Notes

This is to explain further what I said in class about the laws of vector spaces. Since the kind of reasoning involved is a little advanced, this won't be on the midterms or final, but it's worth looking at to see what the issues are.

The laws used in the definition of a vector space are *some* of the laws true in F^n for every n.

Additional laws not part of the definition include basic ones such as $0v = \mathbf{0}$ but also miscellaneous ones that could be proved, such as $(r + s)(v + w) = rw + (sv + (sw + rv))$.

Near the beginning of the course I mentioned this fact:

Proposition. The defining laws of vector spaces over R are enough to imply *all* the laws that hold in $\mathbb{R}^n$ for all n, in the sense that any vector space does satisfy all these laws.

(The same is true for any field F in place of $\mathbb{R}$, but let's stick to $\mathbb{R}$.)

At the time it was not clear how a statement like this could be proved. But now it can be. First, a proof that doesn't quite work:

?? "Proof." If V is a vector space over $\mathbb{R}$, then $V \cong \mathbb{R}^n$, so all the laws of $\mathbb{R}^n$ hold in V.

The trouble with this attempted proof is that a vector space might not be finite dimensional and so might not be isomorphic to $\mathbb{R}^n$, no matter how n is chosen. (Notice that in the attempted proof, n comes out of nowhere.)

Valid proof. Given any law that holds in $\mathbb{R}^n$ for all n, we want to try to verify it in V. The law has a certain number of variables, say m. We could write the law using variable symbols $v_1, \ldots , v_m$. To show that the law holds in V, we need to check that the law is true for all possible ways of putting specific vectors for $v_1, \ldots , v_m$. With specific vectors $v_1, \ldots v_m$, we can take the subspace W that they span. We know that W, being the span of a finite list of vectors, *is* finite-dimensional, say of dimension n (with $n \leq m$). Then $W \cong \mathbb{R}^n$, so the law in question, being true in $\mathbb{R}^n$, is true in W for those specific vectors.

That's the proof. Notice that each time a different law and list of specific vectors is checked, a different W is used.