Some tools from linear geometry

1. Review: Describing straight lines

You have in the past seen three ways to express a straight line in $\mathbf{R}^2$ algebraically:

Functionally, as the graph $y = mx + b$ of a linear function.

Relationally, as the graph of an equation $ax + by + c = 0$ giving a linear relation between x and y. (Not both of a and b are 0.)

Parametrically, by an equation $\mathbf{x}(t) = P_0 + t\mathbf{v}$ giving the path of a moving point. (Here $\mathbf{v} \neq \mathbf{0}$.)

For the *functional* form, a disadvantage is that vertical lines cannot be expressed. In fact, any graph of this kind is closely tied to the orientation of the axes; it would be messy to rotate the graph and then re-express it in the same way, for example. Advantages of this form are that there is only one way to express each non-vertical line and that there is no restriction on the numbers m and b that can be used.

For the *relational* form, an advantage is that any line can be expressed. For example, $1x + 0y + 2 = 0$ gives a vertical line. A disadvantage is that there is more than one way to express each line. For example, multiplying the equation through by 2 gives the same line.

The *parametric* form can be viewed as the path of a moving point with velocity vector $\mathbf{v}$ and with $P(0) = P_0$. An advantage is that any line can be expressed. A disadvantage is that there are many ways to express each line: P_0 can be changed to any other point on the line, and $\mathbf{v}$ can be multiplied by any nonzero scalar, without changing the line represented.

In computer graphics, there are many applications for the relational and parametric forms, as you will see. The functional form is rarely used. It's easy to take a line written functionally and re-express it either relationally or parametrically. Therefore, in these notes we'll consider only these last two forms.

2. More on the parametric form

Suppose two different points P and Q are given. A parametric equations for the line through P and Q is found by letting $\mathbf{v} = Q - P$, so that you get the **two-point parametric form**:

$\mathbf{x}(t) = P + t(Q - P)$.

If you multiply out and condense the terms a different way, you get another way of writing the same thing:

$\mathbf{x}(t) = (1 - t)P + tQ$,

which expresses points on the line as linear combinations of the two given points, as t changes. Notice that $\mathbf{x}(0) = P$ and $\mathbf{x}(1) = Q$. The expression can be regarded as a weighted average of the two points, with weights $1 - t$ and t. If $t = \frac{1}{2}$, you have the usual average, the midpoint of the line segment. If $t = \frac{1}{4}$ (say), you get the point $\frac{3}{4}P + \frac{1}{4}Q$, which is a fourth of the way from P to Q. In particular, notice that this point is closer to P, which has the larger of the two weights.

Observe also that points on the line segment $\overline{PQ}$ are those for which $0 \le t \le 1$; if $t < 0$ or $t > 1$ then you get points on the line outside the segment.

The parametric form is useful even for describing individual points on a line segment. If you want the point that is 0.30 of the way from P to Q, for example, you can describe it as $P + 0.30(Q - P)$.

Although we're only considering lines in $\mathbf{R}^2$ for the moment, the parametric form works in exactly the same way to describe lines in $\mathbf{R}^3$. In fact, in $\mathbf{R}^3$, the parametric form is by far the best way to describe a line or line segment.

3. More on the relational form

Let's write $f(x, y) = ax + by + c$, and let's always assume that at least one of a and b is not zero. Let L be the line with equation $f(x, y) = 0$.

Observation 1. For any such f with not both a and b equal to 0, the plane $\mathbf{R}^2$ is divided into three subsets: L itself, with equation $f(x, y) = 0$; the half-plane $f(x, y) > 0$, and the half-plane $f(x, y) < 0$.

Observation 2. For any constant $k \neq 0$, the equation $f(x, y) = k$ describes a line parallel to L. (In terms of concepts from calculus, $f(x, y) = k$ describes a level curve of f. Thus, for such an f the level curves happen to be parallel lines.)

Observation 3. The vector $\mathbf{N} = [a, b]$ is a normal (i.e., a perpendicular) to L.

(This is the two-dimensional version of what you are used to with normals for planes in $\mathbf{R}^3$.)

Observation 4. The normal $\mathbf{N} = [a, b]$ points in the direction of the *positive* half-plane $f > 0$. (This makes it easy to tell which side is the one with $f > 0$.)

Observation 5. If P_0 is any point on L, the equation of L can be written as $\mathbf{N} \cdot (\mathbf{x} - P_0) = 0$ (the **point-normal** form).

Observation 6. The absolute value of f at each point equals the perpendicular distance from L times $|\mathbf{N}|$ (the length of the normal).

Remark. A function of the form $f(x, y) = ax + by + c$ is really an affine transformation $f : \mathbf{R}^2 \to \mathbf{R}$. Let's call such a function an *affine function.*

Because of the possibility of scaling a, b, c, there are many choices of f that give the same line L. Which choice is best? Actually, there are several appropriate choices of f, depending on circumstances. One is the two-point form, discussed next; another is barycentric coordinates, discussed in §8; and a third is the signed distance function, discussed in the exercises.

4. The two-point relational form

Suppose we want to find a relational description $f(x, y) = 0$ for the line through two given points P and Q. There is a particular choice of f that does this nicely. It can be derived in either of two ways.

For the first way, take any third point R, and recall the affine transformation that takes the standard triangle to P, Q, R. Let the determinant of its extended matrix be $\Delta(P, Q, R)$. By adding the third row to the first two, we get a simpler description. Thus

$$\Delta(P, Q, R) = \det \begin{bmatrix} (P-R) & 0 \\ (Q-R) & 0 \\ R & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{bmatrix} = \det \begin{bmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{bmatrix}.$$

Here are some facts about $\Delta(P, Q, R)$. You have seen the first three; the last two follow from (a).

Proposition 1. For any points P, Q, R in $\mathbf{R}^2$:

(a) the area of the triangle formed by P, Q, R is $\frac{1}{2}\Delta(P, Q, R)$ in absolute value;

(b) if $\Delta(P, Q, R) > 0$ then P, Q, R have the same orientation as the standard triangle, namely, as you go from P to Q to R and back to P you are traversing the triangle *counterclockwise*; equivalently, Q is to the *left* of P as seen from R, just as $(0, 1)$ is to the left of $(1, 0)$ as seen from the origin;

(c) similarly, if $\Delta(P, Q, R) < 0$, then the traversal P to Q to R to P is *clockwise*; equivalently, Q is to the *right* of P as seen from R;

(d) $\Delta(P, Q, R) = 0$ when P, Q, R are collinear (lie on a line) and so form only a degenerate triangle of zero area;

(e) $\Delta(P, Q, R) = \Delta(Q, R, P) = \Delta(R, P, Q)$ (i.e., Δ doesn't change when the three points are permuted cyclically).

By (d), the points R for which $\Delta(P, Q, R) = 0$ form the line we're looking for. To emphasize which point is varying, let's put $\mathbf{x}$ for R:

Proposition 2. The line L in $\mathbf{R}^2$ through two points P and Q has equation $\Delta(P, Q, \mathbf{x}) = 0$.

Let's call this the *two-point relational form* of L. To see that the function $\Delta(P, Q, \mathbf{x})$ is really affine, expand $\Delta(P, Q, \mathbf{x})$ by cofactors of the third row; you get

$$\Delta(P, Q, \mathbf{x}) = \det \begin{bmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ x_1 & y_2 & 1 \end{bmatrix} = (p_2 - q_2 x) + (q_1 - p_1 y) + (p_1 q_2 - p_2 q_1).$$

The second way of deriving the two-point relational form is to use P and Q to make a normal to the desired line L:

For the the normal use $Q - P$ rotated counterclockwise 90°. Thus if $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, let $[a, b] = (q_1 - p_1, q_2 - p_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [-(q_2 - p_2), q_1 - p_1]$. Since P is on L, $ap_1 + bp_2 + c = 0$, so $c = -(ap_1 + bp_2)$ $= p_1 q_2 - p_2 q_1$. Thus our affine function is

$(p_2 - q_2 x) + (q_1 - p_1 y) + (p_1 q_2 - p_2 q_1)$, as before.

Note 1. From the second method of deriving the two-point relational form, it is clear that if you walk along the line L from P towards Q, the half-plane with $\Delta(P, Q, \mathbf{x}) > 0$ will be *on your left.* This is also clear from (b) of Proposition 1, if you put $R = \mathbf{x}$ again.

Note 2. Because the two-point relational form does not involve division, it is especially good for use in a computer without built-in floating-point operations, if you use only integers for coordinate values of points.

5. Intersections of lines and line segments

Let's say that a line segment $\overline{AB}$ *intersects* (or *touches*) a line L if $\overline{AB}$ and L have at least one point in common, and, more specially, that $\overline{AB}$ *crosses* L if A and B are not on L but some other point of $\overline{AB}$ is on L. Similarly, one line segment could intersect or cross another line segment. Thus, intersecting is a little more general than crossing. In this discussion, let's concentrate on crossing and then see what changes are necessary to handle intersecting. There are different cases, depending on whether we're dealing with lines or line segments and depending on how they are expressed. Two cases are the most important:

Case 1. Does a **line segment** $\overline{AB}$ cross a **line** L expressed in the relational form $f(x,y) = 0$, and if so, where?

This case is the most useful and also the easiest. **Method**. $\overline{AB}$ crosses L if $f(A)$ and $f(B)$ are nonzero and of opposite signs. To find the crossing point C, let $t = -\dfrac{f(A)}{f(B) - f(A)}$ and then let $C = A + t(B - A)$. See Figure 1.

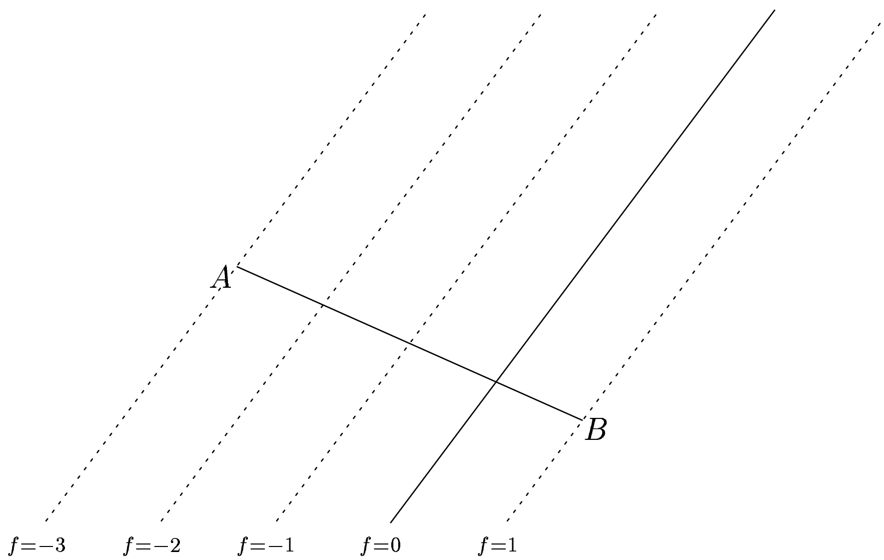

Figure 1: A crossing case

Explanation: The key is to observe that if you follow the line segment from A to B, the value of f changes at a constant rate. So, for example, if $f(A) = -3$ and $f(B) = 1$, then since $f(C) = 0$, C ought to be $\frac{3}{4}$ of the way from A to B, i.e., $t = \frac{3}{4}$. The $\frac{3}{4}$ is the change in value from A to C divided by the change from A to B, i.e., $t = \dfrac{f(C) - f(A)}{f(B) - f(A)} = \dfrac{0 - f(A)}{f(B) - f(A)}$ $= -\dfrac{f(A)}{f(B) - f(A)}$.

Case 2. Do line segments $\overline{AB}$ and $\overline{PQ}$ cross each other, and if so, where?

Method: Use the method of Case 1 both ways around: $\overline{AB}$ should cross the line containing P and Q, and vice versa. For each of the two lines, you can use the two-point relational form.

In other words, if the segments are $\overline{PQ}$ and $\overline{AB}$, find all four of $\Delta(P,Q,A)$, $\Delta(P,Q,B)$, $\Delta(A,B,P)$, and $\Delta(A,B,Q)$. The line segments cross each other when all four of these numbers are nonzero, the first two have opposite signs, and the last two have opposite signs. If they do cross each other, the crossing point is $C = A + t(B - A)$, where $t = -\frac{\Delta(P,Q,A)}{\Delta(P,Q,B) - \Delta(P,Q,A)}$. See Figure 2.

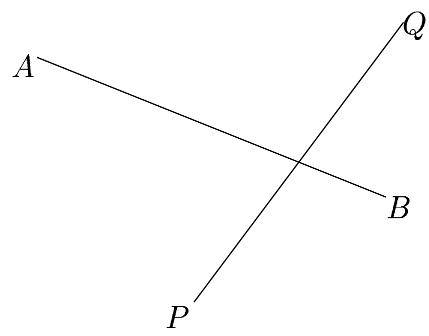

Figure 2: Another crossing case

Note. It is not enough just to check that, say, $\overline{AB}$ crosses the line containing P and Q. That can happen even if the two line segments are each an inch long but are a mile from one other. A less extreme example is shown in Figure 3.

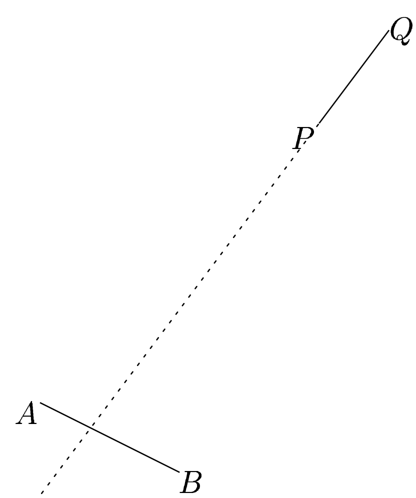

Figure 3: Another crossing case

What if we are interested in intersecting instead of just crossing, for the case of two line segments $\overline{AB}$, $\overline{PQ}$? The answer is that the circumstances will always be clear from the values of $\Delta(P,Q,A)$, $\Delta(P,Q,B)$, $\Delta(A,B,P)$, and $\Delta(A,B,Q)$, provided not all four of these numbers are zero. For example, if $\Delta(P,Q,A) > 0$, $\Delta(P,Q,B) = 0$, $\Delta(A,B,P) < 0$, $\Delta(A,B,Q) > 0$, then $\overline{AB}$ intersects $\overline{PQ}$ at B, somewhere between P and Q.

If all four of the $\Delta()$ values are zero, then A, B, P, Q all lie on one line, and a different method is needed: Look at the x-coordinates a_1, b_1, p_1, q_1 or at the y-coordinates a_2, b_2, p_2, q_2 to see the situation. For example, if $a_1 < q_1 < p_1 < b_1$, then $\overline{PQ}$ is contained entirely in $\overline{AB}$. (Which should you use, the x-coordinates or the y-coordinates? A good choice is to use whichever are the more spread out. The "spread" of the x-coordinates is the maximum of the four x-coordinates minus their minimum, and similarly for the "spread" of the y-coordinates; check which is the greater.)

6. Intersections of a line segment and a plane

Now let's apply the ideas of the last sections in three dimensions:

(1) The *three-point relational form* of a plane in $\mathbf{R}^3$ is given by the equation $\Delta(P, Q, R, \mathbf{x}) = 0$, where

$$\Delta(P, Q, R, \mathbf{x}) = \begin{bmatrix} P & 1 \\ Q & 1 \\ R & 1 \\ \mathbf{x} & 1 \end{bmatrix}.$$

(2) $\Delta(P, Q, R, \mathbf{x}) > 0$ when $\mathbf{x}$ is in the same relation to P, Q, R that the origin is to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, namely, P to Q to R to P is clockwise as seen from $\mathbf{x}$.

(3) A line segment $\overline{AB}$ crosses the plane given by an affine function $f(x, y, z)$ if $f(A)$ and $f(B)$ are nonzero and have opposite signs; the crossing point C is given by $C = A + t(B - A)$, where $t = -\dfrac{f(A)}{f(B) - f(A)}$.

7. When is a point inside a polygon? The convex case.

What is a good computer method for deciding whether a point Q in $\mathbf{R}^2$ is in the interior of a triangle, or more generally, any convex polygon?

If the polygon has n vertices, call them $P_0, \ldots, P_{n-1}$ and

let $P_n = P_0$. Then the sides are $\overline{P_{i-1}P_i}$ for $i = 1, \ldots, n$. Here is a method:

(*) Q is in the interior of the polygon $\Leftrightarrow$ for each $i = 1, \ldots, n$, P_{i-1} is to the right of P_i as seen from Q, or else for each $i = 1, \ldots, n$, P_{i-1} is to the left of P_i as seen from Q. Computationally:

(**) Q is in the interior of the polygon $\Leftrightarrow$ all determinants $\Delta(P_{i-1}, P_i, Q)$ have the same sign.

The reason why (*) is equivalent to (**) is contained in Proposition 1 of §4. Of course, if any determinant is zero, Q is on the boundary.

8. How to tell when a point is inside an arbitrary polygon.

Again, label the polygon $P_0, \ldots, P_{n-1}$ with $P_n = P_0$, and consider a point Q.

Method #1: Choose a point S that you know is outside the polygon. Find how many times the line segment $\overline{QS}$ crosses the edges of the polygon. If this count is odd, then Q is inside the polygon; if it is even, then Q is outside.

Details: For the x coordinate of S, you could, say, take the max of the x coordinates of the points P_i and add 1. Then S is definitely outside. For the y coordinate of S, use a random number between 0 and 1; that way, there is essentially no possibility that $\overline{QS}$ will go through one of the P_i, an undesirable case. If Q is on the boundary, you'll discover that fact as part of looking for the crossings.

Method #2 (the winding number method): Add up the angles subtended at P by the edges of the polygon, with appropriate signs, and divide the total by 2π. The result is called the *winding number* (the number of times that the polygon winds around P). If the winding number is 1 or -1, P is inside; if it is 0, P is outside.

In Method #2, for each i you will need to find the signed angle between the vector $P_{i-1} - Q$ and the vector $P_i - Q$, using the method of Handout C, §5, (vi). It is a good idea to check separately that Q is not equal to any of the points P_i, so that the difference vectors are both nonzero. You can check that Q is not on the boundary when you are finding the angles.

9. Barycentric coordinates for a triangle

Proposition 3. Let P, Q, R be three noncollinear points in $\mathbf{R}^2$. Then every point S in the plane can be written uniquely in the form $S = c_1P + c_2Q + c_3R$, where $c_1 + c_2 + c_3 = 1$.

Definition. In the Proposition, c_1, c_2, c_3 are called the *barycentric coordinates* of S with respect to P, Q, R. Let's write $(c_1, c_2, c_3)_{bary}$ for the barycentric coordinates of a point.

For example, the center of mass C of the triangle PQR is the obtained by averaging the three vertices, so has barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})_{bary}$. In fact, "barycentric" means "weight-centered". As another example, P itself has barycentric coordinates $(1, 0, 0)_{bary}$. See Figure 4.

Barycentric coordinates are often handy when you need to do something with a triangle that treats all three vertices the same way. The points inside the triangle are easily identified:

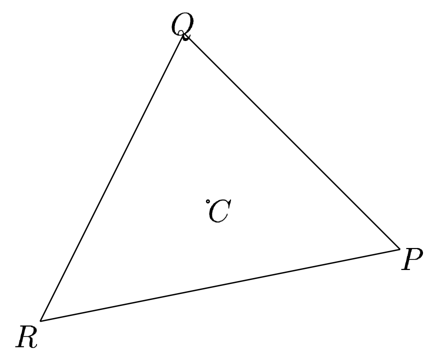

Figure 4: Triangle and its center of mass

Proposition 4. A point S is inside the triangle PQR (or on an edge) if and only if its three barycentric coordinates with respect to P, Q, R are all nonnegative.

Recall that a *convex combination* is a linear combination in which the coefficients are nonnegative and have sum 1. Thus the points inside the triangle are the convex combinations of P, Q, and R.

Proposition 5. The barycentric coordinates of the point $\mathbf{x}$ with respect to P, Q, and R have the values

$$c_1 = \frac{\Delta(Q, R, \mathbf{x})}{\Delta},$$

$$c_2 = \frac{\Delta(R, P, \mathbf{x})}{\Delta},$$

$$c_3 = \frac{\Delta(P, Q, \mathbf{x})}{\Delta},$$

where $\Delta = \Delta(Q, R, P) = \Delta(R, P, Q) = \Delta(P, Q, R)$.

Thus c_3 (for example) is obtained by scaling the two-point relational form for the line through P and Q, so is an affine function of position itself. We could write $c_3(\mathbf{x})$. The scale factor of $\frac{1}{\Delta}$ is just what is needed to have $c_3(R)$ = 1, as it should be. In particular, c_3 is zero on the line through P and Q, and is constant on each line parallel to that line.

10. Problems

KK-1. Explain how any line written functionally can be easily rewritten in (a) the usual relational form; (b) in parametric form.

KK-2. Describe how to draw a diagram of n circles, each of radius r, whose centers are on a circle of radius R (not drawn) centered at the origin, with connecting line segments as shown. Say explicitly how to find the coordinates

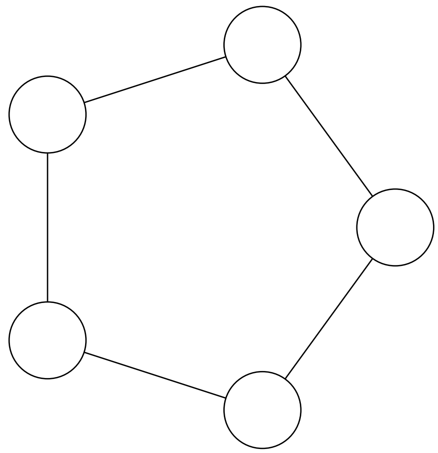

Figure 5: A diagram with circles

of the centers of the circle and of the end points of the line segments. See Figure 5.

KK-3. (a) For an affine function $f(x,y) = ax + by + c$, what is the gradient? (b) For most functions, the gradient depends on the position (x,y). Is this the case for affine functions? (c) What does the graph $z = f(x,y)$ look like?

KK-4. Verify Observation 4 of §3. (Method: What is the relation between the gradient of a function of two variables and the direction of increase?)

KK-5. Verify Observation 3 of §3, using each of these two methods. First method: Consider two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on the line and check that (a, b) is perpendicular to $P - Q$. Second method: The gradient of a function of two variables at any point is perpendicular to the level curve of the function through that point. (See the preceding problem.)

KK-6. If the point-normal equation mentioned in Observation 5 of §3 is rewritten as $ax + by + c = 0$, what is c in terms of $\mathbf{N}$ and P_0?

KK-7. Explain Observation 6 of §3 by using the point-normal form. (For given $\mathbf{x}$, choose P_0 to be at the foot of the perpendicular from $\mathbf{x}$.)

KK-8. The relational and parametric representations of a line can each be interpreted in terms of affine functions between spaces. The *relational* form really expresses the line as the set of points where a certain affine transformation $f : \mathbf{R}^2 \to \mathbf{R}^1$ has value 0 (as mentioned in Observation 6 of §3). The *parametric* form really expresses the line as the image (set of all values) of an affine transformation $\mathbf{P}{:}\mathbf{R}^1 \to \mathbf{R}^2$. In each case, find the extended matrix of the affine transformation involved.

KK-9. Suppose you want to rotate a line 30° counterclockwise about the origin. (a) If the line is given in relational form, how can you describe the new coefficients a', b', c' in terms of the old a, b, c? (b) If the line is given parametrically, how can you describe the new P_0 and $\mathbf{v}$ in terms of the old P_0 and $\mathbf{v}$? (Leave the answers in terms of products of matrices and vectors, without multiplying out.) (c) If the line is given in functional form $y = mx+b$, express the functional form of the rotated line in terms of m and b.

KK-10. Given two points P, Q, suppose we started from the equation $\Delta(P, Q, \mathbf{x}) = 0$. In §4 it is explained how we could be sure the equation has the form $ax + by + c = 0$. (a) In this equation, how could we be sure that not both a and b are zero, so that it does represent some line L? (An equation $0x + 0y + 0 = 0$ would represent the whole plane; an equation $0x + 0y + c = 0$ with $c \neq 0$ would represent the empty set.) (b) From the definition of $\Delta(P, Q, \mathbf{x})$, why is it obvious that both P and Q are on L?

KK-11. Show that the circle in $\mathbf{R}^2$ through three noncollinear points $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ is given by

$$\det \begin{bmatrix} a_1^2 + a_2^2 & a_1 & a_2 & 1 \\ b_1^2 + b_2^2 & b_1 & b_2 & 1 \\ c_1^2 + c_2^2 & c_1 & c_2 & 1 \\ x^2 + y^2 & x & y & 1 \end{bmatrix} = 0.$$

Use these three steps: (a) Show that all three points satisfy the equation. (b) By expanding the determinant in a suitable way, show that the equation has the form $Ex^2 + Ey^2 + Fx + Gy + H = 0$, where $E \neq 0$. (c) Explain why the graph of any equation of this form is a circle or a single point or the empty set. (By (a), though, the graph is not a single point or empty, so it's a circle.)

KK-12. (a) (A high-school problem) Inscribe a triangle in a semicircle of radius 1 so that one side of the triangle is the diameter of the circle. What choice of the third vertex of the triangle minimizes the area outside the triangle? (b) (The relevance for us) Given a fixed base $\overline{PQ}$ for a triangle and specified area A, describe the set of all points R for which the triangle PQR has area A, and say what this answer has to do with $\Delta(P, Q, R)$.

KK-13. (a) If you are looking at two points P and Q from a point R and $\Delta(P, Q, R) > 0$, is Q to the right or to the left of P, as seen from R? (b) In $\mathbf{R}^2$, is Q to the right or to the left of P as seen from R, where $P = (2, 3)$, $Q = (3, 4)$, $R = (8, 6)$?

KK-14. Show that if P, Q, R are three points in $\mathbf{R}^2$, then $\Delta(P, Q, R)$ equals the third component of the cross product in $\mathbf{R}^3$ of $(P - R, 0)$ and $(Q - R, 0)$. (Use properties of determinants.)

KK-15. For points in $\mathbf{R}^3$, suppose we write $\Delta(P, Q, R, \mathbf{x}) = ax + by + cz + d$. The corresponding normal vector to the plane through P, Q, and R is (a, b, c). To see if this normal vector is slanted up (so that the "positive half-space" is the half-space above the plane rather than below), we need to check that $c > 0$. The problem: Show that $c = -\Delta(\overline{P}, \overline{Q}, \overline{R})$, where $\overline{P}$, $\overline{Q}$, $\overline{R}$ are orthographic projections on the x, y-plane. (Method: Expansion by cofactors.)

KK-16. Show that it is not possible to have exactly three of the four numbers $\Delta(P, Q, A)$, $\Delta(P, Q, B)$, $\Delta(A, B, P)$, $\Delta(A, B, Q)$ be zero.

KK-17. (a) Consider the three line segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, where $A = (4, 1)$, $B = (1, 6)$, $C = (4, 4)$, $D = (3, 2)$, $E = (3, 5)$, $F = (2, 4)$. Find explicitly the corresponding affine functions from the two-point form. (Give coefficients numerically.) (b) Using (a) determine which pairs among these line segments cross. (There are three possible pairs to consider.) (c) Find the points at which the pairs you listed in (b) cross.

KK-18. In Case 2 of §5, list as many different ways as possible for how the four numbers $\Delta(P, Q, A)$, $\Delta(P, Q, B)$, $\Delta(A, B, P)$, $\Delta(A, B, Q)$ could be negative, zero, or positive. Make a sketch to illustrate each way. To eliminate some cases that are similar to others, assume that $\Delta(P, Q, A) \leq \Delta(P, Q, B)$, that $\Delta(A, B, P) \leq \Delta(A, B, Q)$, that $\Delta(P, Q, A) \leq \Delta(A, B, P)$, and that not all four numbers are zero.

KK-19. Explain explicitly step-by-step how a computer could determine whether the line segments $\overline{AB}$ and $\overline{PQ}$ intersect, where $A = (-10, 0)$, $B = (30, 2)$, $P = (50, 3)$, $Q = (10, 1)$.

KK-20. Is the point $P = (6, 5)$ inside the non-convex polygon with vertices $A = (7, 5.5)$, $B = (9, 4)$, $C = (7, 8)$, $D = (5, 4.5)$, $E = (3, 6)$, $F = (5, 2)$ (in order)?

Do this problem twice, by using Method #1 and then Method #2 of §8. (For this particular polygon and choice of P, all angles should come out to be multiples of $\frac{1}{4}\pi$.)

KK-21. If line segments $\overline{AB}, \overline{CD}$ in $\mathbf{R}^3$ are viewed from above at infinity , does one appear to pass above the other, and if so, which, where $A = (1, 2, 0)$, $B = (4, -1, 3)$, $C = (4, 1, 1)$, and $D = (3, -1, 2)$? Use a method suitable for a computer.

(Suggested: First determine whether the projections of $\overline{AB}, \overline{CD}$ on the x, y-plane cross. If they do, then use the same t, u for the original segments to find the two points that appear to the viewer to be on top of one another. See which of the two points has the greater z-value.)

KK-22. Consider any line L in $\mathbf{R}^2$. Designate the half-plane on one side as "positive" and the other half-plane as "negative". The corresponding *signed distance function* $D(x, y)$ is simply the perpendicular distance of (x, y) from L if (x, y) is in the positive half-plane, or minus that distance if (x, y) is in the negative half-plane. The signed distance function ought to be an affine function defining L. Is it? (To see, start with any affine function defining L and scale it, using Observation 6 of §3.)

KK-23. *Proposition.* The distance of the point (x, y) from the line with equation $ax + by + c = 0$ is the absolute value of $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}}$.

Prove this proposition by using Observation 6 of §3. (The fraction is an affine function. What is the length of its normal?)

KK-24. In $\mathbf{R}^2$, consider the triangle PQR, with $P = (8, 1)$, $Q = (3, 6)$, $R = (13, 11)$. Find the barycentric coordinates of these points: Q itself, $A = (8, 6)$, $B = (5, 4)$, $C = (9, 7.5)$.

KK-25. Sketch a triangle PQR in the plane. (a) Indicate on it the point with barycentric coordinates $(\frac{1}{2}, 0, \frac{1}{2})_{bary}$ and the point with $(0, \frac{1}{2}, \frac{1}{2})_{bary}$. (b) Indicate *all* points with $c_3 = \frac{1}{2}$. (c) Indicate all points with $c_3 = .25$ and all points with $c_3 = .75$.

KK-26. Show that barycentric coordinates in a triangle are invariant under translation, and in fact are invariant under any affine transformation. (Method: Starting from a relation $S = c_1 P + c_2 Q + c_3 R$ with $c_1 + c_2 + c_3 = 1$, apply an affine transformation $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$ and see if the same relation holds between the image points P', Q', R', S', where $P' = T(P)$ and so on.)

KK-27. Prove Proposition 3 of §9. (Method: Given S, write $S - R$ as a linear combination of the linearly independent vectors $P - R$ and $Q - R$.)

KK-28. Prove Proposition 5 of §9. (Take the vector equation $c_1 P + c_2 Q + c_3 R = S$ and write it as two equations, one for each coordinate. Then with the equation $c_1 + c_2 + c_3 = 1$, you have three equations in three unknowns. Use Cramer's rule to solve them, and show that you get the result wanted. Use the fact that $\det A = \det A^t$.)

KK-29. Take a triangle PQR such that $\Delta(P, Q, R) > 0$. For each point in the plane, each of the three barycentric coordinates could be < 0, $= 0$, or > 0, making $3 \cdot 3 \cdot 3 = 27$ conceivable outcomes. Break the plane into regions (some of which could be single points or pieces of lines) based on outcomes that actually occur. How many regions are there (twenty-seven, or some smaller number)? Make a sketch with labels for each of your regions.

KK-30. Invent barycentric coordinates for a tetrahedron in $\mathbf{R}^3$ and state facts similar to those in §9.

KK-31. Describe in detail an algorithm to draw the pattern shown in Figure 6. Here $P = (1,1)$, $Q = (5,3)$, $R = (3,6)$.

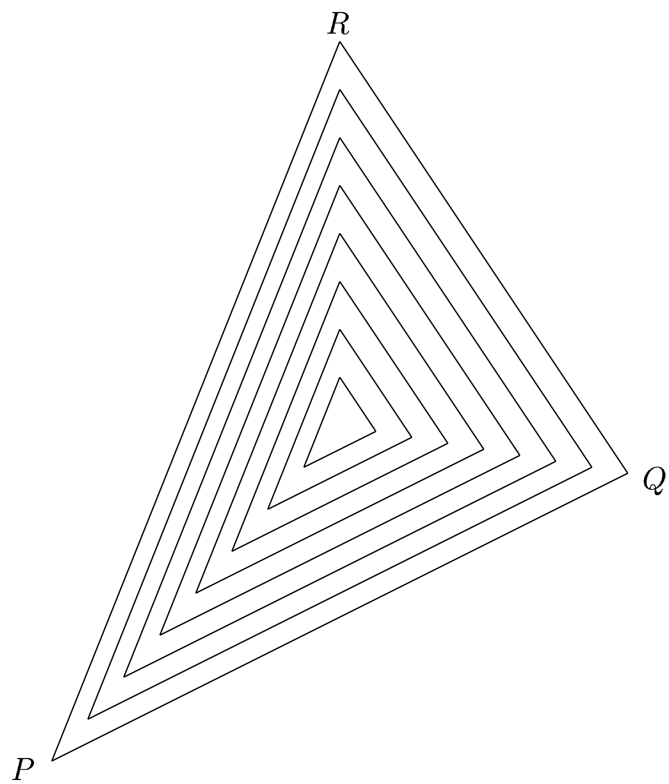

Figure 6: A pattern of telescoping triangles

KK-32. Describe in detail an algorithm to draw the pattern shown in Figure 7. Assume that you have a plotting package that will draw a circle if you tell it the center and the radius.

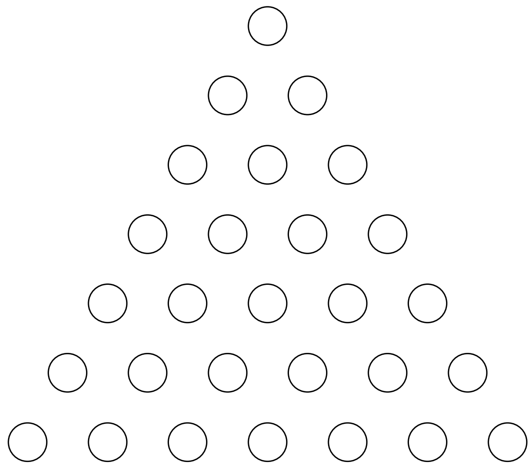

Figure 7: A pattern with circles

KK-32. Is the point $(6, 8)$ inside the triangle with vertices $(6, 7)$, $(3, 4)$, $(7, 10)$? Use a method suitable for a computer.