NOTES

Here is the observation we have been using in class:

Observation 1. For an $m \times n$ matrix M whose columns are column vectors $M_1, \ldots, M_n$, and for $r_1, \ldots, r_n \in F$, the following are equivalent:

(1) $M\mathbf{r} = \mathbf{0}$, i.e., $\mathbf{r}$ is in the null space of M.

(2) $r_1 M_1 + \cdots + r_n M_n = \mathbf{0}$, i.e., $r_1, \ldots, r_n$ are coefficients for a linear relation of the columns of M.

(3) $r_1, \ldots, r_n$ are solutions of the set of homogeneous linear equations whose matrix of coefficients is M

The reason is simply that $r_1 M_1 + \cdots + r_n M_n = M\mathbf{r}$.

For example, $r_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + r_3 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + r_4 \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$.

From this we get:

Observation 2. The linear relations between columns of a matrix remain unchanged when you do row reduction.

The reason is that elementary row operations do not change the solutions to the set of homogeneous linear equations whose coefficient matrix is the matrix in question. According to Observation 1, these solutions give the coefficients for linear relations.

We also get a method:

Observation 3. Given a spanning set for a subspace of a vector space, to thin it out to a basis of the subspace, do this:

Step 1. Make a matrix M with the spanning set as its *columns*.

Step 2. Row-reduce to a matrix E in row-reduced echelon form.

Step 3. Look for the pivot columns of E.

Step 4. Choose the corresponding columns of the original matrix M to be the basis. For example, if the pivot columns in E are 1 and 4, then use columns 1 and 4 of M.

Why this works:

The pivot columns of E are clearly linearly independent, since each has an entry with value 1 where the others have value 0, and the other columns are linear combinations of the pivot columns. Since M has the same linear relations between columns as E (by Observation 2), the same statement is true about the corresponding columns of M.

Example: Find a basis for the subspace of $\mathbb{R}^2$ spanned by $(1,2)$, $(3,4)$, $(5,6)$, and $(7,8)$.

Solution. Make the matrix $M = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$ and row-reduce, getting $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. The first two columns are the pivot columns, so the first two vectors in the original list are a basis for the span of all four.