Projections from three dimensions to two

1. The setup

There are three ingredients: A viewplane, an object, and a viewpoint, which can be at infinity. The viewplane is often taken to be the x, y-plane.

There are several versions:

#1: A right-handed coordinate system with the viewplane thought of as horizontal. This is the usual picture from calculus. The viewpoint is regarded as being in the positive z-direction.

#2: A right-handed coordinate system with the viewplane thought of as vertical, as it would be if the x, y-plane is a display screen. The viewpoint is regarded as being in the positive z-direction.

#3: A left-handed coordinate system with the viewplane thought of as vertical. The viewpoint is regarded as being in the negative z-direction.

Notes.

(a) The first two versions are exactly the same mathematically! The same computer program would give correct output for each.

(b) It doesn't really matter whether the object is on the same side of the viewplane as the viewpoint, or on the opposite side, or straddling it. A formula valid for one case works for the other; it's just that points of the object may have positive z-values in one case and negative in another.

(c) With a left-handed coordinate system, the algebraic computation of cross products stays the same, while the geometric meaning of the cross product obeys the left-hand rule instead of the right-hand rule.

(d) If the viewpoint is at infinity, we usually think of it as being on one side of the viewplane or the other, even though in $\mathbf{P}_3$ it doesn't really make a difference. Otherwise, we couldn't talk later about hidden lines and faces.

(e) Although we usually talk about "an object", of course the object could be a whole complicated scene with many parts.

(f) For the present, we ignore the issue of hidden lines and faces, which is quite complicated when you analyze it in detail. Solids, then, can be wire-frame figures, where only edges are given and faces are not filled in.

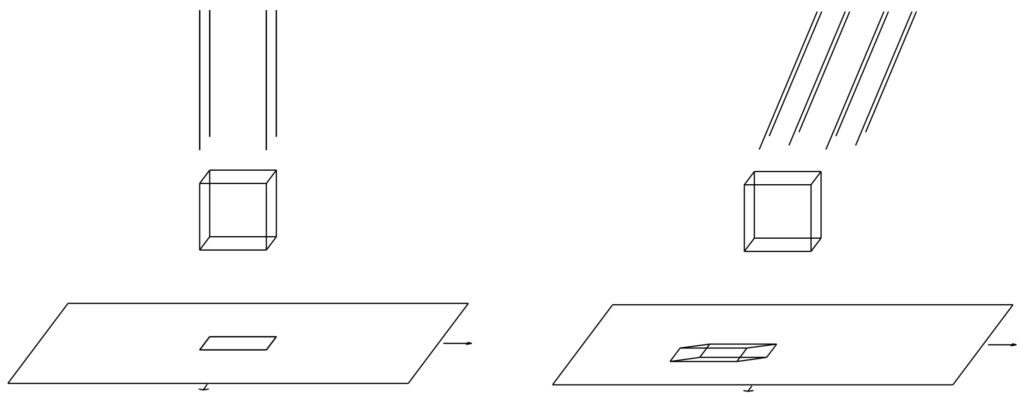

Figure 1: Orthographic and oblique projections

(g) The viewpoint and viewplane are often called the *center of projection* and the *projection plane*.

2. The main classification

Here the only things that matter are the viewplane and the viewpoint. Forget about the object.

(I) **Orthographic**—a projection with parallel rays all perpendicular to the viewplane. The viewpoint is at infinity on the z-axis.

(II) **Oblique**—a projection with parallel rays slanted with respect to the viewplane. The viewpoint is at infinity but *not* on the z-axis.

(III) **Perspective**—a projection from a finite viewpoint.

Here (I) and (II) can be grouped together under the heading "parallel projections."

For pictures produced by each, see Carlbom and Paciorek [1] and Figures 4 and 6 below. In Figures 1 and 2, the rays shown, if extended, actually go through the vertices of the cube and then to their images.

3. Characteristics of the main types

Here are some positive and negative features of each type:

(I) Orthographic

+ Orthographic projections are trivial to calculate.
+ Parallel lines in the object have parallel images.

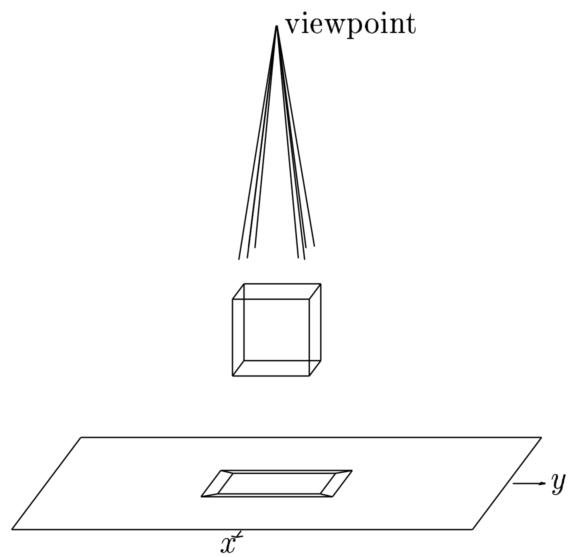

Figure 2: A perspective projection

+ If a face is parallel to the viewplane, then its image is undistorted–you can even measure on it.
− A box-shaped object has a dull and uninformative image if some faces are parallel to the viewplane.

(II) Oblique

+ Oblique projections are easy to calculate.
+ Parallel lines in the object have parallel images.
+ If a face on the object is parallel to the viewplane, then the image of that face is undistorted, as if seen straight-on. This is good for engineering purposes.
+ Oblique projections give a good psychological impression of the object, since you can see some sides of the object even if one face is seen undistorted.
− Oblique projections don't correspond to reality, since in reality you can't see a face of a box undistorted (as if seen straight-on) and also see some sides of the box.

(III) Perspective

+ Perspective projections correspond best to what the eye actually sees.
− Parallel lines in the object do not stay parallel in the image (except those that are parallel to the viewplane to start with).

– Every face of the object is distorted in size or shape during projection.

– Perspective projections are harder to calculate.

4. How to calculate projections

The object is given as points in $\mathbf{R}^3$. For an object described by line segments and polygons, all you need is the vertices; the images of the vertices can be connected up with line segments just as the vertices were. (Again, this ignores the question of hidden lines.)

The image is in the x, y-plane, which really consists of points $(x, y, 0)$ in $\mathbf{R}^3$, but we'd like images as points in $\mathbf{R}^2$ so that they are ready to plot on paper or on a screen.

(I) **Orthographic**

Easy: $(x, y, z) \mapsto (x, y)$.

(II) **Oblique**

Here the rays are parallel and slanted. To describe the direction of the rays, we use a vector $\mathbf{V}$. It turns out to be easiest if $\mathbf{V}$ is multiplied by a scalar to make the third coordinate 1; this will still describe the same rays Thus, we write $\mathbf{V} = (a, b, 1)$. (The viewpoint is at infinity with homogeneous coordinates $(a, b, 1, 0)_h$.)

A good method is to apply a *viewing transformation* to make the rays perpendicular to the viewplane while leaving the viewplane alone. After applying this transformation to the object, we are back in Case (I). See Figure 3.

Since the origin stays fixed and parallel lines stay parallel, the transformation is a homogeneous linear transformation. Thus, the transformation will be given by a 3×3 matrix A taking $(1, 0, 0) \mapsto (1, 0, 0)$, $(0, 1, 0) \mapsto (0, 1, 0)$ (since these are both in the viewplane, which stays fixed), and $(a, b, 1) \mapsto (0, 0, 1)$ (to make $\mathbf{V}$ perpendicular to the x, y-plane).

Notice that it would be easier if the transformation did the opposite, since then we'd have images of standard basis vectors. So, let B be a 3×3 matrix taking

$(1, 0, 0) \mapsto (1, 0, 0)$
$(0, 1, 0) \mapsto (0, 1, 0)$
$(0, 0, 1) \mapsto (a, b, 1)$

Therefore $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$. Since B is the inverse of A, we have $A =$

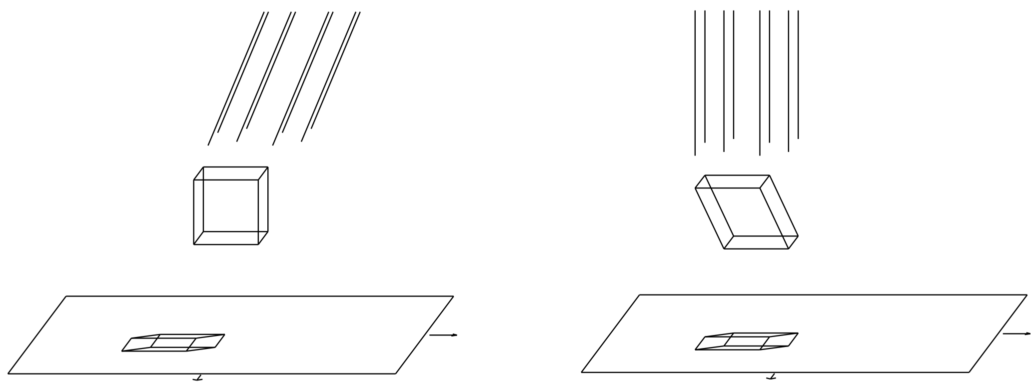

Figure 3: Viewing transformation of oblique projection

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix}.$$

(Here we can use the special method of taking inverses of matrices that are like I except in the off-diagonal entries of one row or one column.)

Now, to find the image of a point, just multiply by A and then project orthographically:

$$(x, y, z) \mapsto (x, y, z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix} = (x - az, y - bz, z) \mapsto (x - az, y - bz),$$

or briefly,

4.1 $$\boxed{(x, y, z) \mapsto (x - az, y - bz)}.$$

In practice, you can either apply this method with the matrix multiplication or else program the final formula $(x, y, z) \mapsto (x - az, y - bz)$. You can think of this formula as saying that the image of (x, y, z) consists of (x, y) but offset linearly somewhat, depending on z. See Figure 3.

Example: (a) Find the viewing transformation for an oblique projection from the direction $(2, 3, 5)$. (b) Under this projection, find the image of the $2\times2\times2$ cube with vertices $(\pm1, \pm1 \pm 1)$.

Solution: (a) Scaling $(2, 3, 5)$ we get $\mathbf{V} = (0.4, 0.6, 1)$, so the viewing matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.4 & -0.6 & 1 \end{bmatrix}.$$

(b) The top face, with points $(\pm1, \pm1, +1)$, goes to points $(\pm1-0.4, \pm1-0.6)$. The bottom face, with points $(\pm1, \pm1, -1)$ goes to points $(\pm1+0.4, \pm1+0.6)$. In $\mathbf{R}^2$, these are 2×2 squares offset from one another. Connect them up and you have the traditional oblique picture of a cube, as shown in Figure 4.

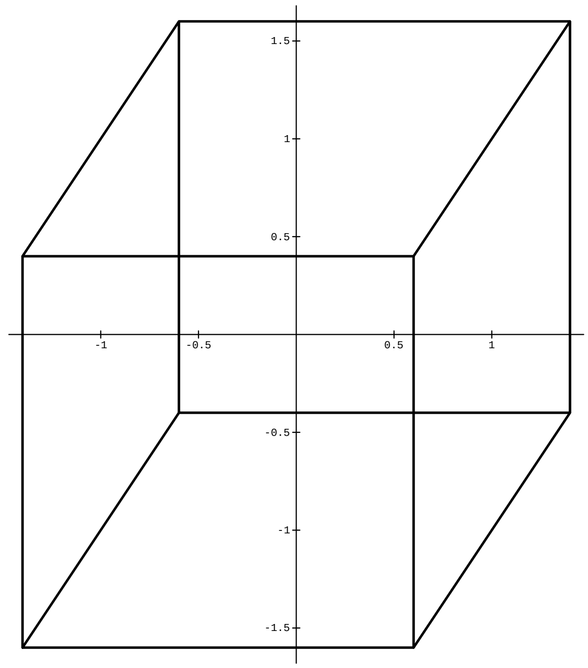

Figure 4: Oblique image of cube in plane

(Notice that in this example the cube straddles the viewplane, but this has no effect on the formulas. Rather, there is one formula, and it works no matter whether z is positive or negative.)

(III) **Perspective**

Let's consider just the case where the viewpoint is on the z-axis, so it has the form $(0, 0, H)$ for some height H. The object should lie entirely below this height.

Following the idea used for oblique projections, let's try to find a "viewing transformation" that changes the picture to an orthographic projection. This time the viewpoint is an ordinary point and needs to be changed to a point at infinity, so we need to use a projective transformation.

Discussion. Recall that the key points for handling projective transformations are
$X = pt(1, 0, 0, 0)_h$, at infinity on the x-axis,
$Y = pt(0, 1, 0, 0)_h$, at infinity on the y-axis,
$Z = pt(0, 0, 1, 0)_h$, at infinity on the z-axis,
$O = pt(0, 0, 0, 1)_h$, the origin.
(There was also E, but we won't need it.)

The given viewpoint is $(0, 0, H) = pt(0, 0, H, 1)_h$, but it turns out to be best to use the equivalent homogeneous coordinates $pt(0,0,1,1/H)_h$. The new viewpoint, at infinity on the z-axis, is to be Z. Since X, Y, O are in the $x.y$-plane, they should stay fixed. Then we want to find a 4×4 matrix A that gives a projective transformation taking

$$X \mapsto X, \quad Y \mapsto Y, \quad pt(0,0,1,1/H)_h \mapsto Z, \quad O \mapsto O.$$

Notice that it is easy to find a 4×4 matrix B to go in the opposite direction, since the coordinates of X, Y, Z, O are standard basis vectors in $\mathbf{R}^4$: Just let the rows of B be the images of these standard basis vectors, so that

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{H} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Then

$$A = B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{H} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

again since B is like I except for the off-diagonal entries in one row. A gives the viewing transformation.

Now, under the viewing transformation, $(x,y,z) = pt(x,y,z,1) \mapsto pt(x,y,z,1)A = pt(x,y,z,1-z/H)_h$. To find the equivalent ordinary point, divide through by the last homogeneous coordinate, to get $pt(x/(1-z/H), y/(1-z/H), z/(1-z/H), 1)_h = (x/(1-z/H), y/(1-z/H), z/(1-z/H))$. Finally, we need to project orthographically by discarding the third coordinate. This gives the final result that

4.2 $$\boxed{(x,y,z) \mapsto \left(\frac{x}{1-z/H}, \frac{y}{1-z/H}\right)}.$$

As before, you can either apply this method as a formula, or actually transform points with a viewing transformation and then project. Since we assumed that the object lies below the height H of the viewpoint, the denominator $(1-z/H)$ is always positive. See Figure 5.

Example. Suppose we want a perspective picture of the cube $(\pm1, \pm1, \pm1)$ on the x, y-plane from the viewpoint $(0,0,4)$. This is the case $H = 4$. Using the perspective transformation, for the vertices $(\pm1, \pm1, +1)$ on the top face we get

$$(\pm1, \pm1, 1) = pt\ (\pm1, \pm1, 1, 1)_h \mapsto pt\ (\pm1 \pm 1, 1, 1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= pt(\pm1, \pm1, 1, \frac{3}{4})_h = pt(\pm\frac{4}{3}, \pm\frac{4}{3}, \frac{4}{3}, 1)_h = (\pm\frac{4}{3}, \pm\frac{4}{3}, \frac{4}{3})$, which projects orthographically to $(\pm\frac{4}{3}, \pm\frac{4}{3})$.

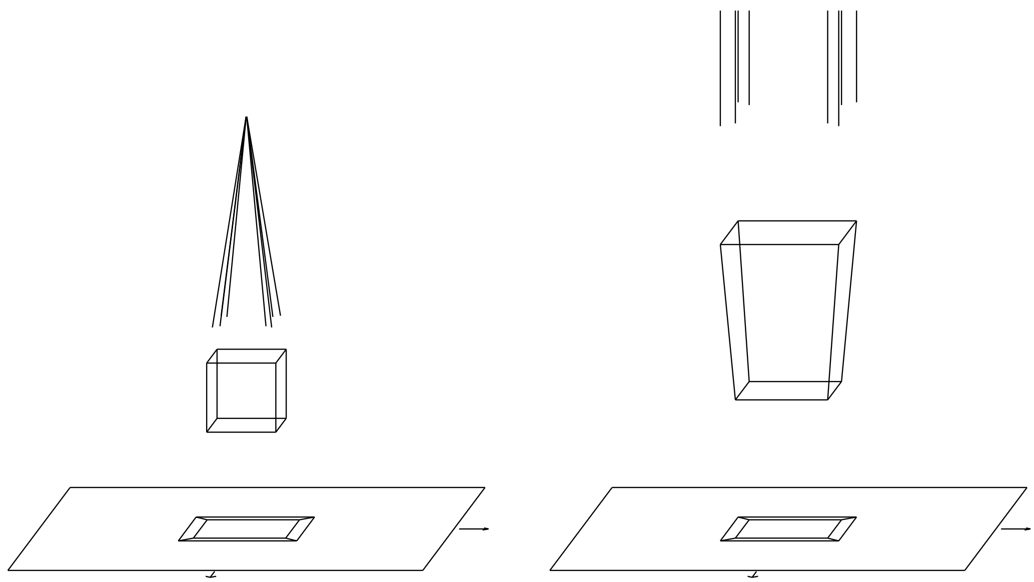

Figure 5: Viewing transformation of perspective projection

Similarly, for the vertices on the bottom face we get

$$(\pm 1, \pm 1, -1) = pt\ (\pm 1, \pm 1, -1, 1)_h \mapsto pt\ (\pm 1 \pm 1, -1, 1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= pt(\pm 1, \pm 1, -1, \frac{5}{4})_h = pt(\pm\frac{4}{5}, \pm\frac{4}{5}, -\frac{4}{5}, 1)_h = (\pm\frac{4}{5}, \pm\frac{4}{5}, -\frac{4}{5})$, which projects orthographically to $(\pm\frac{4}{5}, \pm\frac{4}{5})$. Joining up these eight points, you get the "square inside a square" perspective image of the cube. See Figure 6.

5. Projections on a slanted viewplane

To make an interesting picture of a scene, for example a street with houses, it's often desirable to use a slanted viewplane. Then even an orthographic projection becomes interesting.

To compute a projection on a viewplane numerically, the viewplane has to have some sort of coordinate system. For simplicity, we'll always assume that the viewplane goes through the origin in $\mathbf{R}^3$ and that its coordinate system has the same origin and scale as the ordinary coordinate system in all of $\mathbf{R}^3$. As usual, our goal will be to start with a point in $\mathbf{R}^3$, project it on the viewplane, and get a pair of numbers that can be used for plotting. We are not interested in the actual three-dimensional position of the image points on the viewplane, but rather just the two-dimensional location of the image points in the coordinate system of the viewplane.

A coordinate system for a plane is best described by giving its two stan-

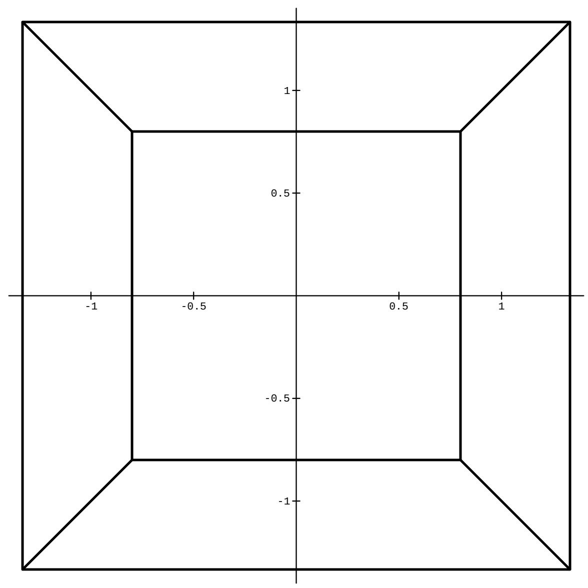

Figure 6: Perspective image of cube in plane

dard basis vectors as vectors in $\mathbf{R}^3$. In other words, we give the unit vector $\mathbf{v}$ in $\mathbf{R}^3$ that is $(1, 0)$ in the coordinate system of the plane, and the unit vector $\mathbf{w}$ in $\mathbf{R}^3$ that is $(0, 1)$ in the coordinate system of the plane. The two vectors $\mathbf{v}$ and $\mathbf{w}$ constitute the **coordinate frame** of the coordinate system for the plane. (The coordinate frame information would normally also include the location in $\mathbf{R}^3$ of the origin of the plane, but we don't need that information since we're assuming the origin of the viewplane is the origin in $\mathbf{R}^3$.)

In notation, let's continue to write vectors in lower case if they are known to be unit vectors, and in upper case otherwise.

Problem: Project an object **orthographically** on a slanted viewplane through the origin.

Case 1: We are given a coordinate frame for the viewplane—orthonormal vectors $\mathbf{v}$ and $\mathbf{w}$.

Method (i): Given a point $\mathbf{x} = (x, y, z)$ in the object, its projection in terms of the viewplane coordinates is $(\mathbf{x} \cdot \mathbf{v}, \mathbf{x} \cdot \mathbf{w})$. Reason: As you may recall, these are the projections of $\mathbf{x}$ in the directions of $\mathbf{v}$ and $\mathbf{w}$.

Method (ii): In outline: We find a 3×3 rotation matrix P taking the viewplane to the x, y-plane, with $\mathbf{v} \mapsto \mathbf{i}$ and $\mathbf{w} \mapsto \mathbf{j}$. We apply P to the object, and then project on the x, y-plane. The two numbers we get are the desired ones, since they are the same ones we would get by projecting on the original slanted viewplane and finding the image points in terms of the coordinate system of the viewplane. See Figure 7.

Details: Let $\mathbf{n} = \mathbf{v} \times \mathbf{w}$. Then $\mathbf{n}$ is a unit normal to the viewplane, and $\mathbf{v}, \mathbf{w}, \mathbf{n}$ are an orthonormal set of vectors in $\mathbf{R}^3$. We want a rotation matrix P taking $\mathbf{v} \mapsto \mathbf{i}$, $\mathbf{w} \mapsto \mathbf{j}$, $\mathbf{n} \mapsto \mathbf{k}$, i.e., $\mathbf{v}P = \mathbf{i}$, $\mathbf{w}P = \mathbf{j}$, $\mathbf{n}P = \mathbf{k}$. The other direction is easier, since then standard basis vectors are going to other vectors. In other words, we find Q taking $\mathbf{i} \mapsto \mathbf{v}$, $\mathbf{j} \mapsto \mathbf{w}$, $\mathbf{k} \mapsto \mathbf{n}$; Q is simply the 3×3 matrix $Q = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \\ \mathbf{n} \end{bmatrix}$. Then $P = Q^{-1} = Q^t = [\mathbf{v}^t | \mathbf{w}^t | \mathbf{n}^t]$. To find the projection of $\mathbf{x}$, we first find $\mathbf{x}P$ and then project on the x, y-plane by discarding the third coordinate.

But notice that $\mathbf{x}P = \mathbf{x}[\mathbf{v}^t | \mathbf{w}^t | \mathbf{n}^t] = (\mathbf{x} \cdot \mathbf{v}, \mathbf{x} \cdot \mathbf{w}, \mathbf{x} \cdot \mathbf{n})$, which projects to $(\mathbf{x} \cdot \mathbf{v}, \mathbf{x} \cdot \mathbf{w})$, the same answer as in the earlier method.

Why bother with the second method? One answer is that if we want to do hidden-line elimination later, the third-coordinate information can be used to tell which points on the object are closer to the viewplane than others. Another answer is that if we want to do, say, a perspective projection, then the first method no longer applies, but the idea of rotating does. See Figure 7.

Case 1′: We are given non-unit vectors $\mathbf{V}$ and $\mathbf{W}$ along the positive axes of the viewplane.

Method: Just normalize $\mathbf{V}$ and $\mathbf{W}$ by letting $\mathbf{v} = \mathbf{V}/|\mathbf{V}|$ and $\mathbf{w} = \mathbf{W}/|\mathbf{W}|$.

Remark: If we need the information that comes from $\mathbf{n}$, instead of finding $\mathbf{v}, \mathbf{w}$ and then $\mathbf{n}$ it is simpler to let $\mathbf{N} = \mathbf{V} \times \mathbf{W}$ and then normalize $\mathbf{N}$, if we are working by hand.

Case 2: We are given a normal vector $\mathbf{N}$ to the viewplane.

Method?? This isn't enough information, since we don't know how the coordinate system of the viewplane is situated. It could be turned various ways and we would still have the same normal. If we make a projection of a house and pick a viewplane coordinate system at random, then when we plot the results on a screen the house might be turned sideways or upside down.

Case 2′: We are given a normal vector $\mathbf{N}$ to the viewplane and an *up-vector*—a vector $\mathbf{U}$ in $\mathbf{R}^3$ whose image is supposed to be "up" in the picture.

Figure 7: Rotating an orthographic view

In other words, the projection of **U** on the viewplane is supposed to be in the positive **w** direction, since that's the "y-axis" of the viewplane.

Method: Observe that the **v** direction of the viewplane is perpendicular to **U**. It is also perpendicular to **N**. So, not worrying about normalizing, do this:

Step (1): find $\mathbf{V} = \mathbf{U} \times \mathbf{N}$. (We should ask why it's this order instead of the opposite, but this order is the one that makes **V** point in the correct direction, as you can see from a picture.)

Step (2): Let $\mathbf{W} = \mathbf{N} \times \mathbf{V}$.

Step 3: Normalize **V** and **W** (and **N**, if you are using the rotation method). We are now in Case 1.

Note: To remember which way around the cross products go, think this way: **V** is like **i**, **W** is like **j**, and **N** is like **k**, and **U** is sort of like **j** also since it is to project to the **W** direction. Then Step 1 is like $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and Step 2 is like $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Also observe that it might seem more natural to normalize vectors as soon as you compute them, but by hand it's easier to save normalization to the end, as in Step 3.

Example: Suppose that a house is described by giving its vertices in $\mathbf{R}^3$, with the x, y-plane as the ground. In order to make an interesting picture, the viewplane $x + y + z = 0$ is to be used. What coordinate frame and rotation matrix should be used?

Solution: Since the $x.y$-plane is the ground, take $\mathbf{U} = \mathbf{k}$. The viewplane normal is $\mathbf{N} = (1, 1, 1)$. Then

$$\mathbf{V} = \mathbf{U} \times \mathbf{N} = \mathbf{k} \times \mathbf{N} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = (-1, 1, 0)$$

$$\mathbf{W} = \mathbf{N} \times \mathbf{V} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = (-1, -1, 2)$$

$\mathbf{v} = \mathbf{V}/|\mathbf{V}| = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

$\mathbf{w} = \mathbf{W}/|\mathbf{W}| = (-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$

$\mathbf{n} = \mathbf{N}/|\mathbf{N}| = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. Then

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

As a check, we could verify that $\mathbf{v}P = \mathbf{i}$, $\mathbf{w}P = \mathbf{j}$, $\mathbf{n}P = \mathbf{k}$, and, since **U** is supposed to project perpendicularly on the positive **w**-axis of the viewplane, $\mathbf{U}P$ should project perpendicularly on the positive y-axis of the x, y-plane.

Figure 8: Rotating a perspective view

Problem. Make a **perspective** projection of an object on a slanted viewplane through the origin.

Method: Again, we are either given a coordinate frame for the viewplane or else we need to find one. This time, it is best to find the rotation P that takes the coordinate frame and normal to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbf{R}^3$. We must rotate the viewpoint as well as the object! Then we do a perspective projection on the x, y-plane. See Figure 8.

Note: With **oblique** projections it is rare to use a slanted viewplane, because it is desirable to have a face of the object be parallel to the viewplane and the object is usually not given slanted. (Remember that if a face is parallel to the viewplane then its projection is undistorted, and this possibility is one of the main virtues of oblique projections.)

6. Specifying viewing direction by angles

6.1 *The use of angles.*

For the moment, consider orthographic projections only, on a possibly slanted viewplane, with up-vector **k**. Then the only piece of information needed is a viewplane normal **N**, which is the same thing as the viewing direction.

It is often handy to be able to specify the viewplane normal by using angles, rather than by a vector such as $(1,1,1)$. For example, suppose you want to show successive views as you walk around the object; in that case, it would be good to give the angle at which the viewing direction is slanted with respect to the ground, and an angle to tell how far around the object you have gone.

To tie angles to vectors, several schemes are possible: latitude-longitude, alt-azimuth, and spherical coordinates. These schemes are practically the same except for the choice of reference directions from which angles are measured. Let's concentrate first on latitude-longitude.

6.2 *The latitude-longitude system*

On the earth, a "great circle" is a circle that divides the earth into equal halves; an example is the equator. Recall that the *latitude* of a point on the earth is its angular distance above the equator, and the *longitude* is its angular distance east of a great circle through the north pole and Greenwich, England ("gren′-itch"). South latitude and west longitude count as negative angles. For example, Los Angeles is approximately at latitude 34°, longitude −118°.

Regard the earth as a unit sphere centered at the origin in $\mathbf{R}^3$, with the north pole at $(0,0,1)$ and the point of latitude zero, longitude zero at $(1,0,0)$. See Figure 9, which however does not indicate the viewplane and whatever object we are trying to project.

Problem. Compute an orthographic projection of an object on the viewplane, with viewplane normal going through the point on the earth with latitude θ and longitude ϕ, and with up-vector **k**.

Method #1. Convert the angular description of the viewplane normal to a Cartesian description, and then use our previous slanted-viewplane method.

Details: The point on the earth with latitude θ and longitude ϕ can be found by starting at $(1,0,0)$ on the x-axis, then rotating by θ upwards towards the z-axis, and then rotating east by ϕ around the z-axis. In other words, $\mathbf{N} = (1,0,0)R_\theta^{x\to z}R_\phi^{x\to y}$, which comes out $\mathbf{N} = \mathbf{n} = (\cos\theta\ \cos\phi, \cos\theta\ \sin\phi, \sin\theta)$.

Exception: Our previous slanted-viewplane method doesn't work when the normal is along the same line as the up-vector. On the earth, this corresponds

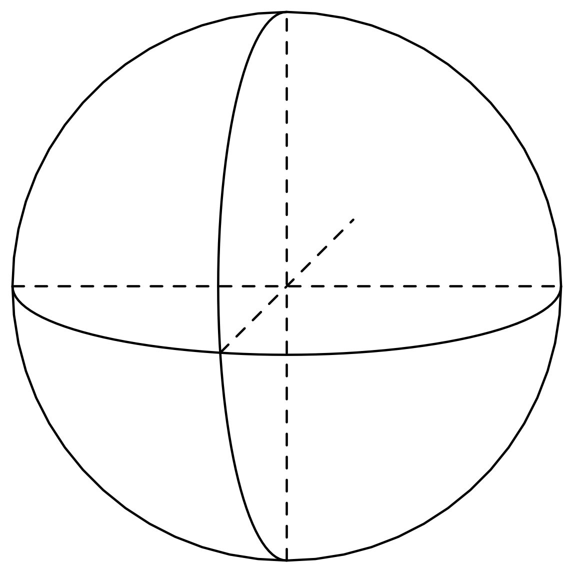

Figure 9: A coordinatization of the earth

to views from above the north pole or below the south pole. For those, just project directly on the x, y-plane.

Method #2. Find directly the rotation that rotates the viewplane to the x, y-plane and then project orthographically on the x, y-plane.

Details: Recall that under this rotation $\mathbf{v} \mapsto \mathbf{i}$, $\mathbf{w} \mapsto \mathbf{j}$, $\mathbf{n} \mapsto \mathbf{k}$. Although we haven't found $\mathbf{v}$ and $\mathbf{w}$, we at least know that the up-vector $\mathbf{k}$ projects orthographically to the positive $\mathbf{w}$-direction in the viewplane. In other words, the specified point on the earth (which is at the end of the vector $\mathbf{n}$) should rotate to where the north pole used to be (i.e., at $(0, 0, 1)$), and the north pole *after* the rotation should project orthographically on the positive y-axis. How can we accomplish this?

First attempt: Compose two standard rotations: Rotate the specified point west by an angle ϕ to the x, z-plane and then north by $\frac{\pi}{2} - \theta$. This does take the specified point to $(0, 0, 1)$, but unfortunately the north pole stays fixed under the first rotation and moves to a position over the x-axis under the second. A picture of the earth made this way would have the north pole at the side of the picture.

Second attempt: Plan ahead better. Rotate the specified point west by an angle $\frac{\pi}{2} + \phi$, to the y, z-plane, and then rotate north, as shown in Figure 10.

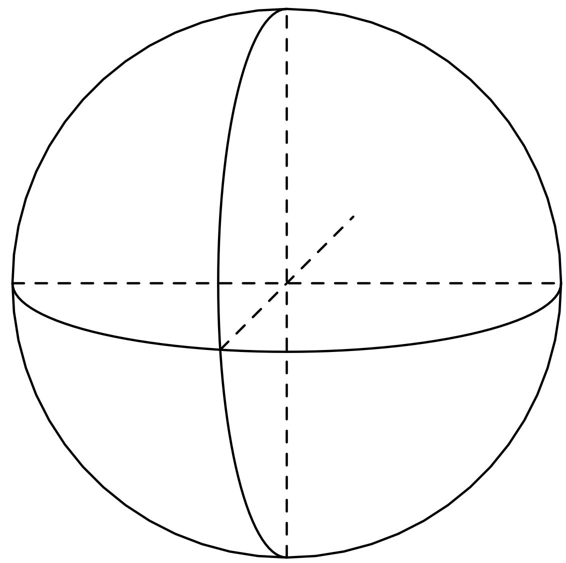

Figure 10: The direct method

This does work, and the rotation is thus

6.3 $$\boxed{R^{y\to x}_{\frac{\pi}{2}+\phi} R^{z\to y}_{\frac{\pi}{2}-\theta}}.$$

Bonus: This method works even for a view from above the north pole, where $\theta = \frac{\pi}{2}$, or from below the south pole, where $\theta = -\frac{\pi}{2}$. Although the view-plane is the x, y-plane in these cases, so that the up-vector does not give any information about which way to orient the picture, you have given that information yourself in specifying the viewing longitude ϕ!

6.4 *Other schemes*

- In the **alt-azimuth** scheme, imagine yourself standing *on* the earth facing north. Stretch out your arm horizontally. Now raise it to an angle of 30° from horizontal. This angle is the *altitude* of your arm. Now, holding your arm steady, rotate 40° clockwise. Your arm now has an *azimuth* of 40°. As you see, any direction can be described in terms of an altitude and azimuth.

  Now, to tie this to Cartesian coordinates, impose a coordinate system with the origin at your shoulder, the x-axis headed north, the y-axis headed east, and the z-axis pointing up.

- In **spherical coordinates**, the earth is a unit sphere again. The location of a point is described exactly the same as in the latitude-longitude system, except that in place of latitude we use *co-latitude*, the angle measured from the north pole down towards the equator, or past the equator. Thus the co-latitude has a value between 0 and π (for the south pole).

6.5 *Perspective projections*

As usual, consider only viewplanes that go through the origin. A perspective projection can be described by giving the direction of the viewpoint from the origin and the distance of the viewpoint from the origin. The direction can be described using angles, just as in the orthographic case. The viewplane is taken to be perpendicular to the line from the origin to the viewpoint.

7. Subclassifications of projections

For each main kind of projection, there is a different question to ask. We'll assume the object is box-like and that one face is designated the top and another as the front.

(I) **Orthographic** projections. First ask: "Is the object shown face-on (a "principal[1] view"), or corner-on ("axonometric")?"

(In other words, is a face parallel to the viewplane, or is some corner closer to the viewplane than any other corner is? If the object is lined up with the x, y, z-axes of $\mathbf{R}^3$, then in terms of a normal $\mathbf{N}$ to the viewplane we are asking whether only one coordinate of $\mathbf{N}$ is nonzero, or all three. We don't consider "edge-on", which would be the case where an edge is parallel to the viewplane but no face is, or equivalently, exactly two of the coordinates of $\mathbf{N}$ are nonzero.)

If the object is shown **face-on**, we ask: "Are we looking towards the front, a side, or the top?" As in architecture, these principal views are called respectively, a "front elevation", a "side elevation", and a "plan". If all three are given, we have "multiview orthographic".

If the object is shown **corner-on** (axonometric projection), then we ask: "At the image of a corner, how many different angles are there?"

If only **one** angle (i.e., three angles of 120° each), the projection is **isometric**, if **two** (so exactly two are the same), **dimetric**, and if **three** (all different), **trimetric**. If the object is lined up with the x, y, z-axes of $\mathbf{R}^3$, you can tell

[1] Notice that "principal" ends in "al", in contrast to the word "principle" that means a rule. To remember which is which, notice that "principle" and "rule" both end in "le".

the subtype by seeing how many different numbers are involved in the three coordinates of **N**—one, two, or three.

(II) **Oblique projections**. Here we'll assume that one face is shown undistorted, which means it's parallel to the viewplane—again, this could be called a "principal view". We ask: What is the "foreshortening ratio"? Consider a line segment perpendicular to the viewplane. The foreshortening ratio is the length of its image divided by its original length.

If the foreshortening ratio is $\frac{1}{2}$, there is a special name: *cabinet projection*. If the ratio is 1, the projection is a *cavalier projection*.

Otherwise, there is no special name. Furthermore, it doesn't make any difference how the object is turned compared to the viewing direction **V**; only the foreshortening ratio matters, in this terminology.

If the viewplane is the x, y-plane and the viewing direction is $\mathbf{V} = (a, b, c)$, or equivalently, $(a/c, b/c, 1)$, then the perpendicular line segment $(0, 0, 0)$ to $(0, 0, 1)$ projects to $(-a/c, -b/c)$, and so the foreshortening ratio is $|(-a/c, -b/c)|/1 = \sqrt{a^2 + b^2}/|c|$.
In terms of trigonometry, if **V** makes an angle θ with the x, y-plane, then the foreshortening ratio is "adjacent over opposite", which is cot θ.

(III) **Perspective projections**. Ask: Of the three families of parallel edges of the box, how many are *not* parallel to the viewplane (so that their images are nonparallel and meet at a finite point)?

If one, it is a **one-point** projection; if two, it is a **two-point** projection, and if all three, it is a **three-point** projection. Each has a distinctive look.

In terms of the viewplane normal **N**, these are equivalent to asking how many of the coordinates are nonzero—one, two, or three?

8. Recognizing kinds of projections

Again, we assume that the object is box-like or at least has three obvious families of parallel lines, mutually perpendicular. We'll also agree in advance that oblique projections will be used *only* for a "principal oblique view"—the situation where one face is parallel to the viewplane, so that the face is shown undistorted and yet you can see some sides of the object.

First, by looking at images of rectangles, decide whether the view is **parallel** or **perspective**.

- If **parallel**: next observe whether one face is seen undistorted (still a rectangle).

- If a face is **undistorted**, the projection is either a principal orthographic view or is oblique. To tell which, see if you can also see a side of the object or not.
  - If you can **see a side**, then the projection is **oblique**. Do you know the sides of the object in $\mathbf{R}^3$? Then you may be able to determine the foreshortening ratio and subtype—
    - cabinet?
    - cavalier?
    - nothing in particular?
  - If you see **no sides**, then the projection is **principal orthographic**. Is it a
    - front elevation?
    - side elevation?
    - plan?
- If all faces are **distorted**, the projection is axonometric. Count the number of different angles to tell the subtype. Is it
  - isometric?
  - dimetric?
  - trimetric?

- If **perspective**: Count the number of families of parallel lines in the object that are *not* parallel in the image. Is the view
  - one-point?
  - two-point? or
  - three-point?

Advice: We are examining a two-dimensional image as lines on paper, and we have to try hard to think of the picture as something two-dimensional rather than three. For example, lines on the paper may not be parallel, but they may represent parallel lines of the object in three dimensions, and so the visual circuitry in our brain tries to regard them as parallel. Useful techniques:

(a) To tell if two lines are parallel, extend them with a ruler.

(b) To tell if two adjacent oblique angles are equal, extend the middle ray across the vertex with a ruler and compare the acute angles you get.

(c) To tell if two line segments are equal, transfer one segment to the other by marking the edge of a piece of paper.

9. Problems

Problem O-1. Give the main classification of the projections used to make

(a) the picture of the earth in Figure 9.

(b) our usual drawing of x, y, z axes (with z vertical, y horizontal, and x slanted);

(c) the 3-d engineering graph paper shown at the end of this handout;

(d) the map of Math Sciences and Boelter Hall displayed by the elevator in the MS 5th floor lobby off the breezeway.

Problem O-2. In the article by Carlbom and Paciorek [1] , two picture captions have accidentally been switched. Which ones?

Problem O-3. Imagine a large cubical storage shed, 10 feet in each dimension and open in the front. It sits on horizontal ground. Sketch views of the shed as seen from these locations, and say what kind of projection each is. (No computations expected. Classify by major kind (orthographic, oblique, perspective) and by subclassification if relevant. Imagine a viewplane perpendicular to the line of sight described.)

(a) with your eye at ground level, 30 feet in front of the shed, looking directly toward it (in other words, along a line on the ground going to the middle of the front of the shed).

(b) Corner-on from 40 feet away, along a line passing directly through the top front left corner and the bottom back right corner. (Here "left" and "right" are interpreted with respect to you.)

(c) From very far away, along the same line as in (b). ("Very far" = "from infinity". Imagine using a telescope, so the shed still does not appear tiny.)

(d) From very far away, along a vertical line of sight.

Problem O-4. Find a right-handed orthonormal coordinate frame $\mathbf{v}, \mathbf{w}, \mathbf{n}$ for a viewplane with normal $\mathbf{N} = (1, 2, 3)$ and up-vector $\mathbf{U} = \mathbf{k} = (0, 0, 1)$.

Problem O-5. Find a right-handed orthonormal coordinate frame $\mathbf{v}, \mathbf{w}, \mathbf{n}$ for a viewplane with normal $\mathbf{N} = (a, b, c)$, using $\mathbf{k} = (0, 0, 1)$ as an up-vector. Assume $a^2 + b^2 \neq 0$. (Be sure to use the *unit* normal $\mathbf{n}$ where needed. Of course, your answer will be in terms of letters. It is easiest to normalize lengths last.)

Problem O-6. (a) For a perspective projection from an arbitrary viewpoint (a, b, c) onto the x, y-plane, by what 4×4 matrix should the homogeneous coordinates of the points of the object be transformed?

(b) Compute and sketch a picture of the standard tetrahedron, with vertices $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}$, and the origin, as projected on the x, y-plane from $(1, 2, 2)$.

Problem O-7. Consider the cube $(\pm 1, \pm 1, \pm 1)$, viewplane $x + 2y + 2z = 0$, and up-vector $\mathbf{k} = (0, 0, 1)$.

(a) Find the coordinate frame and rotation matrix that go with this setup.

(b) Find expressions for the images of the vertices in the x, y-plane, after rotation, using an orthographic projection. Your answer may be left as a matrix product (with the first matrix being a row vector). The matrix on the right can be 3×2, since the missing third column has the effect of discarding the third entry.

(c) Repeat (b) using a perspective projection with viewpoint $(4, 8, 8)$ (before rotation). You may leave the answers in homogeneous coordinates. (In other words, after doing the same rotation as for the orthographic problem, multiply by an additional viewing transformation to take care of the perspective, and finally project orthographically to two dimensions. The last two steps can be combined by taking the 4×4 matrix for the viewing transformation and then deleting the *third* column and use the resulting 4×3 matrix.)

(In all parts, you may combine cases by using $\pm$.)

Problem O-8. Consider the "standard unit cube" in $\mathbf{R}^3$, in other words, the cube whose vertices have coordinates with entries 0 and/or 1 only. Give the main and sub-classification of each of the following projections of the cube. You are not asked to compute images.

(a) viewplane $x + y + z = 0$, viewpoint $pt(5, 5, 5, 0)_h$;

(b) viewplane $x + y + 2z = 0$, viewpoint $pt(1, 1, 2, 0)_h$;

(c) viewplane $x + y + z = 0$, viewpoint $pt(5, 5, 5, 1)_h$;

(d) viewplane $x + 2z = 0$, viewpoint $pt(5, 0, 10, 1)_h$;

(e) viewplane $z = 0$, viewpoint $pt(1, 1, 2\sqrt{2}, 0)_h$;

(f) viewplane $z = 0$, viewpoint $pt(0, 0, 5, 1)_h$.

Problem O-9. The three-dimensional analogue of a window in user (world) coordinates is a box-shaped *viewing volume*. Parts of the object that are in front of the viewing volume or in back of it would not be shown. Suppose your viewing volume is $-4 \leq x \leq 4$, $-3 \leq y \leq 3$, $0 \leq z \leq 1$, and you are making a perspective projection on the x, y-plane with viewpoint $(0, 0, 5)$.

After you apply the projective transformation in $\mathbf{P}_3$ to move the viewpoint to infinity on the z-axis, where has each of the corners of the viewing volume moved to?

Problem O-10. Consider a perspective projection on the x, y-plane from the viewpoint $(0, 0, H)$, and the projective transformation on $\mathbf{P}_3$ that moves the viewpoint to infinity on the z-axis (the *viewing transformation*).

(a) Under the projection, what ultimately happens to the viewpoint itself? (Apply the projective transformation and then the orthographic projection to the viewpoint and interpret what your answer means.)

(b) In (a), what happens to other points (a, b, H) where a, b are not both zero?

(c) Under just the viewing transformation, as the viewpoint is being taken to Z at infinity, the point Z is being taken to some other point. What point?

Problem O-11. In our usual setup for a perspective projection, the viewplane is the $z = 0$ plane and the viewpoint is at $(0, 0, H)$. As in formula 4.2, the result of the projecting is $(x, y, z) \to (x/(1 - \frac{z}{H}), y/(1 - \frac{z}{H}))$.

What formula is obtained instead if the viewplane is the $z = H$ plane and the viewpoint is the origin?

Suggestion: Instead of trying to do this from the beginning, make the transformation $(x, y, z) \to (x, y, H - z)$ that trades the $z = 0$ and $z = H$ planes and then use formula 4.2 (so you put $H - z$ for z in that formula). Simplify algebraically.

Problem O-12. Suppose that a car is represented in three dimensions in a coordinate system for which the positive z-axis is horizontal and points forward on the car, the positive x-axis is horizontal and points to the left, and the positive y-axis is up. For the origin, take some point that is roughly in the center of the car, and for the units use meters.

(a) Is this coordinate system right-handed or left-handed?

(b) Suppose you want to make an a view of the car from the outside, looking toward the right front corner of the car (right as seen by the driver). Specifically, suppose that you use an orthographic projection along rays slanted at 30° to the ground and at 45° sideways from forward. Give an appropriate normal vector for the viewplane and an appropriate up-vector. (You may leave your answer as a matrix product.)

(c) Suppose you want to make successive views as if you are walking around the car in a circle five meters in radius and centered on the ground under the origin you chose. Use a perspective projection with a viewpoint at your

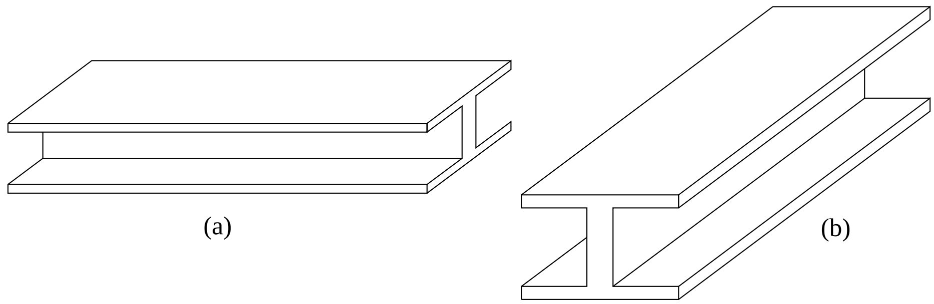

Figure 11: Two oblique projections of an I-beam

eyes, one meter higher than the origin. For a viewplane normal, use the line from your eyes to the origin. Give the viewplane normal, the distance of the viewpoint from the viewplane, and the up-vector to use. (You may leave your answer in the form of a matrix product. The normal will be in terms of some angle or time parameter.)

Problem O-13. A puzzle: Figure 11, similar to one in Carlbom and Paciorek [1], shows two oblique projections of the same I-beam, but with different faces being undistorted and with possibly different foreshortening ratios. Find these foreshortening ratios and (if relevant) give the name(s) of the projection(s). (Measure the pictures with a ruler in millimeters and use what logic you can. A good idea is to find a distance that is shown undistorted in both pictures and use that as a "unit", in terms of which to measure all distances on the picture and on the object. One catch is that the pictures are not to the same scale.)

Problem O-14. *Reconstruction of a cube from its image: the orthographic case.* Consider a cube with one vertex at the origin in $\mathbf{R}^3$ but otherwise in no special position or orientation and of no special size. Suppose that this cube is projected orthographically on the x, y-plane, and consider the problem of reconstructing the cube from just knowing its image. Specifically, let the vertices adjacent to the vertex at the origin be (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Looking at the image, you can see the pairs (x_i, y_i) for each i, but the values of the z_i are not obvious. This problem shows how the z_i can be found.

It is handy to define the three vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$, even though these have no obvious pictorial interpretation in terms of the original cube. (If the cube is a unit cube, they are triples consisting of the cosines of the angles that the axes of the cube make with the x-, y-, and z-axes.)

(a) Show that if the cube is a *unit* cube then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other and of length 1, and that $\mathbf{z}$ is plus or minus $\mathbf{x} \times \mathbf{y}$.

(Method: Write the original vertices as rows of a 3×3 matrix. What is special about this matrix? $\mathbf{x}$ and $\mathbf{y}$ are two of its columns.)

(b) Explain why if the cube is not necessarily a unit cube, $\mathbf{x}$ and $\mathbf{y}$ are still orthogonal to each other and have equal lengths S, where S is the length of the sides of the cube. (Method: $\mathbf{x}/S$, $\mathbf{y}/S$, $\mathbf{z}/S$ are as in (a).)

(c) Invent a way of finding z_1, z_2, z_3 (except for a factor of ± 1) from a knowledge of the three image points (x_i, y_i). (The factor of ± 1 is needed because if you have one solution then its reflection in the x, y-plane is another solution. The method is the same idea as in (b).)

Problem O-15. *Reconstruction of a cube from its image: the general parallel case.* Suppose a picture of a cube with sides of length S is using with a parallel projection on the x, y-plane from the viewpoint $pt(a, b, 1, 0)_h$. If a and b are 0, the projection is orthographic; otherwise, it is oblique. In either case, allow the cube to be tilted with respect to the viewplane. (In the oblique case this is contrary to our usual guarantee that we won't use oblique projections unless one face is parallel to the viewplane.)

As in Problem O-14, let one vertex of the cube be the origin and let the adjacent vertices be (x_i, y_i, z_i) for $i = 1, 2, 3$. This time, since the projection rays might be slanted, the images in the viewplane are not just the first two coordinates of the points, so let the images be called (u_i, w_i) for $i = 1, 2, 3$. As in Problem O-14, define vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

(a) It is a fact that $\mathbf{u} = \mathbf{x} - a\mathbf{z}$ and $\mathbf{w} = \mathbf{y} - b\mathbf{z}$. These sound similar to something you know for oblique projections, but notice that these are vector equations, not scalar equations. Why are they true? (Method: Think coordinatewise.)

(b) For short, let $U = \mathbf{u} \cdot \mathbf{u}$, $W = \mathbf{w} \cdot \mathbf{w}$, $D = \mathbf{u} \cdot \mathbf{w}$. Explain why, if the projection is orthographic, then $U = W$ and $D = 0$. (You may quote what you need from (b) of Problem O-14, which makes this very easy.)

(c) Show that, conversely, *if* $U = W$ and $D = 0$, then the projection *is* orthographic. (Method: You may quote what you need from Problem O-14. Start by finding U, W, D in terms of $\mathbf{x}, \mathbf{y}, \mathbf{z}$, carefully using rules for dot products that are like rules of high-school algebra. Then use what you know about $\mathbf{x}, \mathbf{y}, \mathbf{z}$.)

Note. By (b) and (c), you can tell just from the image of a cube whether a parallel projection is orthographic or oblique, if you're willing to measure the coordinates of the images of the vertices. (In fact, since it's also easy to see if a projection is perspective, all three of the possibilities orthographic, oblique, and perspective can be determined from the image alone.)

Problem O-16. This problem carries Problem O-15 further for the case of a unit cube, $S = 1$. Show how to compute the points (x_i, y_i, z_i) and the oblique projection direction $(a, b, 1)$ just from the image points (u_i, w_i). In other words, from an oblique image of a tilted unit cube show how to reconstruct the whole projection setup precisely.

(It is not required to find a formula for everything in terms of the image points; rather, you can solve for unknowns in stages, with a recipe for a solution at each stage in terms of quantities already computed at preceding stages. Outline: In the notation of Problem O-15, (c) of that problem gives $M = 1 + a^2$, $N = 1 + b^2$, $D = ab$, so you can solve for a and b up to a factor of ± 1. Next concentrate on finding $\mathbf{z}$, the list of z coordinates, as follows. Using the equations of (a) of Problem O-15 and facts about $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, you get $\mathbf{z} \cdot \mathbf{u} = -a$, $\mathbf{z} \cdot \mathbf{w} = -b$. Notice that these are equations of planes in $\mathbf{R}^3$; since $\mathbf{z}$ is of length 1, it lies where the line of intersection of these planes cuts the unit sphere $z_1^2 + z_2^2 + z_3^2 = 1$. [There could be two possibilities. To find them, you'd need to represent the line parametrically, then substitute coordinates in the equation for the sphere and simplify to get a quadratic equation in t; solve.] Then, knowing $\mathbf{z}$, how can you get $\mathbf{x}$, $\mathbf{y}$?)

Problem O-17. This problem generalizes Problem O-16 by allowing a cube of any side S. Show how to compute the points (x_i, y_i, z_i) and the oblique projection direction $(a, b, 1)$ just from the image points (u_i, w_i). In other words, from an oblique image of a tilted cube show how to reconstruct the whole projection setup precisely, even without knowing the length of the side of the cube.

(Method: With notation as in Problem O-15 and Problem O-16 you get $M = S^2(1 + a^2)$, $N = S^2(1 + b^2)$, $D = S^2 ab$. Here M, N, and D are constants known from the coordinates of the images of the vertices, and a, b, S are unknown. For convenience let $A = Sa$ and $B = Sb$, so that $M = S^2 + A^2$, $N = S^2 + B^2$, $D = AB$. Then $M - N = \ldots$ and $D = AB$ are two equations in just the unknowns A, B. Solve the second for B and substitute in the first to get an equation in A alone, which clears to a quadratic equation in A^2. Solve for A^2 using the quadratic formula. Then use $M = \ldots$ to solve for S^2. You should get $S^2 = \frac{1}{2}(M + N - ((M - N)^2 + 4D^2)^{\frac{1}{2}})$. Now, knowing S, try to reduce everything to Problem O-16.)

Problem O-18. You wish to view the earth from above Los Angeles, which is approximately at latitude 34°, longitude −118°. What rotation should you use? (You may leave your answer as a product of standard rotations, without explicit entries. Don't bother to convert to radians.)

Problem O-19. If you use the latitude-longitude method to make a picture of the earth from above a point in the southern hemisphere, does the north pole come out at the top of your picture (i.e., after being rotated does it project to the positive y-axis), or at the bottom (negative y-axis)?

Problem O-20. (a) Find a rotation formula, similar to the one in formula 6.3, for a viewing direction with altitude α and azimuth β. (b) Find a rotation formula for a viewing direction with co-latitude λ and longitude ϕ.

10. *References

[1] I. Carlbom and J. Paciorek, *Planar geometric projections and viewing transformations*, ACM Computing Surveys 10 (1978), 465-502.

[2] F. S. Hill, Jr., *Computer Graphics*, Macmillan, New York, 1990.

with writing, shown in perspective.