Review notes for vectors

We shall be interested principally in two and three dimensions.

1. Notation

A triple of numbers (x, y, z) can be viewed as a point in three-dimensional space, in which case we'll usually write $P = (x, y, z)$. It can also be viewed as a vector $\mathbf{x}$ or $\mathbf{v}$, in which case we'll often write $\mathbf{x} = (x, y, z)$. We may also write $\mathbf{x} = (x_1, x_2, x_3)$, especially when talking about computer algorithms. Another alternative is $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}$. We'll emphasize row vectors, as most computer graphics texts do, rather than column vectors, as many linear algebra texts do. Therefore generally we consider a column vector as the transpose of a row vector and write $\mathbf{x}^t = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Vectors may be drawn starting from any point, rather than just from the origin.

$\mathbf{R}^3$ = the set of all triples = "real 3-space". We usually picture $\mathbf{R}^3$ in the way you are used to, with the x, y-plane horizontal and the z-axis vertical. Sometimes in computer graphics it is better to use other positions, though.

Notation for $\mathbf{R}^2$ and $\mathbf{R}^n$ is similar. Usually $\mathbf{R}^3$ will be used in examples when the dimension doesn't matter.

In $\mathbf{R}^3$, the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, as usual. In $\mathbf{R}^2$, $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

Notice that if $\mathbf{x} = (x, y, z)$, then $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. For this reason, the entries x, y, z can be called the *components* of $\mathbf{x}$, or the *coordinates* of $\mathbf{x}$, or the *coefficients* of $\mathbf{x}$. This equation also illustrates how every vector is a linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (uniquely). Thus $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are a basis for the vector space $\mathbf{R}^3$, called the *standard basis*. In $\mathbf{R}^n$ for general n, we write the standard basis as $\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(n)}$, so that in $\mathbf{R}^3$ we have $\mathbf{e}^{(1)} = \mathbf{i}$, $\mathbf{e}^{(2)} = \mathbf{j}$, $\mathbf{e}^{(3)} = \mathbf{k}$.

2. Vectors

Vectors have length and direction. Often, we'll be using a vector $\mathbf{v}$ to indicate a direction, and will talk of "the direction $\mathbf{v}$". The *length* or *norm* or *magnitude* of $\mathbf{v}$ is $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

If $\mathbf{u}$ has length 1, it is a *unit vector*. If $\mathbf{v}$ is any nonzero vector, then $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector with the same direction as $\mathbf{v}$, and $\mathbf{u}$ is said to be obtained by *normalizing* $\mathbf{v}$. Here the word "normal" means "of length 1".

Be careful—the word "normal" is also used to mean "perpendicular".

A unit vector is often used to describe a direction. Its components are sometimes called its *direction cosines*, for a reason explained under C.

Addition and subtraction of vectors are performed "coordinatewise". So is multiplication of a vector by a scalar.

Handy: If P and Q are points, the vector from P to Q is $Q - P$.

3. Lines

We usually represent lines in parametric form: $\mathbf{x} = P_0 + t\mathbf{v}$, where P_0 is a fixed point, $\mathbf{v}$ is a fixed vector giving the direction, and t is a parameter, which we can regard as time.

This is the same as saying (in three dimensions) $\begin{cases} x = p + at \\ y = q + bt \\ z = r + ct \end{cases}$

where $P_0 = (p, q, r)$ and $\mathbf{v} = (a, b, c)$.

4. The dot product

The *dot product* or *scalar product* or *inner product* of vectors $\mathbf{v}$ and $\mathbf{w}$ (say in $\mathbf{R}^3$) is $v_1w_1 + v_2w_2 + v_3w_3$, denoted $\mathbf{v} \cdot \mathbf{w}$. In geometrical terms, $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. The dot product makes sense in $\mathbf{R}^n$ for any n.

The dot product has many uses:

(1) $|\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}$.

(2) $\mathbf{v} \perp \mathbf{w}$ precisely when $\mathbf{v} \cdot \mathbf{w} = 0$. (The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors.)

(3) The angle θ between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$ satisfies $\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$. Notice, though, that $\cos\theta = \cos(-\theta)$, so the sign of θ is not determined by the dot product, even in $\mathbf{R}^2$, where it is natural to say a counterclockwise angle is positive and a clockwise angle is negative.

(4) By (3), if $\mathbf{v} \cdot \mathbf{w} > 0$ then the angle between the vectors is between $0°$ and $90°$ in absolute value; if $\mathbf{v} \cdot \mathbf{w} < 0$, the angle is between $90°$ and $180°$ in absolute value.

(5) $\mathbf{i} \cdot \mathbf{v}, \mathbf{j} \cdot \mathbf{v}, \mathbf{k} \cdot \mathbf{v}$ are the components of $\mathbf{v}$, so $\mathbf{v} = (\mathbf{i} \cdot \mathbf{v}, \mathbf{j} \cdot \mathbf{v}, \mathbf{k} \cdot \mathbf{v})$.

(6) The dot product of two *unit* vectors is simply the cosine of the angle between them.

(7) By (5) and (6) together, you can see that the components of a **unit** vector $\mathbf{u}$ are the cosines of the angles that $\mathbf{u}$ makes with $\mathbf{i}, \mathbf{j}, \mathbf{k}$, or in other words, the cosines of the angles that $\mathbf{u}$ makes with the x-, y-, and z-axes. For this reason the components of a *unit* vector are called *direction cosines.*

(8) In a matrix product AB, each entry of AB is the dot product of a row of A with a column of B.

(9) Let $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbf{R}^n$, written as row vectors. Then $\mathbf{w}^t$ is a column vector. The matrix product $\mathbf{v}\mathbf{w}^t$ makes sense and its value is a 1×1 matrix, which is the same thing as a scalar. By (8), this scalar is just $\mathbf{v} \cdot \mathbf{w}$. To summarize: $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}\mathbf{w}^t$.

(10) In $\mathbf{R}^3$, the equation of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal $\mathbf{N} = (a, b, c)$ is $\mathbf{N} \cdot (\mathbf{x} - P_0) = 0$, or equivalently,

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, or equivalently,

$ax + by + cz + d = 0$, where $d = -\mathbf{N} \cdot \mathbf{P}_0$.

Similarly, in $\mathbf{R}^2$, the equation of the line through the point $P_0 = (x_0, y_0)$ perpendicular to the vector $\mathbf{N} = (a, b)$ is $a(x - x_0) + b(y - y_0) = 0$. However, usually we deal with lines parametrically, as in §3.

Note: For clarity, in this course let's try to use $\mathbf{n}$ when we mean a *unit* normal and $\mathbf{N}$ when we mean a normal of any nonzero length.

5. The projection of a vector on a line

Suppose a vector $\mathbf{v}$ is projected onto a line.

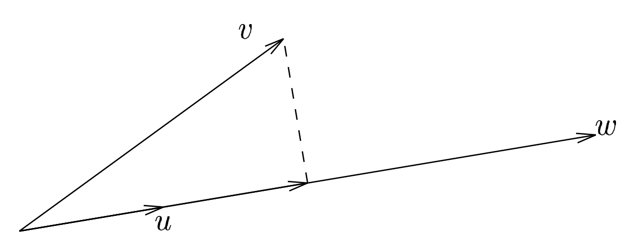

Figure 1: A vector projected onto a line

The *direction* of the line can be expressed two ways: By giving a *unit* vector $\mathbf{u}$ along the line, or by giving an arbitrary nonzero vector $\mathbf{w}$ along the line, in which case we can take $\mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|}$.

(1) The *length* of the projection of $\mathbf{v}$ on the line is $|\mathbf{v}|\cos\theta = \mathbf{v}\cdot\mathbf{u}$, where a negative sign means that the projection has direction opposite to $\mathbf{u}$. The length can also be expressed as $\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}$.

(2) The *vector* projection of $\mathbf{v}$ on the line is therefore $(\mathbf{v}\cdot\mathbf{u})\mathbf{u}$ (scalar times vector). This vector projection can be called the *vector component* of $\mathbf{v}$ in the direction of the line.

(3) If the direction of the line is expressed with an arbitrary nonzero vector $\mathbf{w}$, the vector projection of $\mathbf{v}$ on the line becomes $\frac{(\mathbf{v}\cdot\mathbf{w})\mathbf{w}}{|\mathbf{w}|^2}$ or $\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$.

(4) The vector component of $\mathbf{v}$ *perpendicular* to the line is just $\mathbf{v}$ minus the vector component *along* the line, so it is equal to $\mathbf{v} - (\mathbf{v}\cdot\mathbf{u})\mathbf{u}$ or $\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$.

(5) A plane is described using a normal $\mathbf{N}$ or unit normal $\mathbf{n}$. The vector component of $\mathbf{v}$ *perpendicular* to the plane is $(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ or $\frac{\mathbf{v}\cdot\mathbf{N}}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N}$ and the vector component of $\mathbf{v}$ *along* the plane (i.e., the projection of $\mathbf{v}$ on the plane) is $\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ or $\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{N}}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N}$. These sound the other way around from (3) and (4) because $\mathbf{N}$ is already perpendicular to the plane.

6. The cross product

The *cross product* or *vector product* of $\mathbf{v}$ and $\mathbf{w}$, denoted $\mathbf{v}\times\mathbf{w}$, is defined only in $\mathbf{R}^3$. Geometrically, $\mathbf{v}\times\mathbf{w}$ is a vector with direction perpendicular to $\mathbf{v}$ and $\mathbf{w}$, of length $|\mathbf{v}||\mathbf{w}||\sin\theta|$ (the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$). So far, this description fits two vectors (if the length is nonzero), one the negative of the other. If vectors are drawn from the origin, $\mathbf{v}\times\mathbf{w}$ is the one from whose end $\mathbf{w}$ appears to be counterclockwise from $\mathbf{v}$. To remember this relationship, just remember that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, and relate that picture to your $\mathbf{v}$ and $\mathbf{w}$. Algebraically,

$$\mathbf{v}\times\mathbf{w} = \left(\det\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix}, -\det\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix}, \det\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}\right),$$

(or you can write the middle term as $\det\begin{bmatrix} v_3 & v_1 \\ w_3 & w_1 \end{bmatrix}$, so that the three coordinates change cyclically without a minus sign). There is an easy way of remembering these relations: Use $\mathbf{v}\times\mathbf{w} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ and expand by cofactors of the first row. This doesn't make very good sense mathematically, as you usually don't take determinants of matrices with vectors as entries, but it is handy as an aid to remembering.

Cross products have several uses in this course:

(1) They are just what is needed to find a vector perpendicular to two given vectors in $\mathbf{R}^3$.

(2) The angle θ between two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbf{R}^3$ satisfies $|\sin\theta| = \frac{|\mathbf{v}\times\mathbf{w}|}{|\mathbf{v}||\mathbf{w}|}$.

(3) For vectors in $\mathbf{R}^2$, let us denote by $D(\mathbf{v}, \mathbf{w})$ the useful determinant

$$D(\mathbf{v}, \mathbf{w}) = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}.$$

If you make $\mathbf{v}$, $\mathbf{w}$ into vectors in $\mathbf{R}^3$ by giving each a third component of 0, their cross product becomes $(0, 0, D(\mathbf{v}, \mathbf{w}))$. Thus in $\mathbf{R}^3$, it is easy to compute the cross product of two vectors in the x, y-plane, and the result lies along the z-axis.

(4) Two nonzero vectors have the same direction if and only if their cross product is 0. (If they are drawn from the origin, this would mean they lie along the same line.)

(5) In $\mathbf{R}^2$, the area of the parallelogram with sides given by $\mathbf{v}$ and $\mathbf{w}$ is $A = |D(\mathbf{v}, \mathbf{w})|$.

(6) In (3), the cross product of $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{R}^2$ (or better, in the x, y-plane of $\mathbf{R}^3$) points *up* in $\mathbf{R}^3$ if $\mathbf{w}$ is counterclockwise from $\mathbf{v}$ and *down* if $\mathbf{w}$ is clockwise from $\mathbf{v}$. Thus in $\mathbf{R}^2$, $\mathbf{w}$ is counterclockwise from $\mathbf{v}$ precisely when $D(\mathbf{v}, \mathbf{w}) > 0$.

(7) In $\mathbf{R}^2$, the sine of the angle θ between two vectors is given by

$\sin\theta = \frac{D(\mathbf{v},\mathbf{w})}{|\mathbf{v}||\mathbf{w}|}$.

$\sin\theta$ does distinguish between positive and negative θ, but notice that $\sin 80° = \sin 100°$, for example, so $\sin\theta$ is not enough information to determine θ uniquely.

7. Signed angles between vectors in the plane—a practical guide

In $\mathbf{R}^3$, given two vectors $\mathbf{v}$, $\mathbf{w}$ at the origin, in talking about the angle θ between them you might as well allow only values $0 \le \theta \le \pi$, because the same angle can look clockwise or counterclockwise when seen from different directions. To find θ, it's enough to use the dot product, which works in $\mathbf{R}^n$ for any n. As in Section **3** above:

(1) $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$.

In $\mathbf{R}^2$, though, it makes sense to ask for θ with a sign. $\theta > 0$ means a counterclockwise angle and $\theta < 0$ means a clockwise angle. Possible values would be $-\pi < \theta \leq \pi$. This time dot products alone are not enough, because with cosines you can't tell the difference between θ and $-\theta$; for example, $\cos(-30°) = \cos 30°$.

You might at first consider using $\sin\theta$ instead. In fact, there is a corresponding formula, as noted in section **5** above:

(2) $\sin\theta = D(\mathbf{v},\mathbf{w})/|\mathbf{v}||\mathbf{w}|$, where $D(\mathbf{v},\mathbf{w}) = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

However, with sines you can't tell the difference between θ and $\pi - \theta$; for example, $\sin 30° = \sin 150°$.

What about the tangent function? From (1) and (2) we would get

(3) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{D(\mathbf{v},\mathbf{w})}{\mathbf{v}\cdot\mathbf{w}}$,

where the denominators have been canceled. But the tangent has a similar problem: it can't distinguish between θ and $\theta + \pi$.

One way to get an exact θ would be to use *two* trig functions. Since you're solving for θ you would use two of the arc functions, which in C and C++ are `acos( )`, `asin( )`, and `atan( )`. The first would give you an angle in a specific range such as $0 \leq \theta \leq \pi$ for `acos( )`, and then you would use the second to change the angle if warranted (say, negating the angle if `asin( )` is negative).

Fortunately, though, there is a single function in C and C++ that is designed expressly for this kind of application: the `atan2(y,x)` function. This function finds $\arctan\frac{y}{x}$ while paying attention to whether x and y are positive, negative, or zero; it returns a value between $-\pi$ and π, as desired. It works even if the denominator `x` is zero!

Thus in C or C++ (using subscripts 0,1 for 1,2), you can find θ by

(4) `theta = atan2(v[0]*w[1]-v[1]*w[0], v[0]*w[0]+v[1]*w[1]);`

There will be an error if both arguments are zero.

Note. In some versions of C and C++ this function may be built in; in others you may need to use `#include<math.h>` and to compile with a flag `-lm`. This kind of function is also available in other languages, but check which argument is for `x` and which for `y`.

8. Problems

[not to hand in unless assigned]

Problem B-1. Suppose P, Q, R, S are vertices of a parallelogram in $\mathbf{R}^2$, with the line PQ parallel to RS and SP parallel to QR. Find an algebraic expression for S in terms of P, Q, and R. (Notice that if P, Q, R are given, there's only one possible place for S.)

Problem B-2. Find the equation of the plane through points $P = (1, 2, 1)$, $Q = (3, 4, 3)$, $R = (2, 2, 3)$ in $\mathbf{R}^3$.

Method: Find the plane through P with normal $(Q - P) \times (R - P)$.

Problem B-3. Find the angle between the planes $x + 2y + z = 0$ and $3x + y + z = 0$.

Method: Find the angle between their normals. The angle between two planes can be looked at so that $0 \leq \theta \leq 90°$, so adjust your answer if necessary to achieve this.

Problem B-4. Find the line of intersection of the two planes in the preceding problem.

Method: The line wanted is perpendicular to both normals and goes through the origin.

Problem B-5. Find the line of intersection of the two planes $x+2y+z-3 = 0$ and $3x + y + z - 4 = 0$.

Method: The line wanted has the same direction as in the preceding problem, but does not go through the origin. To find a point on the line, one way is to pick an arbitrary value for one of x, y, and z and solve the resulting two-variable equations simultaneously to get values for the other two variables.

Problem B-6. In $\mathbf{R}^2$, find the cosine of the angle between the lines $3x + 4y + 7 = 0$ and $4x + 3y - 2 = 0$. (Do like the similar problem above for planes in $\mathbf{R}^3$.)

Problem B-7. If P, Q, R are as in Problem B-2, are the points A, B on the same side of the plane through P, Q, R, where $A = (1, 2, 3)$ and $B = (1, 2, 5)$?

Method #1: Find the equation of the plane in the form $ax + by + cz + d = 0$, and see whether the left-hand side has the same sign when evaluated at A and at B.

Method #2: Let $\mathbf{N} = (Q-P) \times (R-P)$ and see whether the angles between $A-P$ and $\mathbf{N}$ and between $B-P$ and $\mathbf{N}$ are both between $0°$ and $90°$ or both between $90°$ and $180°$.

Problem B-8. Find the vector projection of $\mathbf{v} = (1, 1, 1)$ on the line in the direction of $\mathbf{w} = (1, 2, 2)$.

Problem B-9. Find the vector that is the projection of the vector $\mathbf{v} = (1, 1, 1)$ on the plane $x + 2y + 2z = 0$.

Problem B-10. Find the signed angle between $(1, 2)$ and $(2, -1)$ using the method of §7. For `atan2`, just evaluate it as a computer would.