a regular octahedron As follows: The "golden ratio" (a favorite number of the ancient Greeks) is the number

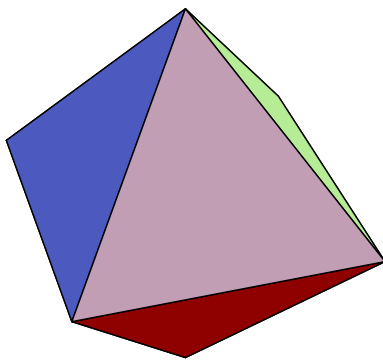


Figure 5: An octahedron

$\rho = (1 + \sqrt{5})/2 \approx 1.618034\dots$, which is the positive root of the equation $x^2 = x + 1$. Take the regular octahedron described in Problem S-5. Each vertex of the icosahedron will be a point on the edge of the octahedron, dividing the edge in the ratio $\rho : 1$. Of course, there are two choices of such a point, and you need to choose just one.

(b) Give an example of such a vertex. (Method: Choose an edge, think of it as a line segment, and notice that the two choices involve $t = 1/(\rho + 1)$ and $t = \rho/(\rho + 1)$. But choose just one vertex.)

(c) Find two more such points, so that the three are on the three edges of one face of the octahedron.

(d) What is the length of each edge of the icosahedron?

(e) If you can, list all vertices.

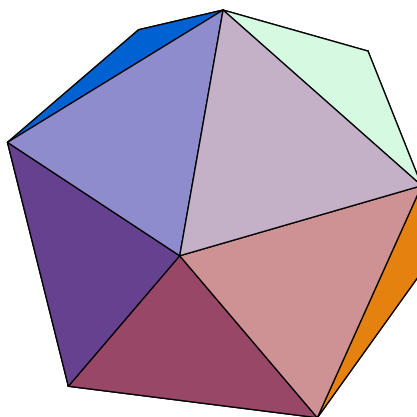


Figure 6: An icosahedron

Problem S-20. A *regular dodecahedron* is a polyhedron with twelve sides, each a pentagon. Three sides meet at each vertex. See Figure 7.

(a) How many vertices and how many edges does a regular dodecahedron have? (Do as in (a) of Problem S-19.)

The regular dodecahedron and regular icosahedron are *dual* to each other. This means that if you start with one of these shapes, and then take the center point of each face, those points make the vertices of the other shape.

(b) If you have done Problem S-19 or if you have been provided with the solution to it, then you can construct a regular dodecahedron as the dual of the icosahedron. Find two vertices of this regular dodecahedron.

(c) What is the length of a side of the regular dodecahedron made this way?

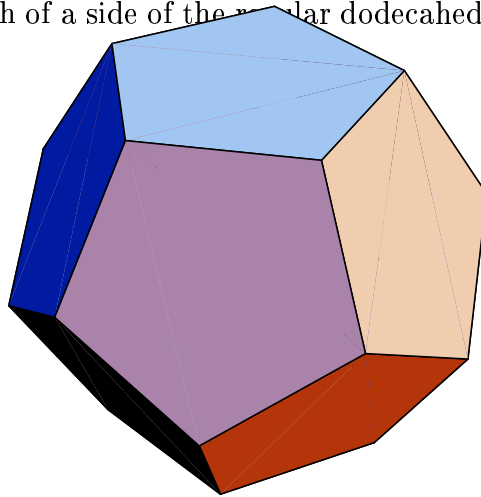


Figure 7: A dodecahedron

Problem S-21. In Problem S-6, (c) states that affine transformations preserve barycentric combinations. Show that, conversely, if a function $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ preserves barycentric combinations then T must be an affine transformation. (This is a strong statement, since T is not stated to have any special property other than preserving barycentric combinations. Method: Let $\mathbf{b} = T(\mathbf{0})$ and show that the related function $U : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a homogeneous linear transformation, where $U(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$. Keep in mind the possibility of making a linear combination such as $\mathbf{v} + \mathbf{w}$ barycentric by rewriting it as $\mathbf{v} + \mathbf{w} - \mathbf{0}$, where the coefficients add to 1.)

Problem S-22. (a) An interesting convex polyhedron is the rhombic dodecahedron, with twelve sides, each a rhombus, and fourteen vertices. (An important property of rhombic dodecahedra is that they can be stacked with no space between, just as cubes can.) One way to make one is to use vertices $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, $(0, 0, \pm 2)$, and $(\pm 1, \pm 1, \pm 1)$, as shown. Is the face with vertices $(2, 0, 0)$, $(1, 1, 1)$, $(0, 0, 2)$, $(1, -1, 1)$ visible from the viewpoint $(3, 5, -2)$? (Use a computer method. See Figure 8.)

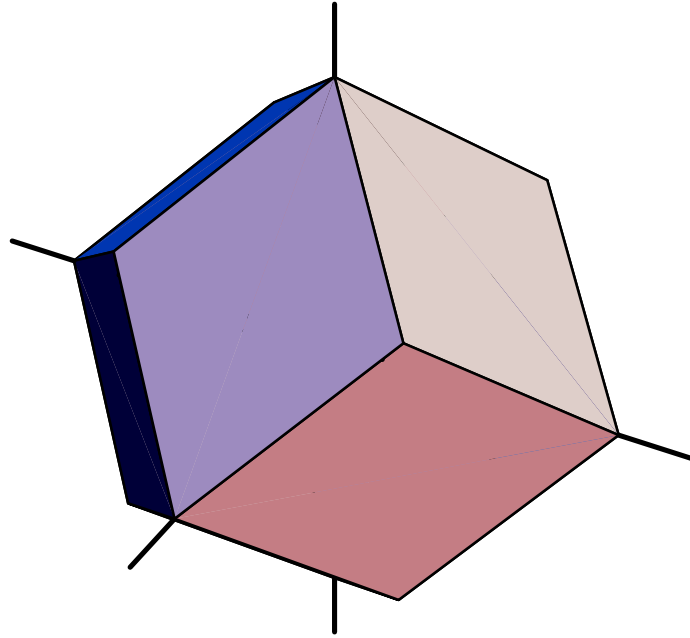


Figure 8: A rhombic dodecahedron

(b) Explain how a rhombic dodecahedron can be obtained by taking the union of a suitable octahedron (Figure 5) and a suitable cube, both centered at the origin, and then taking the convex hull. (You may use without proof the fact that the rhombic dodecahedron is convex, being the convex hull of its vertices.)

Problem S-23. For any kind of polyhedron, the *Euler characteristic* (Euler = “oiler”), is $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces.

(a) Find the Euler characteristics of a tetrahedron, cube, octahedron, dodecahedron (regular), icosahedron, and rhombic dodecahedron. (See various Figures.) They should all have the same value! (But indicate your computation in each case.)

(b) If you take a polyhedron and on some face draw a new edge between two vertices that are not already connected by an edge, what is the effect on the Euler characteristic?

(c) If you take a polyhedron and on some face make a new vertex (somewhere in the middle) and draw edges from it to the other vertices of the face, what is the effect on the Euler characteristic?

Actually, all polyhedra that are ball-like have the same Euler characteristic, and from (b) and (c) you can start to see why. In contrast, polyhedra that are like a torus (doughnut-shaped) have a different Euler characteristic.