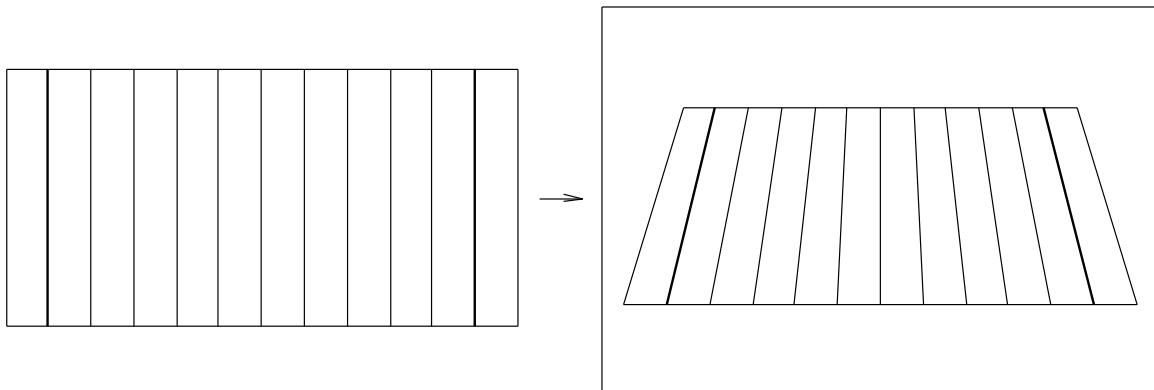


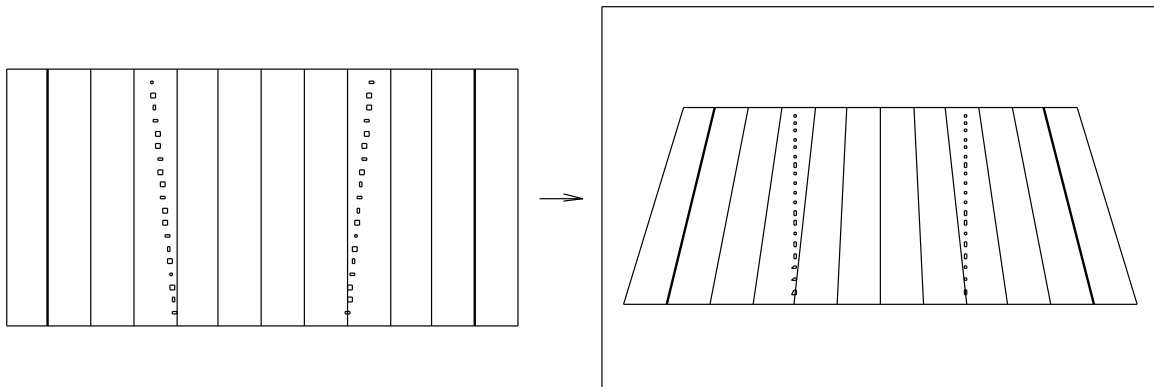
## Homogeneous coordinates and the real projective plane

### 1. Overview

The transformations involved in making perspective images do not preserve parallelism of lines. For example, a football field is flat and its yard lines are parallel, but on a television screen it might look like the picture on the right:



On the other hand, members of a band on the field might be standing in two non-parallel lines, but the lines could become parallel on the screen:



The kind of transformations that can handle distortions of this kind, while keeping straight lines straight, are called *projective transformations*. To describe them, we first need a new system of coordinatizing the plane.

### 2. Homogeneous coordinates for points in the plane

The ordinary system for naming points in the real plane is called *Cartesian coordinates* (after Descartes, who invented analytic geometry). In this

system, each point of the plane has exactly one name, consisting of a pair of numbers. (Often a point is even thought of as *being* a pair of numbers, as when we write  $\mathbf{R}^2$  for the real plane.)

You know one other system for naming points in the real plane: polar coordinates. In polar coordinates, each point has many names of the form  $(r, \theta)$ ; for example, the ordinary point  $(-1, 0)$  has the polar name  $(1, \pi)$  and also  $(1, 3\pi)$ ; the origin is  $(0, \theta)$  for every  $\theta$ .

In computer graphics it is often helpful to use *homogeneous coordinates*. With homogeneous coordinates, each point of the plane has many names, each one being a *triple* of numbers. You have already seen one example of homogeneous coordinates, although we didn't call it that: In using extended vectors for affine transformation, the ordinary name  $(3, 2)$  became  $(3, 2, 1)$ .

Let's put a subscript  $_h$  after a triple when it means the homogeneous coordinates of a point in the plane. That way it won't be confused with a triple meaning a point in  $\mathbf{R}^3$ . The subscript doesn't do anything; it's just a reminder.

What are all the names in homogeneous coordinates of the point whose ordinary name is  $(3, 2)$ ? They are simply the non-zero scalar multiples of the extended-vector name. Some examples of names for  $(3, 2)$  are

$(3, 2, 1)_h$ ,  $(30, 20, 10)_h$ ,  $(300, 200, 100)_h$ ,  $(0.3, 0.2, 0.1)_h$ ,  $(-3, -2, -1)_h$ ,  $(12, 8, 4)_h$ , and so on.

*Problem.* What is the ordinary name of  $(8, 20, 4)_h$ ?

*Solution.* First scale so the 4 becomes 1. Thus  $(8, 20, 4)_h$  describes the same point as  $(2, 5, 1)_h$ , which is the point with ordinary name  $(2, 5)$ .

Because one point has many names in homogeneous coordinates, it is a good idea to distinguish between a name of a point and the point itself. Let's write *pt* for "the point whose name is". Thus in the solution to the last problem, we could have written  $pt(8, 20, 4)_h = pt(2, 5, 1)_h = (2, 5)$ . (It wouldn't really have made sense just to say the names themselves are equal, because they aren't. This notes will use *pt* when it's appropriate, but you don't have to.)

*Problem.* What is the ordinary name of  $(x, y, s)$  (assuming that  $s \neq 0$ )?

*Solution.*  $pt(x, y, s)_h = pt(\frac{x}{s}, \frac{y}{s}, 1)_h = (\frac{x}{s}, \frac{y}{s})$ .

### 3. Projective transformations

A *projective transformation* of the plane is simply a transformation that is a homogeneous linear transformation *for homogeneous coordinates*. Thus a projective transformation corresponds to a  $3 \times 3$  matrix  $A$  so that the point

whose name in homogeneous coordinates is  $(x, y, s)_h$  is mapped to the point whose name in homogeneous coordinates is  $(x, y, s)_h A$ . In symbols,

$$T(pt\mathbf{x}_h) = pt\ \mathbf{x}_h A.$$

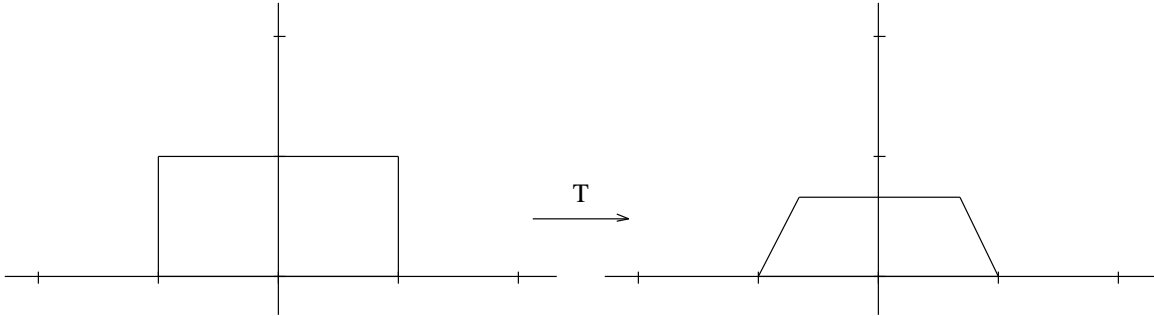
Actually, this definition of a projective transformation requires a few clarifications. First,  $A$  should be nonsingular. The others can wait until Section 4 below.

*Example 3.1* . Let's transform the corners of the rectangle with vertices  $(1, 1)$ ,

$(-1, 1)$ ,  $(-1, 0)$ ,  $(1, 0)$  using  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ . First,

$$(1, 1) = pt(1, 1, 1)_h \rightarrow pt(1, 1, 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = pt(1, 1, \frac{3}{2})_h = pt(\frac{2}{3}, \frac{2}{3}, 1)_h = (\frac{2}{3}, \frac{2}{3}).$$

Thus  $T(1, 1) = (\frac{2}{3}, \frac{2}{3})$ . The same sort of calculation gives  $T(-1, 1) = (-\frac{2}{3}, \frac{2}{3})$ ,  $T(-1, 0) = (-1, 0)$ ,  $T(1, 0) = (1, 0)$ . This gives a picture somewhat like that of the football field:



## 4. Points at infinity

So far we have discussed homogeneous coordinates for ordinary points such as  $(3, 4)$ . Because these coordinates result from multiplying an extended vector such as  $(3, 4, 1)_h$  by a nonzero scalar, the third number is never 0.

On the other hand, applying a projective transformation could very well result in a triple ending in 0. In Example 3.1 above, for instance, we could compute

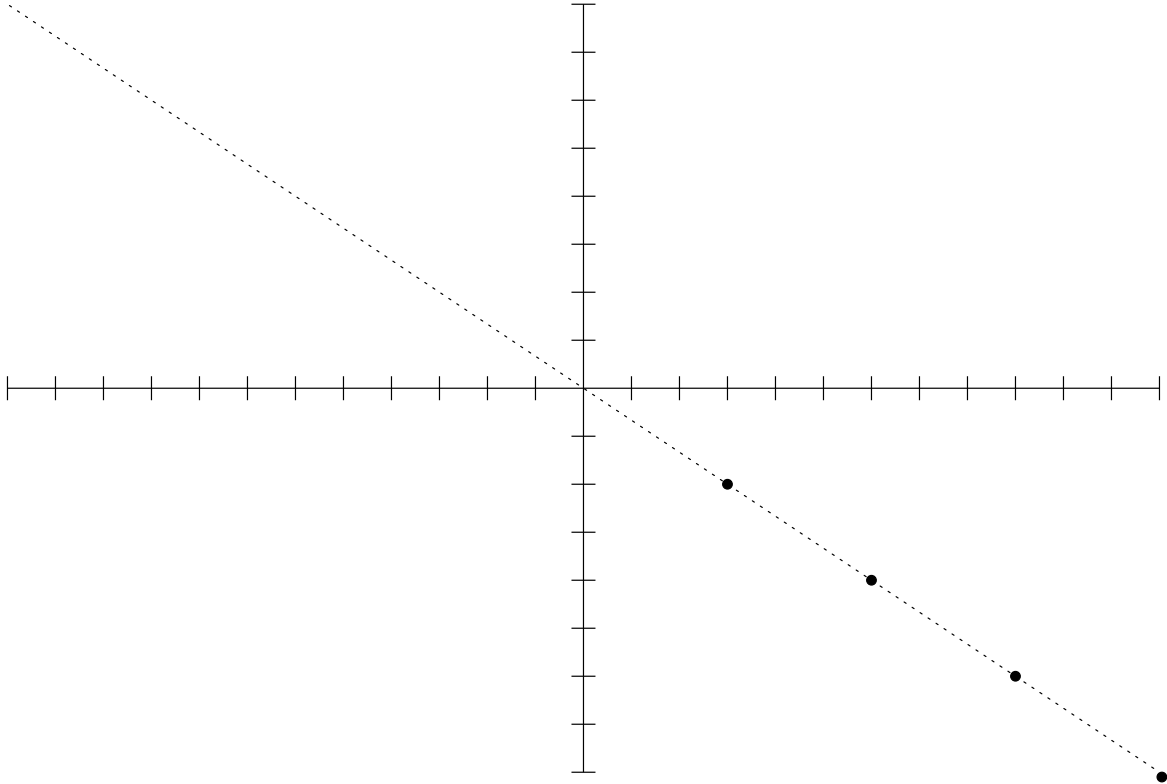
$$T(3, -2) = pt(3, -2, 1)_h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = pt(3, -2, 0)_h,$$

but it's not clear what  $pt(3, -2, 0)_h$  means, since there's no such point in the usual sense.

Can a meaning be attached to a triple ending in 0? The answer is yes (except for the hopeless case  $(0, 0, 0)$ ). Let's sneak up on  $pt(3, -2, 0)_h$  by looking at the sequence of points

$$pt(3, -2, 1)_h, pt(3, -2, \frac{1}{2})_h, pt(3, -2, \frac{1}{3})_h, pt(3, -2, \frac{1}{4})_h, pt(3, -2, \frac{1}{5})_h, \\ pt(3, -2, \frac{1}{6})_h, \dots$$

Algebraically, these triples have limit  $(3, -2, 0)$ . To picture them graphically, just represent them in Cartesian coordinates. For example,  $pt(3, -2, \frac{1}{2})_h = pt(6, -4, 1)_h = (6, -4)$ . We get the points  $(3, -2)$ ,  $(6, -4)$ ,  $(9, -6)$ ,  $(12, -8)$ ,  $(15, -10)$ ,  $(18, -12)$ , and so on:



As you see, this sequence of points is headed off the real plane, in the direction given by the vector  $\mathbf{v} = (3, -2)$ . We get the following idea:

The homogeneous coordinates  $(a, b, 0)_h$  represent a **point at infinity** along the line through the origin with direction vector  $(a, b)$ .

Of course, a point at infinity is in one sense an imaginary invention, but it does have a concrete reality in that it is representable by numbers and corresponds to a line through the origin. For these reasons, we can talk about points at infinity and be sure that we will not run into trouble.

Let us call the usual points of the Cartesian plane *ordinary points*, in contrast to the points at infinity.

Several interesting observations:

*Observation 4.1* . Because  $(3, -2, 0)_h$  and  $(-3, 2, 0)_h$  differ by a scalar factor, namely  $-1$ , they represent the *same* point at infinity. Thus the point at infinity corresponds more to the line itself than to one direction on the line. You can picture the line as somehow “wrapping around” at its point at infinity.

*Observation 4.2* . *Every* line with direction  $(3, -2)$  goes towards the same point at infinity. For example, the line  $P(t) = (4, 5) + t(3, -2)$  has points  $(4 + 3t, 5 - 2t)$ , which in homogeneous coordinates are  $pt(4 + 3t, 5 - 2t, 1)_h = pt(\frac{4}{t} + 3, \frac{5}{t} - 2, \frac{1}{t})$ . As  $t \rightarrow \infty$ , this triple has limit  $(3, -2, 0)_h$ .

Thus a point at infinity corresponds to *a family of parallel lines* (i.e., the set of all lines parallel to a given line). In some sense, the parallel lines meet at that point at infinity.

*Problem 4.3* . What point at infinity lies on the line through  $(2, 3)$  and  $(3, 5)$ ?

*Solution*. The vector between the two points is  $(1, 2)$ . This vector gives the direction of the family of all lines parallel to the given line. Therefore the given line goes through the point  $pt(1, 2, 0)_h$  at infinity.

*Observation 4.4* . A projective transformation can take points at infinity to ordinary points, and vice-versa. For example, in going from the real football field to the television screen, the point at infinity where the yard-lines “meet” is mapped to the ordinary point where their images meet, off the top of the page. Also, the ordinary point off the field where the two lines of the band would meet (if extended), is mapped to a point at infinity (because the images of the two lines are parallel).

## 5. The real projective plane

This last observation means that a projective transformation is *not* just a function on  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ . After all, the function notation  $f : A \rightarrow B$  is supposed to mean that one is considering the domain of  $f$  to be  $A$  and that all values of  $f$  are in  $B$ . The following definition is therefore handy:

**Definition.** The (real) **projective plane**, denoted  $\mathbf{P}_2$ , is the set of all ordinary points *and* points at infinity.

To summarize:

**Fact 1.** The projective plane  $\mathbf{P}_2$  has two kinds of points: ordinary points and points at infinity.

**Fact 2.** Each point of the projective plane  $\mathbf{P}_2$  can be represented by homogeneous coordinates, in many ways.

**Fact 3.** Every triple  $(a, b, c)_h$ , except  $(0, 0, 0)$ , represents a point of  $\mathbf{P}_2$ . If  $c \neq 0$  then the point is the ordinary point  $(\frac{a}{c}, \frac{b}{c})$ ; if  $c = 0$  then the point is a point at infinity in the direction given by the direction vector  $(a, b)$ .

**Fact 4.** If  $(a, b, c)_h$  is one name for a point, its other names have the form  $(ra, rb, rc)_h$ , where  $r$  is any nonzero scalar.

**Fact 5.** A *projective transformation* of the projective plane is a transformation  $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$  that is a nonsingular homogeneous linear transformation in homogeneous coordinates. A projective transformation has the form  $T(pt\mathbf{x}_h) = pt\mathbf{x}_hA$ , where  $A$  is a nonsingular  $3 \times 3$  matrix.

## 6. Geometry of the real projective plane

The real projective plane is useful in geometry, specifically *projective geometry*. You already know that its points are the ordinary points together with the points at infinity (which may be regarded as corresponding to families of parallel lines in the ordinary plane). Its lines consist of (a) ordinary lines, except that each ordinary line is considered to contain its corresponding point at infinity, and (b) the “line at infinity” consisting of all points at infinity.

With this definition, geometry in the real projective plane obeys very simple rules:

*Rule 1.* Every two lines meet in exactly one point.

*Rule 2.* Every two points lie on exactly one line.

Rule 1 says that in projective geometry there are *no* parallel lines. Rule 2 is the same as in ordinary plane geometry.

There is one theorem in projective geometry that shows why projective transformations are important and is useful as background in computer graphics.

**The Fundamental Theorem of Real Projective Geometry:** *Any one-to-one function  $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$  that takes lines to lines is a projective transformation.*

(In other words,  $T$  actually comes from some nonsingular  $3 \times 3$  matrix  $A$ .)

## 7. Making projective transformations

First, we need to choose names for special points. The first three have homogeneous coordinates that are standard basis vectors and so will be handy for working with matrices.

Let  $X = pt(1, 0, 0)_h$  (at infinity on the  $x$ -axis);

let  $Y = pt(0, 1, 0)_h$  (at infinity on the  $y$ -axis);

let  $O = pt(0, 0, 1)_h$  (the ordinary point  $(0, 0)$ , i.e., the origin);

let  $E = pt(1, 1, 1)_h$  (the ordinary point  $(1, 1)$ ).

It is important not to confuse  $X$  with  $(1, 0)$ , which is *not* at infinity and has homogeneous coordinates  $(1, 0, 1)_h$ .

*Problem 7.1 .* Find a projective transformation  $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$  for which  $T(X) = P$ ,  $T(Y) = Q$ , and  $T(O) = R$ , where  $P = (2, 4)$ ,  $Q = (4, 1)$ , and  $R = (6, 3)$ .

*Solution.* Just write down a matrix whose rows are  $P, Q, R$  in homogeneous coordinates:  $A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 1 & 1 \\ 6 & 3 & 1 \end{bmatrix}$ . Then  $T(X) = pt(1, 0, 0)A = P$ , and so on.

*Problem 7.2 .* In Problem 7.1, was that the only solution?

*Solution.* No: Any choice of homogeneous coordinates for  $P, Q, R$  would work. In other words, if  $r, s, t$  are any nonzero scalars, any matrix of the form  $\begin{bmatrix} r \cdot 2 & r \cdot 4 & r \cdot 1 \\ s \cdot 4 & s \cdot 1 & s \cdot 1 \\ t \cdot 6 & t \cdot 3 & t \cdot 1 \end{bmatrix}$  is an answer to Problem 7.1.

Because there is some freedom in the answer, let's see if we can use that freedom to specify where another point goes:

*Problem 7.3 .* Find a projective transformation  $T$  for which  $T(X) = P$ ,  $T(Y) = Q$ ,  $T(O) = R$ , and also  $T(E) = S$ , where  $P, Q, R$  are as in Problem 7.1 and  $S = (6, 8)$ .

*Solution.* Write down the more general solution to Problem 7.1 with  $r, s, t$ , as described in the solution to Problem 7.2. We would like

$$(1, 1, 1)_h \begin{bmatrix} r \cdot 2 & r \cdot 4 & r \cdot 1 \\ s \cdot 4 & s \cdot 1 & s \cdot 1 \\ t \cdot 6 & t \cdot 3 & t \cdot 1 \end{bmatrix} = (6, 8, 1)_h.$$

This is the same as the set of linear equations

$$2r + 4s + 6t = 6$$

$$4r + 1s + 3t = 8$$

$$1r + 1s + 1t = 1.$$

In matrix form: 
$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}.$$

Here we see that the coefficient matrix has *columns* that are the extended-vector forms of  $P, Q, R$ . (In other words, it is the transpose of the simplest answer to Problem 7.1.) These equations also say that  $\hat{S} = r\hat{P} + s\hat{Q} + t\hat{R}$  (extended vectors).

Calculation (say by Gauss-Jordan) shows that the solution is  $r = 1$ ,  $s = -2$ ,  $t = 2$ . In the general solution to Problem 7.1, substitute these

values. We get the answer 
$$\begin{bmatrix} 2 & 4 & 1 \\ -8 & -2 & -2 \\ 12 & 6 & 2 \end{bmatrix},$$

*Definition.* Four points in  $\mathbf{R}^2$  or  $\mathbf{P}_2$  are said to be *in general position* if no three are collinear (i.e., on the same line). Let's say that four points in general position form a *quartet*<sup>1</sup>.

For example, in Problem 7.3,  $P, Q, R, S$  are in general position and so make a quartet.  $X, Y, O, E$  also make a quartet, which we can call the *standard quartet*.

*Fact 7.4.* There is a projective transformation taking  $X, Y, O, E$  to any given quartet  $P, Q, R, S$  in  $\mathbf{P}_2$ , with  $X \rightarrow P$  and so on.

The reason is that having no three of the four points be in a line is just what is needed to guarantee that in the method of Problem 7.3, the coefficient matrix is nonsingular and none of  $r, s, t$  come out zero.

*Problem 7.5.* Describe a general method of finding a projective transformation that takes one given quartet  $P, Q, R, S$  to another  $P', Q', R', S'$ , with  $P \rightarrow P'$  and so on.

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<sup>1</sup>Some people say "quadrangle", but that's misleading because angles don't make sense in  $\mathbf{P}_2$ .



*Solution.* We can use the same idea as for affine transformations, but with the standard quartet in place of the standard triangle. First find a  $3 \times 3$  matrix  $A$  that gives a projective transformation taking  $X, Y, O, E$  to  $P, Q, R, S$ . Then find another matrix  $B$  that gives a projective transformation taking  $X, Y, O, E$  to  $P', Q', R', S'$ . The answer to the problem is  $A^{-1}B$ . (By Problem 7.5,  $A$  and  $B$  are guaranteed to be nonsingular, so  $A^{-1}B$  exists and is nonsingular, as it should be for a projective transformation.)

**Problem 7.6 .** Find a projective transformation taking the four points  $(1, 0)$ ,  $(0, 1)$ ,  $(0, 0)$ ,  $(1, 1)$  (the vertices of the standard square) to  $P, Q, R, S$  of Problem 7.3.

*Solution.* Use the method of Problem 7.5. To take the standard quartet to the square, we must solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The solution is easily found to be  $r = 1$ ,  $s = 1$ ,  $t = -1$ . Therefore  $A =$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

We can invert  $A$ , and for this particular  $A$  it turns out

that  $A^{-1} = A$ .  $B$  was already found in Problem 7.3. We get  $A^{-1}B =$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ -8 & -2 & -2 \\ 12 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 10 & 3 \\ 4 & 4 & 0 \\ -12 & -6 & -2 \end{bmatrix}.$$

**8. How much freedom** in making projective transformations of the plane?

A homogeneous linear transformation is a special instance of an affine transformation, and an affine transformation is a special instance of a projective transformation.

With a *homogeneous linear transformation* we can map two given vectors, not along the same line, to two other given vectors. In terms of points, the origin must go to the origin, but we could take two other points, not on the same line through the origin, to two given points.

With an *affine transformation* we can take a triangle to a triangle. (The three points of the first triangle must not be collinear.)

With a *projective transformation* of  $\mathbf{P}_2 \rightarrow \mathbf{P}_2$ , we can take a quartet to a quartet.

*Fact.* The projective transformation taking one given quartet to another is unique.

*Note.* Its matrix is not unique, since multiplying the whole matrix by a nonzero scalar has no effect on the transformation. That's the *only* way the matrix can vary, though. In particular, you *cannot* multiply each row of the matrix by a separate nonzero scalar. That would not change the images of  $X$ ,  $Y$ , and  $O$ , but it *does* change the images of other points, such as  $E$ .

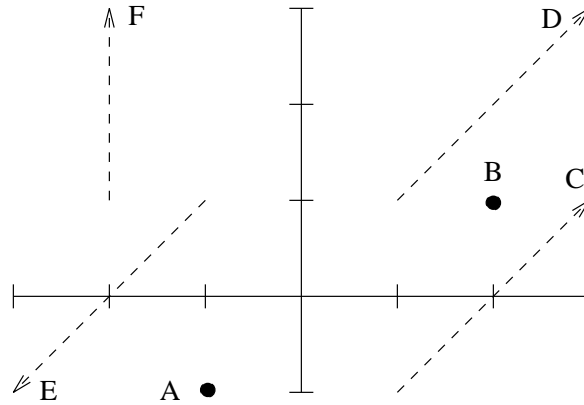
## 9. Three-dimensional projective space

All the ideas discussed above can be adapted to three-dimensional space. An overview:

- (1) Homogeneous coordinates are  $(x, y, z, s)_h$ . The names of each point differ by nonzero scalar factors.  $pt(x, y, z, s)$  is an ordinary point if  $s \neq 0$  and is a point at infinity if  $s = 0$ .
- (2) Each point at infinity corresponds to a family of parallel lines in  $\mathbf{R}^3$ . These lines meet in  $\mathbf{P}_3$  at their point at infinity.  
(In  $\mathbf{R}^3$ , two lines are parallel if *they lie in the same plane* and do not intersect. Two lines that do not intersect and do not lie in the same plane are said to be *skew*.)
- (3) Real projective 3-space  $\mathbf{P}_3$  consists of  $\mathbf{R}^3$  together with points at infinity.
- (4) A projective transformation  $T : \mathbf{P}_3 \rightarrow \mathbf{P}_3$  is a transformation obtained by multiplying the homogeneous coordinates of each point by a nonsingular  $4 \times 4$  matrix  $A$ .
- (5) Five points in  $\mathbf{P}_3$  are said to be in general position if no four are coplanar. If  $P_1, \dots, P_5$  are in general position and  $Q_1, \dots, Q_5$  are in general position, then there exists a projective transformation  $T : \mathbf{P}_3 \rightarrow \mathbf{P}_3$  such that  $T(P_1) = Q_1, \dots, T(P_5) = Q_5$ .

## 10. Problems

**Problem M-1.** Give homogeneous coordinates for each of the points indicated below. Points indicated with arrows are at infinity. (Remember, the point at infinity is determined by the *direction* of the line to it.)



**Problem M-2.** Recall the definition of a projective transformation: To transform a point, you choose some triple representing the point in homogeneous coordinates, multiply the triple by the matrix to get a new triple, and then take the point represented by the new triple. Why doesn't it matter which triple you choose to represent the first point?

**Problem M-3.** (a) Is the standard square the same thing as the standard quartet? (b) If it is, explain why. If it is not, say whether the standard square is even a quartet at all, and why.

**Problem M-4.** (a) In  $\mathbf{P}_2$ , find a matrix for the projective transformation that takes the ordinary points  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(0, 1)$  (the “standard square”) to the ordinary points  $(0, 0)$ ,  $(2, 0)$ ,  $(3, 3)$ ,  $(0, 2)$ , respectively. (If you wish you may list the vertices in another order, as long as you list the image vertices in the corresponding order. You may be able to take advantage of Problem 7.6.)

(b) Give all possible answers to (a).

(c) If one person took advantage of the possibility of listing the vertices in another order and another person did not, would their answers to (a) necessarily be the same? Their answers to (b)?

**Problem M-5.** Is there a projective transformation that takes  $X, Y, O, E$  respectively to  $(1, 0)$ ,  $(1, 1)$  (ordinary points),  $X, Y$ ? Why or why not?

**Problem M-6.** Check that the matrix given as the solution to Problem 7.6 above does give a projective transformation that does what it is supposed to.

**Problem M-7.**

Show that the only projective transformation of  $\mathbf{P}_2$  that leaves each of  $X, Y, O, E$  fixed is the identity transformation, i.e., the transformation that leaves *all* points fixed.

(Method: If  $T$  is such a transformation, then  $T$  comes from a  $3 \times 3$  matrix  $M$ . Show that  $M$  must be a nonzero scalar matrix; i.e.,  $M = rI$ , by using the fact that  $x(1, 0, 0)_h M$  is a scalar times  $(1, 0, 0)_h$ , etc. What transformation is produced by a scalar matrix?)

**Problem M-8.** (a) Show that if  $P, Q, R, S$  is a quartet in  $\mathbf{P}_2$ , then the only projective transformation that leaves each of  $P, Q, R, S$  fixed is the identity transformation.

(Method: Suppose  $T$  leaves  $P, Q, R, S$  fixed. Choose a projective transformation  $W$  taking  $X, Y, O, E$  to  $P, Q, R, S$  respectively. Apply the result of Problem M-7 to show  $W^{-1}TW = \mathbf{1}$ , the identity transformation. Here the composition  $W^{-1}TW$  means to apply  $W$ , then  $T$ , then  $W^{-1}$ . Solve for  $T$ , just as you would using matrices.)

(b) Show that if  $P, Q, R, S$  and  $P', Q', R', S'$  are two quartets in  $\mathbf{P}_2$ , there is only *one* projective transformation that takes  $P$  to  $P'$  and so on.

(Method: Suppose both  $T$  and  $U$  take  $P$  to  $P'$  and so on. Apply (a) to  $T^{-1}U$ .)

**Problem M-9.** (a) Explain why it is not possible to define the value of the determinant for a projective transformation. (Method: As in Problem M-7, there is more than one possible matrix for the transformation. Do all possible matrices have the same determinant? Be careful in saying what happens to the determinant of a matrix when *all* entries are multiplied by the same scalar.)

(b) Explain why in  $\mathbf{P}_2$  it is not even possible to talk about the *sign* of the determinant. (Method: If all entries of the  $3 \times 3$  matrix are multiplied by  $-1$ , what happens to the sign of the determinant?)

(c) Explain how it *is* possible to define the sign of a projective transformation in  $\mathbf{P}_3$  using determinants, even though the value of the determinant itself cannot be defined.

(For homogeneous linear transformations, the sign of the determinant tells whether the transformation preserves orientation or reverses it. The same is true for affine transformations. The facts (b) and (c) say in effect that orientation of objects cannot be defined in  $\mathbf{P}_2$  but can be defined in  $\mathbf{P}_3$ .)

**Problem M-10.** In the one-dimensional space  $\mathbf{R}$ , a *linear fractional function* is a function of the form  $f(x) = \frac{ax+c}{bx+d}$ , where  $ad - bc \neq 0$ . Of course, the domain of  $f$  may not be all of  $\mathbf{R}$ . Explain how a linear fractional function is really the same as a projective transformation in  $\mathbf{P}_1$ . (Method: Make the extended vector  $[x1]_h$ , transform it by an appropriate matrix, and scale to put the answer in the form  $[y1]_h$ . What is  $y$  in terms of  $x$ ? Linear fractional functions are important in the theory of complex variables, where  $x$  and  $y$  can be complex.)

**Problem M-11.** Find all  $4 \times 4$  matrices that give projective transformations on  $\mathbf{P}_3$  leaving the  $x, y$ -plane fixed. (In other words,  $T(Q) = Q$  for every point  $Q$  in the  $x, y$ -plane.)

**Problem M-12.** Find an example to show that a projective transformation in  $\mathbf{P}_2$  does not preserve ratios of line segments on the same line. In other words, if  $PR$  is a line segment and  $Q$  is between  $P$  and  $R$  on the segment, the ratio of the length of  $PQ$  to the length of  $PR$  may not be preserved. (In contrast, affine transformations do preserve such ratios; see the exercises of the first handout on affine transformations.)

**Problem M-13.** Let  $T$  be the projective transformation on  $\mathbf{P}_2$  that takes  $X, Y, O, E$  to the standard square  $\mathbf{e}_1, \mathbf{e}_2, (0, 0), (1, 1)$ , in that order. A matrix for  $T$  was found as part of the solution to Problem 7.6 above, namely,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$
 (a) Sketch the images under  $T$  of the edges of the standard square (between vertices in the usual order). (Suggestion: Represent each edge parametrically and transform. Indicate just ordinary points.) (b) On your sketch, indicate the image under  $T$  of the whole standard square region, with interior. (c) What is the image of the diagonal line  $x + y = 1$ ? (d) Sketch the image of the line  $x + y = 2$ .

**Problem M-14.** Describe the image of the hyperbola  $y = \frac{1}{x}$  under the projective transformation with matrix 
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(Method: Express the hyperbola parametrically as  $(t, \frac{1}{t})$  for  $t \neq 0$ . Rewrite in homogeneous coordinates, transform, and find the Cartesian coordinates again in terms of  $t$ . Finally, try to re-express the answer as a curve  $y = \dots$ )

*Note.* Actually, projective transformations can map any kind of conic to any other, for instance, a circle to a parabola.

**Problem M-15.** Verify Rules 1 and 2 of Section 6 above, by discussing the different cases that can occur. (For example, in considering two lines, one might be an ordinary line and the other the line at infinity.)

**Problem M-16.** Prove Fact 7.4. In other words, prove that the method of Problem 7.3 works for any quartet with vertices  $P, Q, R, S$ .

(Method: You may assume this useful fact: If  $P, Q, R$  are points of  $\mathbf{P}_2$  whose homogeneous coordinates are linearly dependent as three vectors in  $\mathbf{R}^3$ , then  $P, Q, R$  are collinear. Using this fact, explain why, in the method of Problem 7.6 above, the coefficient matrix is nonsingular and none of  $r, s, t$  can be zero.)

**Problem M-17.** Explain how the extended matrix of an affine transformation on  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$  can be regarded as the matrix of a projective transformation on  $\mathbf{P}_2 \rightarrow \mathbf{P}_2$ . What does each row of the matrix mean, in terms of  $X, Y, O$ ?

**Problem M-18.** As you now know, the extended vectors used for affine transformations were really homogeneous coordinates. Further, the  $3 \times 3$  extended matrices used for an affine transformation in  $\mathbf{R}^2$  can be applied even to points at infinity, so that the affine transformation becomes a projective transformation.

Show that nonsingular affine transformations, if applied in  $\mathbf{P}_2$ , have the special property that they take points at infinity only to points at infinity. Do this two ways:

- (a) algebraically, by transforming points with coordinates  $(a, b, 0)_h$ ;
- (b) geometrically, by using the fact that affine transformations on  $\mathbf{R}^2$  take parallel lines to parallel lines.

**Problem M-19.** Let  $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$  be a projective transformation. This problem shows how to tell if  $T$  is really an affine transformation.

- (a) For  $X, Y$  as usual, show that if  $T(X)$  and  $T(Y)$  both are points at infinity, then  $T$  is affine. (Method: Narrow down possibilities for the matrix of  $T$ . See if you can get the desired entries to be zero. Scale the *whole matrix* by a nonzero scalar to get the desired entry to be a 1.)
- (b) Explain why if  $T$  takes points at infinity to points at infinity only (in other words, the line at infinity stays at infinity), then  $T$  must be affine.
- (c) Explain why if  $T$  takes even two points at infinity to points at infinity, then  $T$  must be affine. (Part (a) showed one example of this fact. Method:  $T$  takes lines to lines. Where does the line at infinity go?)

(d) Explain why, if  $T$  takes even one parallelogram to a parallelogram, then  $T$  must be affine. (Method: Use (c).)

This last part shows that even with a little information you can tell that  $T$  is affine. An equivalent statement is this: If  $T$  is *not* affine, then  $T$  distorts *every* parallelogram into a non-parallelogram.

**Problem M-20.** Consider all projective transformations on  $\mathbf{P}_2$  that leave  $X, Y, O$  fixed. These are the same as some transformations you knew about before ever hearing of  $\mathbf{P}_2$ . Which ones, exactly?

**Problem M-21.** (a) Explain why each real eigenvector of a nonsingular real  $3 \times 3$  matrix gives a fixed point of the corresponding projective transformation on  $\mathbf{P}_2$ .

(b) Show that every projective transformation on  $\mathbf{P}_2 \rightarrow \mathbf{P}_2$  has at least one fixed point. (Notice that the characteristic polynomial is cubic, and recall that every cubic polynomial has at least one real root, since its graph goes from the third quadrant to the first quadrant and so crosses the  $x$  axis. Mention why the eigenvalue 0 can't occur.)

(c) For a translation, regarded as a projective transformation, describe one fixed point. Are there any others?

**Problem M-22.** In  $\mathbf{P}_3$ , (a) define the standard points  $X, Y, Z, O, E$ ; (b) find a projective transformation taking  $X, Y, Z, O, E$  to  $(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0), (1, 1, 1)$  respectively.

**Problem M-23.** Surprisingly, the intersection of two lines in  $\mathbf{P}_2$  can be found by using a cross product. Consider the two lines whose equations in ordinary coordinates are  $x + 3y + 4 = 0$  and  $4x + y + 5 = 0$ .

(a) Find equations for these two lines in homogeneous coordinates. (Method: Given a line  $ax + by + c = 0$  in ordinary coordinates, put  $x/z$  for  $x$  and  $y/z$  for  $y$  and then clear the denominator; you get simply  $ax + by + cz = 0$ .)

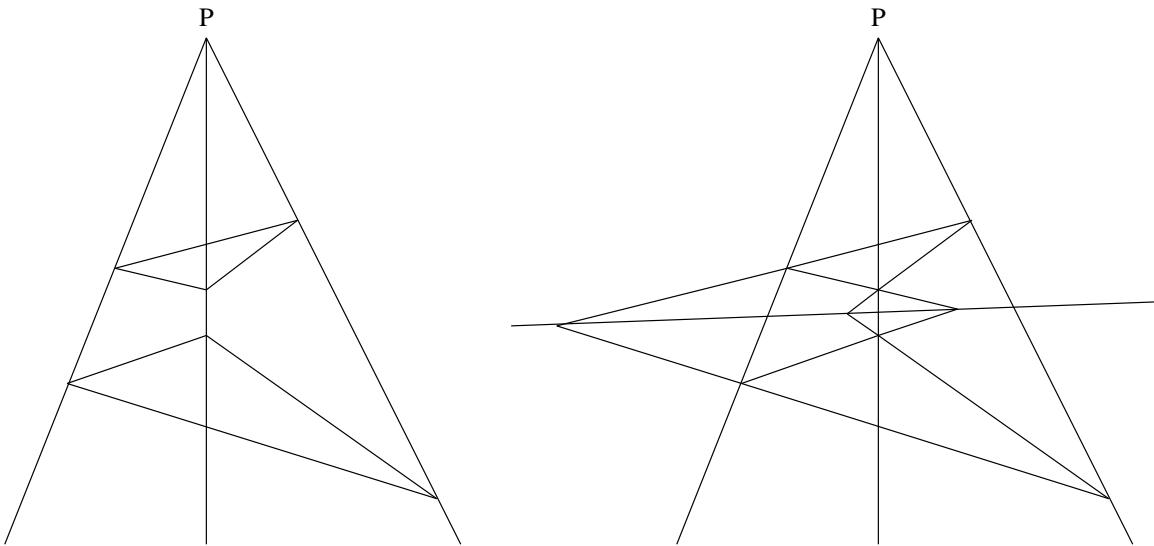
(b) Use a cross product to find the homogeneous coordinates of the point where the two lines intersect. (Method: Solving the two equations simultaneously will give the desired homogeneous coordinates. If you regard the triples as being in  $\mathbf{R}^3$  instead, you are finding the line of intersection of two planes, and you know how to do this using a cross product.)

(c) Express the answer in ordinary coordinates.

(d) Does this method work even if one line is the line at infinity? Give an example.

**Problem M-24.** Projective geometry is concerned with theorems that mention only which lines meet at which points, and not with angles and distances. There are actually some good theorems of this type that could have been understood in high school geometry but were probably not mentioned. Here is one [with problems following the diagram]:

*Desargues' Theorem.* Choose a point  $P$  in the ordinary plane. Draw three lines from  $P$ . Choose two triangles, each with a vertex on each line (but not using  $P$ ), as in the left diagram. If corresponding sides are not parallel, extend the corresponding sides until they meet. Then the three points of intersection are collinear (i.e., they all lie on one line).



There are also versions of the theorem covering cases where one pair of corresponding sides is parallel but the other two pairs are not; where at least two pairs of corresponding sides are parallel; and similar cases for a drawing where the original three lines are chosen to be parallel instead of going through a point  $P$ .

- (a) Write down statements for five of these cases. (Choose interesting ones.)
- (b) Explain how in the *projective* plane there is only one case, of which all your cases are really instances. (This is an example of how using the projective plane can actually make some kinds of geometry simpler and yet more powerful at the same time.)

**Problem M-25.** The diagram of Desargues' Theorem looks somewhat three-dimensional, even though it isn't. However, you can invent a three-dimensional version of Desargues' Theorem in which the lines through  $P$  may not be coplanar.

- (a) Give such a three-dimensional statement.



(b) Actually prove your statement in the case where the lines through  $P$  are not coplanar. To keep things simple, assume that no two lines in the whole diagram are parallel in  $\mathbf{R}^3$ .

*Note.* One good proof in the two-dimensional case is this: Given the diagram, imagine constructing a three-dimensional diagram whose perpendicular projection in two dimensions is the given diagram. The line in three dimensions containing the three intersection points of the sides of the triangles projects to a line in two dimensions with the same property.

**Problem M-26.** An affine transformation in extended coordinates had a “hidden” geometrical interpretation as a map on  $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ , in which the transformation acted on the  $z = 1$  plane. The extended matrix of an affine transformation always takes the  $z = 1$  plane to itself. Develop a similar “hidden” geometrical interpretation for how you use projective transformations when you take  $(x, y) \rightarrow (x, y, 1)_h \rightarrow (x, y, 1)_h A$  and then normalize the result to the form  $(u, v, 1)_h$ . (This time the matrix will usually take the  $z = 1$  plane to another plane. However, the final step of normalizing projects points back to the  $z = 1$  plane. Make a sketch.)

**Problem M-27.** In the survey article by Carlbom and Paciorek it is stated that various entries of  $4 \times 4$  matrix represent a homogeneous linear transformation, a translation, a projection, and a scaling. This would seem to suggest that an arbitrary nonsingular  $4 \times 4$  matrix  $\begin{bmatrix} A & \mathbf{p}^t \\ \mathbf{b} & s \end{bmatrix}$  is the product of the matrices  $\begin{bmatrix} A & \mathbf{0}^t \\ \mathbf{0} & 1 \end{bmatrix}$ ,  $\begin{bmatrix} I & \mathbf{0}^t \\ \mathbf{b} & 1 \end{bmatrix}$ ,  $\begin{bmatrix} I & \mathbf{p}^t \\ \mathbf{0} & 1 \end{bmatrix}$ ,  $\begin{bmatrix} I & \mathbf{0}^t \\ \mathbf{0} & s \end{bmatrix}$ , in some order. Is this true?

**Problem M-28.** Suppose we take a  $4 \times 3$  matrix  $A$  and attempt to define a transformation on  $\mathbf{P}_3 \rightarrow \mathbf{P}_2$  by setting  $T(\mathbf{x}_h) = \mathbf{x}_h A$ . (a) Why doesn’t this define a transformation on *all* of  $\mathbf{P}_3$ , even if  $A$  has rank 3 (the largest possible)? (Method: Is  $\mathbf{x}_h A$  always a nonzero vector?) (b) Even so, this does define a transformation whose domain is *contained in*  $\mathbf{P}_3$ . Describe how you could find  $A$  taking each of  $X, Y, Z, O$  to given points of  $\mathbf{P}_2$ .

(Essentially, you have made a two-dimensional picture of [most of]  $\mathbf{P}_3$  in which the images of  $X, Y, Z$  are “vanishing points” for the corresponding families of parallel lines in  $\mathbf{R}^3$ .)

**Problem M-29.** Suppose you want to make a perspective picture of a box-shaped building whose families of parallel lines are lined up with the coordinate axes. Suppose you want a 3-point perspective projection, with vanishing

points on the viewplane being  $Q = (4, -2)$ ,  $R = (-4, -2)$ ,  $S = (0, 6)$ , with  $X \rightarrow Q, Y \rightarrow R, Z \rightarrow S, O \rightarrow O$  (where  $O$  means an origin in each dimension).

(a) By a direct method as in Problem M-28, find a 4-by-3 matrix  $M$  that accomplishes this projection, when homogeneous coordinates are used in  $\mathbf{P}_3$  and  $\mathbf{P}_2$ .

(b) Observe that  $M$  makes a transformation that is not defined on all of  $\mathbf{P}_3$ , because some legitimate points are mapped to a triple  $(0, 0, 0)_h$ . Find such a point. Geometrically, why should there be such undefined points for a perspective projection?

(c) Is there more than one transformation possible, depending on the choice of  $M$ ? (In other words, are there choices of (a) that are not just scalar multiples of each other?)