Affine transformations, Part I

In most practical problems, you need transformations that are like homogeneous linear transformations except that they can move the origin. These are the affine transformations.

1. Translations

Definition. A *translation* is a transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ of the form $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$, where $\mathbf{b}$ is a fixed vector.

Example. $T(\mathbf{x}) = \mathbf{x} + (1, 2)$. (See Figure 1.)

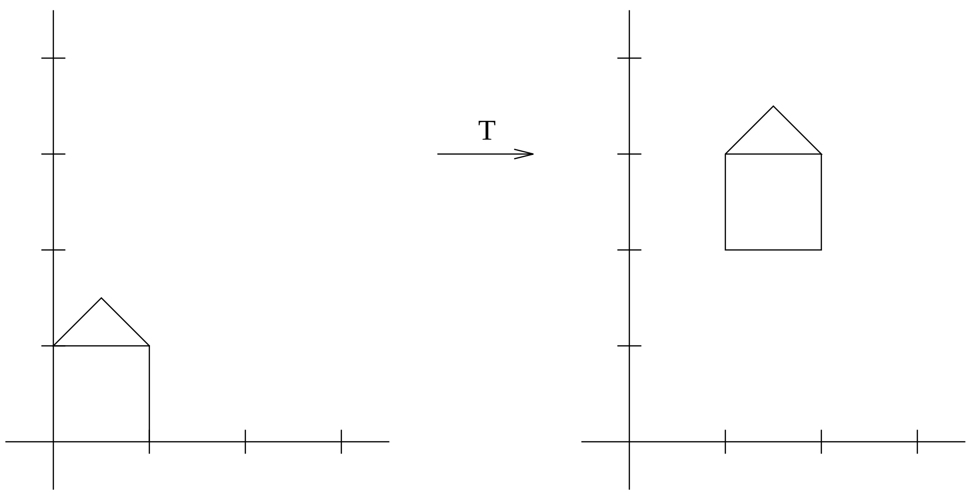

Figure 1: A translation

Observe that because a translation moves the origin, it is definitely not a homogeneous linear transformation (unless $\mathbf{b} = \mathbf{0}$).

2. Affine transformations

Definition. $T : \mathbf{R}^n \to \mathbf{R}^n$ is an *affine transformation* if T is a homogeneous linear transformation followed by a translation. In other words, there are a matrix A and vector $\mathbf{b}$ such that $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$ for all $\mathbf{x}$.

Example 2.1 . $T(\mathbf{x}) = \mathbf{x}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + (5, -1)$.

In other words, $T(x, y) = (2x + 3y + 5, x + 4y - 1)$.

Example 2.2 . Any translation: $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ (the case $A = I$).

Example 2.3 . Any homogeneous linear transformation $T(\mathbf{x}) = \mathbf{x}A$ (the case $\mathbf{b} = \mathbf{0}$).

Example 2.4 . A rotation about an arbitrary center in $\mathbf{R}^2$, say by 90° about the center $(2, 1)$. (It will be explained later how to represent such a rotation as affine. Another example is a rotation about an arbitrary axis in $\mathbf{R}^3$.)

Example 2.5 . A reflection in an arbitrary mirror line in $\mathbf{R}^2$, say in the mirror $x + y = 2$. (It will be explained later how to represent such a reflection as affine. Another example is a reflection in an arbitrary plane mirror in $\mathbf{R}^3$.)

Example 2.6 . If you want to transform world coordinates with a window $-2 \leq x_{world} \leq 6$ and $-1 \leq y_{world} \leq 5$ to device coordinates with $0 \leq x_{dev} \leq 400$ and $0 \leq y_{dev} \leq 300$, you can see by fiddling that the appropriate conversion is $x_{dev} = 50(x_{world} + 2)$ and $y_{dev} = 50(y_{world} + 1)$. This is the same as the affine transformation $\mathbf{x}_{dev} = \mathbf{x}_{world} \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} + (100, 50)$. (For more complicated problems of this kind it will be better to use affine transformation methods directly.)

Example 2.7 If one affine transformation is followed by another, the result (their composition) is still affine: If $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$ and $U(\mathbf{x}) = \mathbf{x}C + \mathbf{d}$, then $T(U(\mathbf{x})) = (\mathbf{x}C + \mathbf{d})A + \mathbf{b} = \mathbf{x}(CA) + (\mathbf{d}A + \mathbf{b})$, the form of an affine transformation. (Notice that $\mathbf{d}A + \mathbf{b}$ is constant. The resulting composition is called $T \circ U$.)

Remarks 2.8

(1) In the definition of an affine transformation, it is important to realize that in the expression $\mathbf{x}A + \mathbf{b}$, the homogeneous linear transformation is applied *before* the translation. If you did the translation first, you would be evaluating $(\mathbf{x} + \mathbf{b})A$, which equals $\mathbf{x}\mathbf{A} + \mathbf{b}'$ for $\mathbf{b}' = \mathbf{b}A$. Thus the result is affine but is not the *same* affine transformation.

(2) In a transformation $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$, let us call A the *homogeneous part* and $\mathbf{b}$ the *translational part*. Almost all of the possible complexity is in the homogeneous part: The determinant of A tells the expansion factor for T, T is one-to-one if A is nonsingular, and so on. Thus affine transformations are easy to understand if you already know the

properties of homogeneous linear transformations. This is why linear algebra courses usually do not mention affine transformations.

3. Extended matrices

For a vector $\mathbf{x} = (x, y)$, the corresponding *extended vector* is $\hat{\mathbf{x}} = (x, y, 1)$. For an affine transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$, the corresponding *extended matrix* is the 3×3 matrix $\hat{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ b_1 & b_2 & 1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ \mathbf{b} & 1 \end{bmatrix}$.

As you see, $\hat{A}$ contains all the information needed for T. A key observation is that if you use extended vectors and matrices, affine transformations can be computed with just a single matrix multiplication:

$$\hat{\mathbf{x}}\hat{A} = (x, y, 1) \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ b_1 & b_2 & 1 \end{bmatrix} = (a_{11}x + a_{21}y + b_1, a_{12}x + a_{22}y + b_2, 1) = (\mathbf{x}A + \mathbf{b}, 1) = T(\mathbf{x})\hat{}.$$

Application 3.1 . If you have a matrix multiplication routine, you can use it directly to compute affine transformations. For example, for T as in Example 2.1, $T(7, 8)$ can be found by computing $(7, 8, 1) \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 5 & -1 & 1 \end{bmatrix} = (43, 38, 1)$, so $T(7, 8) = (43, 38)$.

Application 3.2 . If T and U are composed as in Example 2.7, then $\hat{C}\hat{A}$ gives the extended matrix for $T \circ U$. (Here $\hat{A}$ is the extended matrix for T and $\hat{C}$ for U.)

Application 3.3 . If T has extended matrix A then the inverse transformation T^{-1} has extended matrix $\hat{A}^{-1}$ (if A is invertible).

Example. Find the extended matrix for the rotation by 90° about the center $(2, 1)$.

Solution. Use the three-step method of moving to an easy location (moving the center to the origin), rotating by 90°, and then moving back. The middle step is a homogeneous linear transformation, so its extended matrix has a zero translation part. The first and third steps are translations, so their homogeneous part is I. Thus the answer is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

(Notice that in the three-step method it is most natural to find the third matrix first, the one that takes the easy position to the harder position. The first matrix is the inverse of the third. Here, where the first and third are translations, the inverse is obvious.)

4. Affine transformations in three dimensions

Everything works exactly the same: For $\mathbf{x} = (x, y, z)$, $\hat{\mathbf{x}}$ is $(x, y, z, 1)$. The extended matrix of an affine transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ is 4×4.

Application 4.1 . Rotations in $\mathbf{R}^3$ about axes that do not go through the origin.

Application 4.2 . Reflections in $\mathbf{R}^3$ in plane mirrors that do not go through the origin.

5. Preservation properties

First, recall that if T is a *homogeneous linear transformation* then T preserves linear combinations: $T(r\mathbf{v} + s\mathbf{w}) = rT(\mathbf{v}) + sT(\mathbf{w})$. In fact, a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is homogeneous linear if and only if T preserves all linear combinations.

Affine transformations do not preserve usually linear combinations. For example, if T is a translation, given by $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$, then $T(\frac{1}{2}\mathbf{v} + \frac{1}{4}\mathbf{w}) = \frac{1}{2}\mathbf{v} + \frac{1}{4}\mathbf{w} + \mathbf{b}$, but $\frac{1}{2}T(\mathbf{v}) + \frac{1}{4}T(\mathbf{w}) = \frac{1}{2}(\mathbf{v} + \mathbf{b}) + \frac{1}{4}(\mathbf{w} + \mathbf{b}) = \frac{1}{2}\mathbf{v} + \frac{1}{4}\mathbf{w} + \frac{3}{4}\mathbf{b}$, not the same thing.

However, there is one kind of linear combination that *is* preserved:

Theorem 5.1 . Affine transformations preserve linear combinations in which the sum of the coefficients is 1.

Summary of proof. First check the case of a translation. Then combine that case with the case of a homogeneous linear transformation to get the conclusion for all affine transformations.

5.2 . *Examples* of such linear combinations:

(a) $\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$, the average of two vectors $\mathbf{v}$ and $\mathbf{w}$.

(a*) $\frac{1}{2}P + \frac{1}{2}Q$, the midpoint of the line segment joining two points P, Q. (Remember that points and vectors are essentially the same thing: pairs of numbers, or triples of numbers, etc.)

(b) $(1-t)P + tQ$ for $0 \le t \le 1$, the line segment joining P and Q.

(b*) $(1-t)P + tQ$ for all t, the whole line through P and Q.

(c) $P - Q + R$.

Corollary 5.3 . An affine transformation takes line segments to line segments and lines to lines, if it is nonsingular. (If it is singular, it can take a line to a point.)

Actually, Corollary 5.3 can be improved to an "if and only if" statement, i.e., a $\Leftrightarrow$ statement, in the nonsingular case:

Theorem 5.4 . A one-to-one transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ takes lines to lines $\Leftrightarrow$ T is affine and nonsingular.

(The proof of the "$\Rightarrow$" direction is quite difficult.)

6. Problems

Problem H-1. (a) Write down the extended matrices of (i) a translation by $\mathbf{b}$ in $\mathbf{R}^2$ and (ii) the homogeneous linear transformation $\mathbf{x} \to \mathbf{x}A$ where A is 2×2.

(b) Multiply matrix (i) by matrix (ii). Do you get $\hat{A}$?

(c) Multiply matrix (ii) by matrix (i). Do you get $\hat{A}$?

This problem shows how to write the extended matrix of an affine transformation as the product of the extended matrices of its constituent homogeneous linear transformation and translation. Notice that it matters in which order you do the two constituents!

Problem H-2. Pick two examples of translations in $\mathbf{R}^2$ and multiply their extended matrices in both orders. Does the order make a difference in this case?

Problem H-3. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by
$T(x, y) = (3x + 4y + 7, x - 2y + 5)$. Find the extended matrix of T.

Problem H-4. Show that if you take the product of two extended matrices, the product of their homogeneous parts gives the homogeneous part of the answer.

Problem H-5. Actually, it makes sense to consider an affine transformation between spaces of different dimensions, $T : \mathbf{R}^n \to \mathbf{R}^m$, of the form $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$.

(a) What are the sizes of A and $\mathbf{b}$, in terms of m and n?

(b) Describe the extended matrix of T.

(c) Explain how a linear function such as $f(x, y) = ax + by + c$ is really affine.

(d) In fact, explain how the equation of a line $y = mx + b$ is really affine.

Problem H-6. Suppose that $T : \mathbf{R}^m \to \mathbf{R}^n$ is an affine transformation. (a) Show that U given by $U(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$ is a homogeneous linear transformation. (b) How can you find a matrix for U in terms of the extended matrix for T?

Problem H-7. (a) Prove Theorem 6.1. (b) Prove the Corollary to Theorem 6.1.

Problem H-8. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the affine transformation given by $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$.

(a) Suppose L is a line with parametric equation $\mathbf{x} = P + t\mathbf{v}$. Show that $T(L)$, the image of L when transformed by T, is again a line, the line through $T(P)$ with direction $\mathbf{v}A$, unless $\mathbf{v}A = \mathbf{0}$, in which case $T(L)$ is a single point.

(b) Show that if L and L' are parallel lines (or the same line), then $T(L)$ and $T(L')$ are either parallel lines or the same line or two points or the same point.

Problem H-9. Are (a) and (b) of Problem H-8 valid for $T : \mathbf{R}^n \to \mathbf{R}^n$? For $T : \mathbf{R}^m \to \mathbf{R}^n$?

Problem H-10. Suppose that P, Q, R lie on a line in $\mathbf{R}^2$, and suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is affine. Let $P' = T(P)$, $Q' = T(Q)$, $R' = T(R)$. Show that the ratio of the lengths of the line segments $\overline{PQ}$ and $\overline{PR}$ is the same as the ratio of the lengths of the line segments $\overline{P'Q'}$ and $\overline{P'R'}$. (Assume $P \neq R$. Suggestion: Use the parametric equation of the line, with $t = 0$ at P and $t = 1$ at R.)

Problem H-11. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$. (a) Write down and expand an expression for $T^8(\mathbf{x})$, or in other words, $T(T(T(T(T(T(T(T(\mathbf{x}))))))))$. (Not too pretty.)

(b) In contrast, if $\hat{A}$ is the extended matrix of T, what expression in terms of $\hat{A}$ gives the extended matrix of T^8?

(c) Find the extended matrix of T^8 if $T(x, y) = (y + 1, x + y)$. (Use (b) and the idea $\hat{A}^8 = ((\hat{A}^2)^2)^2$.)

Problem H-12. If A is an invertible 2×2 matrix and $\mathbf{b}$ is a vector, give an expression for the inverse of $\hat{A} = \begin{bmatrix} A & \mathbf{0}^t \\ \mathbf{b} & 1 \end{bmatrix}$. Do this problem three ways:

(a) by writing down another block matrix $\hat{C}$, with unknown pieces, and seeing what the pieces have to be in order to have $\hat{A}\hat{C} = I$ (if possible working with matrices in blocks instead of writing individual entries);

(b) by writing $\mathbf{y} = \mathbf{x}A + \mathbf{b}$ and then solving for $\mathbf{x}$ and seeing what affine transformation you have in terms of $\mathbf{y}$;

(c) by writing $\hat{A}$ as the product of the extended matrices of the constituent homogeneous linear transformation and translation (as in Problem H-1) and then inverting.

(d) A naïve person might think that the inverse should be $\begin{bmatrix} A^{-1} & \mathbf{0}^t \\ -\mathbf{b} & 1 \end{bmatrix}$. Is it? (Here we write $\mathbf{0}^t$ instead of $\mathbf{0}$ to emphasize that $\mathbf{0}$ has been made into a column vector from a row vector.)

Problem H-13. Suppose $T : \mathbf{R}^n \to \mathbf{R}^n$ is an affine transformation, $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$. (a) Show that A and the extended matrix $\hat{A}$ of T have the same determinant. (Therefore if we say "the determinant of T" there is no ambiguity.)

(b) Explain why both the absolute value and the sign of the determinant of $\hat{A}$ have the same interpretations as for the homogeneous linear transformation determined by A.

Problem H-14. Find the affine transformation on $\mathbf{R}^2 \to \mathbf{R}^2$ that gives a rotation of 60° counterclockwise about the center $(1, 1)$. (Give explicit matrix entries.)

Problem H-15. Find the image of the origin under the affine transformation on $\mathbf{R}^2 \to \mathbf{R}^2$ that gives a rotation of 90° *clock*wise about the center $(3, 4)$. (Give explicit vector entries.)

Problem H-16. In $\mathbf{R}^2$, let P be a point that is not the origin, and let T be the translation that moves the origin to P. This problem contrasts different ways of combining rotations and translations.

(a) What matrix expression describes the rotation by angle θ with center P? (Use the three-step method, involving R_θ and T. Leave your answer as a matrix product.)

(b) What matrix expression describes T followed by the rotation of (a)?

(c) What matrix expression describes a rotation by an angle θ about the origin, followed by the translation T?

(d) What matrix expression describes performing T and then rotating about the *origin*?

(e) Take θ to be $45°$ and P to be $(2,1)$. Make a rough sketch, as follows. Draw axes and a scale of perhaps one unit = one inch. Draw the two standard basis vectors starting from the origin. Then for each of (b), (c), (d), indicate the images of the standard basis vectors under the transformation described. Be especially careful in (d).

Problem H-17. Find the affine transformation on $\mathbf{R}^2 \to \mathbf{R}^2$ that gives a reflection whose mirror is the line $y = 5$. (To move the line to the origin, you can move *any* point on the line to the origin.)

Problem H-18. For a rotation in $\mathbf{R}^2$ about an arbitrary center as in §3, is the extended matrix necessarily a rotation matrix?

Problem H-19. An affine transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be *rigid* if it preserves distances. Show that an affine transformation is rigid $\Leftrightarrow$ its homogeneous part is an orthogonal matrix.

(Method: Since T is affine, one can write (i) $T = H \circ U$, where H is the homogeneous part of T and U is the translation; then also (ii) $H = T \circ U^{-1}$. A translation is certainly rigid, the inverse of a translation is a translation, and the composition of two rigid transformations is rigid. Use these principles with (i) to show $\Leftarrow$, and with (ii) to show $\Rightarrow$.)

Note. If the affine transformation preserves orientation as well as being rigid, then the homogeneous part will have positive determinant and so will be a rotation.

Problem H-20. (a) Prove that if A is an $n \times n$ matrix with no nonzero fixed vector (i.e., no nonzero $\mathbf{x}$ with $\mathbf{x}A = \mathbf{x}$), then $I - A$ is invertible. (For a method, combine these facts from linear algebra: Having a nonzero fixed vector is the same thing as having 1 as an eigenvalue. The eigenvalues of $I - A$ are the numbers $1 - \lambda$ for eigenvalues λ of A. If 0 is *not* an eigenvalue of a square matrix, then the matrix is invertible. Assume A has real entries. Although A could still have complex eigenvalues and eigenvectors, in this problem it's enough to deal just with real ones.)

(b) Use (a) to prove this fact: For an affine transformation $T(\mathbf{x}) = \mathbf{x}A + \mathbf{b}$ in $\mathbf{R}^n$, if the homogeneous part A has no nonzero fixed vector, then T does have a fixed vector, and it is unique. In other words, if A has just one fixed vector (the origin), then so does T. (Method: If $\mathbf{c}$ is a fixed vector of T, find a formula for $\mathbf{c}$ in terms of A and $\mathbf{b}$. For an application, see the next problem.)

Problem H-21. As you know, a rotation about the origin in $\mathbf{R}^n$ is the same thing as a rigid homogeneous linear transformation (i.e., orthogonal transformation) that preserves orientation (i.e., has positive determinant). For affine transformations, a rotation about any origin is rigid and preserves orientation, but any translation also has these properties.

(a) Show that in $\mathbf{R}^2$, these are the only possibilities. In other words, a rigid affine transformation in $\mathbf{R}^2$ that preserves orientation is either a translation or else is a rotation about some origin. (You may use the results of Problem H-19.)

(b) In $\mathbf{R}^3$, give an example of a rigid affine transformation that is neither a translation nor a rotation about some axis. (Think in terms of Problem H-19, where the condition about the homogeneous part now fails. What if the translational part is along the axis of the homogeneous part?)

Problem H-22. Show that if P is a 2×2 rotation matrix and is not the identity matrix, and if $P^m = I$, then for any vector $\mathbf{b}$, also $\begin{bmatrix} P & \mathbf{0}^t \\ \mathbf{b} & 1 \end{bmatrix}^m = I$. (Method: View the second matrix as the augmented matrix of an affine transformation and use (a) of Problem H-21.)

Problem H-23. Find the affine transformation that represents a rotation in $\mathbf{R}^3$ about an axis going through the points $P = (2, 3, 4)$ and $Q = (3, 4, 5)$, $90°$ counterclockwise as viewed from Q looking towards P. Leave the answer as a product of matrices with explicit entries, but compute any inverses. (Method: Transform the axis to an easy position; rotate; undo.)

Problem H-24. Stand balanced on one foot. Using your heel as a center, turn your shoe by any nonzero angle θ. Then using your toe as a new center, turn your shoe by an angle -2θ. Then using your heel as a new center again, turn by an angle θ. What has happened? What does this have to do with affine transformations? (Do this problem experimentally, not by writing formulas.)

Problem H-25. (a) Give an example of two rotations in $\mathbf{R}^2$, about different centers, whose composition is a translation (by a non-$\mathbf{0}$ vector). (b) How can this idea be used if you need to move a heavy four-legged table across a room, by yourself?

Problem H-26. Here is a problem that sometimes arises in Engineering: You have a solid object, such as a rectangular plate. You want it to be able to move smoothly between two positions in space, the initial and final positions. Maybe this can be accomplished by a rotation or maybe not, but suppose it can be, for these two particular positions. The axis might be somewhere else in space away from the object, or it might go through the object. How can the axis be found geometrically?

(Here is an idea that may suggest a solution: Suppose A is one point on the object in its initial position, and $T(A)$ is the final position of the same point. Let P be any point on the axis. Since a rotation is rigid, A and $T(A)$ have the same distance from P. Think of this the other way around: P is equidistant from $T(A)$ and A. Therefore P is on the plane that perpendicularly bisects the line segment from A to $T(A)$.)