Notes

Sample problem and solutions for Quiz 3: If u, w, v are linearly independent vectors in a vector space over $\mathbb{R}$, show that $u + w, u + 2v + 3w, u + v$ are linearly independent.

Solution #1: Suppose $r(u + w) + s(u + 2v + 3w) + t(u + v) = 0$. Expanding and collecting coefficients we see this is the same as $(r+s+t)u+(2s+t)v+(r + 3s)w = 0$. Since u, v, w are linearly independent, we have $r + s + t = 0, 2s+t = 0, r+3s = 0$, or $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Row-reducing we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $r = s = t = 0$. Therefore any linear relation between the given vectors is trivial and so they are linearly independent.

Solution #2: u, w, v are a basis for the subspace W they span, so there is an isomorphism of $\mathbb{R}^3 \cong W$ taking the standard basis vectors e_1, e_2, e_3 to u, w, v. Via this isomorphism the question becomes whether $e_1+e_3, e_1+2e_2+3e_3, e_1+e_2$ are linearly independent—in other words, $(1, 0, 1)$, $(1, 2, 3)$, and $(1, 1, 0)$. Make a matrix with these as columns (or rows) and row reduce. We get the identity matrix, so these vectors are linearly independent. Therefore the original vectors were linearly independent.

Comment on problem done in lecture:

The question asked by one of you was whether the following transformation is an isomorphism:

$f : \text{Pols}(\mathbb{R}, 3) \to \text{Pols}(\mathbb{R}, 4)$ given by

$f(a + bx + cx^2 + dx^3) = (a + bx + cx^2 + dx^3)(x + 1)$.

This can't be true, since the transformation is between spaces of different dimensions, but let's see if we can get a better understanding of f by using isomorphisms. This is a little more complicated than some problems because there are two vector spaces involved, of dimensions 4 and 5, and so we are replacing them by two spaces, $\mathbb{R}^4$ and $\mathbb{R}^5$.

First, we can multiply out the product of polynomials and collect coefficients of powers to get

$f(a + bx + cx^2 + dx^3) = a + (a + b)x + (b + c)x^2 + (c + d)x^3 + dx^4$.

Now let's use the fact that $\mathbb{R}^4 \cong \text{Pols}(\mathbb{R}, 3)$ via

$T(r, s, t, u) = r + sx + tx^2 + ux^3$. We also use the fact that $\mathbb{R}^5 \cong \text{Pols}(\mathbb{R}, 4)$ via

$S(q, r, s, t, u) = q + rx + sx^2 + tx^3 + ux^4$.

So we are replacing the spaces on both ends of f, and f itself is replaced by f by $\bar{f} : \mathbb{R}^4 \to \mathbb{R}^5$ given by

$\bar{f}(a, b, c, d) = (a, a + b, b + c, c + d, d)$.

Is $\bar{f}$ linear? Yes.

Is $\bar{f}$ one-to-one? We can check whether $f(a, b, c, d) = f(a', b', c', d') \Rightarrow (a, b, c, d) = (a', b', c', d')$, and the answer turns out to be yes.

Is $\bar{f}$ onto? We can check whether, given any $(p, q, r, s, t) \in \mathbb{R}^5$, it is possible to find (a, b, c, d) with $f(a, b, c, d) = (p, q, r, s, t)$: This is the same as the equations $a = p, a + b = q, b + c = r, c + d = s, d = t$, where a, b, c, d are variables and p, q, r, s, t are constants. Either directly or by matrices, we get $a = p, b = q - p, c = r + p - q, d = s - p + q - r = t$, so $p - q + r - s + t = 0$. Therefore we can't choose just any $(p, q, r, s, t) \in \mathbb{R}^5$ and $\bar{f}$ is not onto. Therefore f is not onto, as expected.

Note: $\bar{f}$ can be expressed in matrix form as

$$\bar{f}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a \\ a+b \\ b+c \\ c+d \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

This gives a clearer picture of f than the polynomial version.