# Solutions to Assignment #6

p. 21, Ex. 3.

This problem can be solved by experimenting, but it's better to use an organized approach. I'll list several options.

(i) Just try matrices that have three zeros. You will find that good solutions are $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and its transpose.

(ii) Notice that $A$ should not be invertible, since a product of invertible matrices is invertible (rule $(AB)^{-1} = B^{-1}A^{-1}$) and the zero matrix is not invertible. Then $A$ doesn't have rank 2 and we also know it doesn't have rank 0 (which would be the zero matrix). So the rank of $A$ is 1. This should help experiments.

(ii′) Continuing with the rank 1 idea and recalling a previous problem, $A = uv^t$ for column vectors $u, v$. $A^2 = uv^tuv^t$, which is column·row·column·row. The middle row and column multiply to make a $1 \times 1$ matrix, which is just a scalar, and it had better be 0 or else $A^2$ won't be the zero matrix. So just pick any two column vectors $u, v$ with $v^tu = 0$ (which really says the dot product of the two is 0) and let $A = uv^t$. Example: $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; then $A = N$ above.

(iii) Plan what $\tau_A$ should do to standard basis vectors. If $\tau_A$ takes $e_1$ to $e_2$ and $e_2$ to $\mathbf{0}$, then doing $\tau_A$ twice will take both standard basis vectors to 0, which says $\tau_{A^2}$ is the zero transformation, as hoped. What is $A$? Its first column is $e_2$ and its second column is $\mathbf{0}$, so $A = N$.

(iv) [for later, after you have had the ingredients] If $A^2$ is the zero matrix, then the eigenvalues of $A$ also must obey $\lambda^2 = 0$, so the determinant and trace of $A$ will be 0. (The trace is the sum of the diagonal entries.) So $A$ should look like $\begin{bmatrix} a & \cdot \\ \cdot & -a \end{bmatrix}$. If $a \neq 0$ then to make the determinant come out 0 the missing entries should be $b$ and $-a^2/b$ for some $b \neq 0$. If $a = 0$ then to make the determinant come out 0 the missing entries should be $b$ and 0 either way around. To summarize, $A = \begin{bmatrix} a & b \\ -a^2/b & -a \end{bmatrix}$ (with $a, b \neq 0$) or $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ (with $b \neq 0$).

p. 21, Ex. 4.

I get

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$, $E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $E_7 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$.

Of course, someone else might use some other order of row-reduction and get other matrices.

Notice that the text uses $E$ for elementary matrices, while I have been using $E$ for a matrix in row-reduced echelon form.

This is really what Theorem 9 on p. 20 is saying for the case of elementary matrices.

p. 21, Ex. 7.

Here we are evidently supposed to do more than just to quote a theorem like the one on Handout S. The more we know, the easier it is to find a good solution. Some methods:

(i) The straightforward method is to write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ and solve $AB = I$ for B in terms of $A$. Then we can check directly that $BA = I$.

Details: As a warmup, think how we would do the problem if $A$ had numbers instead of letters, e.g., $\begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is equivalent to two separate linear systems: $\begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} r \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. They can be made into a single problem with an augmented matrix to row-reduce: $\begin{bmatrix} 3 & 7 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix}$. The answer for $B$ will be the right-hand part after row reduction. This is the same as a common method of computing an inverse, which is of course what we're doing. Starting with $\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$ and row-reducing, we eventually will get $\begin{bmatrix} 1 & 0 & \frac{d}{\Delta} & -\frac{b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$, where $\Delta =$

$ad - bc$ (Greek capital Delta). (If we divide by $a$, say, during the calculation then we have to split cases into $a \neq 0$ and $a = 0$, but the answer will come out the same either way.) $\Delta$ itself can't be 0 since there is a solution. Then we know $B = \begin{bmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$ or more simply, $\frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We can check directly that $BA = I$.

(ii) $A$ has to be of rank 2 since $I$ has rank 2 and rank can't increase under matrix multiplication. So row-reducing $A$ we get $I$. In other words, $E_k \dots E_1 A = I$. Multiplying by $B$ on the right we get $E_k \dots E_1 I = B$, so $B = E_k \dots E_1$ and also we have $BA = I$.

(iii) If $AB = I$ then $\tau_A \tau_B = \tau_{AB} = \mathbf{1}$ so $\tau_A$ is one-to-one so the nullity of $A$ is 0. Then the rank of $A$ is 2, so $\tau_A$ is onto. Since $\tau_A$ is an isomorphism, it has a two-sided inverse, which is $\tau_B$. Then $B$ is a two-sided inverse for $A$.

*Note.* This problem is related to a simple observation: If $f : X \to Y$ has both a left inverse $g$ and a right inverse $h$ then $g = h$. The reason: $g \circ f \circ h$ can be thought of as $(g \circ f) \circ h = \mathbf{1}_X \circ h = h$ and as $g \circ (f \circ h) = g \circ \mathbf{1}_Y = g$, so $h = g$.

In the special case of linear transformations on a finite-dimensional vector space to itself, a corollary of the dimension theorem was that if $T$ is one-to-one then $T$ is onto, so if $T$ has a left inverse then it has a right inverse also, and so a two-sided inverse.

p. 26, Ex. 1. It's simplest to augment $A$ with an identity matrix to keep track of the cumulative effect of the elementary row operations. The idea is that by doing row operations we are computing $PA$, and by doing the same row operations on $[A|I]$ we get $[PA|P]$ so we can see $P$ as well as the reduced version of $A$.

$$[A|I] = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \dots \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

This says that $PA = \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix}$ and $P = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$.

p. 27, Ex. 5.

Yes, scaling the last three rows by $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively and sweeping out columns using the leading 1's, we see that $A$ is elementarily equivalent to the identity matrix and so is invertible.

Other ways to see this, for the future:

The determinant of a triangular matrix is the product of the diagonal entries, so $\det A \neq 0$, which means $A$ is invertible.

The eigenvalues of a triangular matrix are the diagonal entries, and since none are 0 the matrix is nonsingular.

p. 27, Ex. 6.

The rank of $A$ is no more than 1, and the rank of $C = AB$ can't be larger than the maximum of the rank of $A$ and the rank of $B$, so rank $C = 1$. But $C$ is $2 \times 2$ so if $C$ were invertible its rank would have to be 2.

p. 74, Ex. 11.

This problem says to show that $\tau_A$ is the zero transformation $\Leftrightarrow$ $A$ is the zero matrix. Solution:

For "implies": if $\tau_A(v) = \mathbf{0}$ for all $v$ then in particular $\tau_A(e_j) = \mathbf{0}$ for all $j$, which is the same as saying that each column of $A$ is the zero vector, i.e., all entries of $A$ are 0.

For "impliedby": If $A$ is the zero matrix then $Av = \mathbf{0}$ for all $v$, which says $\tau_A$ is the zero transformation.

p. 74, Ex. 12.

Since the range and null space of $T$ are identical, the rank and nullity are the same, say $k$, and we know $\dim V = k + k = 2k$, so $\dim V$ is even.

Technically, the simplest example would be to have $V = \{\mathbf{0}\}$ and let $T$ be the zero transformation. A more meaningful example would be to take $V = F^2$ (since we know the dimension is even!), and let $\text{Range}(T) = \text{Nullspace}(T) = \text{span}(e_1)$. This can be accomplished by having $T(e_2) = e_1$ and $T(e_1) = \mathbf{0}$. In other words, $T = \tau_A$ for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

p. 84, Ex. 7.

A "linear operator" is a transformation on a vector space to itself. We know linear transformations on $\mathbb{R}^2 \to \mathbb{R}^2$ are all of the form $\tau_A$ for a suitable $2 \times 2$ matrix $A$, so we are looking for $2 \times 2$ matrices $A, B$ with $AB = 0, BA \neq 0$ (where 0 means the $2 \times 2$ zero matrix).

One way is to experiment with matrices that have lots of 0 entries. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ provide an example.

Another way is to realize that the nilpotent matrix $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and its transpose are often good examples of bad behavior, so try $A = N$, $B =$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and try to solve for the desired condition. You get an example similar to the one in the preceding paragraph.

p. 85, Ex. 1. $\mathbb{C} \cong \mathbb{R}^2$ by $a + bi \mapsto (a, b)$.

p. 86, Ex. 4.

We need to set up a correspondence between pairs of indices $i, j$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$) and indices running from 1 to $mn$. In other words, we need to specify an order for the pairs of indices and say how to list them in that order. Where does $(i, j)$ go?

One way is to list the matrix by row 1, then row 2, etc.:
$(1, 1), \ldots, (1, n), (2, 1), \ldots, (2, n)$, etc.
Each new row adds $n$ to the position, but row 1 adds nothing, so the contribution of the row index is off by 1. We get that the position of $(i, j)$ in the list is $n(i - 1) + j$. Then $F^{m\times n} \cong F^{mn}$ by having each matrix $A \mapsto (r_1, \ldots, r_{mn})$ given by $r_{n(i-1)+j} = A_{i,j}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

p. 86, Ex. 6.

For "$\Rightarrow$": If $V$ and $W$ are isomorphic we know that a basis of one maps to a basis of the other, so they have the same dimension.

For "$\Leftarrow$": If they have the same dimension, say $n$, we know they are both isomorphic to $F^n$ and so are isomorphic to each other.

**For Problem Q-1:**

(a) $(\forall y)(\exists x)(y = x^3 - x)$.

(b) $(\forall x)(\forall y)(x + y = y + x)$.

(c) $(\forall x)(x > 0 \Rightarrow x^2 > 0)$.

(d) $(\exists x)(1 - 8x + x^2 = 5)$.

(e) $(\exists x)(\exists y)(17x + 25y = 6 \text{ and } 101x - 37y = 13)$.

(f) $(\exists a)(\nexists x)(ax = x + 1)$.

(g) $(\exists a)(\forall x)(ax \neq x + 1)$ (or in other words, $(\exists a)(\forall x)(\text{not } ax = x + 1)$).

**Note:** In logic, $\forall$ and $\exists$ are called **quantifiers**. $\forall$ is the **universal quantifier** (since it says something is true universally) and $\exists$ is the **existential quantifier**.

Normally we don't write mathematics in such a condensed form, but it is important always to notice what the underlying quantification is.

**For Problem Q-2:**

The question is whether each example describes a partition. Reasons weren't required for positive answers, but I'll include them.

(a) No, because the problem says any two blocks are to have a plant in common rather than to be pairwise disjoint. The union of the blocks is the whole set, though.

(b) Yes, because the planes together fill up $\mathbb{R}^3$ and yet any two are disjoint (since they're parallel).

(c) Yes, like (b).

(d) No. The union of the 1-dimensional subspaces is the whole space, but two 1-dimensional subspaces intersect in $\{\mathbf{0}\}$ rather than the empty set.

(e) Yes; the sets of even integers and odd integers are disjoint and their union is all integers.

(f) Yes; any two blocks are disjoint and any integer is in one of the blocks.

(g) Yes; the union of the singleton subsets is the whole space and any two distinct singleton subsets have empty intersection.

(h) Yes. Recall that $f^{-1}(t)$ really means $f^{-1}(\{t\})$. The union of these inverse images is all of $S$ since $x \in f^{-1}(y)$ for $y = f(x)$. The inverse images are disjoint since each $x \in S$ goes to only one element of $T$ by the definition of a function.

Notice that we could be fancier and say that we are first partitioning the image of $f$ into singletons as in (g) and then using rules (iii) and (iv) of Problem O-4.

Also notice that (b) and (c) can be viewed as an application of (h) in any particular example if we can find a linear transformation on $S$ whose null space is the 1- or 2-dimensional subspace.

(i) No; blocks $A_r$ and $A_s$ (with $r \neq s$) are not disjoint since the constant function 0 is in both. There are other functionsn too, such as $(x - r)(x - s)$.

(j) Yes, this is a "tiling of the plane". Notice that it's important to use "half-open intervals" so the tiles fit together with no overlap and no holes between.

**Note:** An example not included that could have been is

(k) There is just one block, consisting of all of $S$. This does count as a partition.

**For Problem Q-3:**

(a) The union of the nonzero parts of 1-dimensional subspaces is the nonzero part of the vector space, since every nonzero vector is in the nonzero part of the 1-dimensional subspace it generates.

Any two 1-dimensional subspaces are disjoint except for $\mathbf{0}$, since if a nonzero vector $v$ is in both, then both subspaces consist of the nonzero scalar multiples of $v$ and so are the same.

(b) Let $F = \mathrm{GF}(q)$. Then $V \cong F^3$ and so $V$ has $q^3$ elements, so the nonzero part of $V$ has $q^3 - 1$ elements. A 1-dimensional subspace is isomorphic to $F^1 = F$ and so has $q$ elements; its nonzero part has $q - 1$ elements. Since $V \setminus \{\mathbf{0}\}$ is partitioned as in (a) and the blocks have equal size, the number of blocks is $(q^3 - 1)/(q - 1) = q^2 + q + 1$. (If you don't know how to factor $q^3 - 1$ as a polynomial in $q$, use long division of polynomials.)

For the case $q = 2$ this gives 7, which checks with the answer to Problem G-4.

**For Problem Q-4:**

(a) For $b = qa + r$, let's follow the suggestion and show that $a$ and $b$ have the same divisors as $r$ and $b$: If $d|a$ and $d|b$ then $d|qa$ and $d|b - qa$, i.e., $d|r$. Therefore any divisor of $a$ and $b$ is also a divisor of $r$ and $b$.

If $d|a$ and $d|r$ then $d|qa$ and $d|qa + r$, i.e., $d|b$. Therefore any divisor of $r$ and $b$ is a divisor of $a$ and $b$.

Here we have use the pretty obvious idea that if $d$ divides $a$ then $d$ divides any integer multiple of $a$ and also the idea that if $d$ divides two integers then it also divides their sum and difference. Since "$d$ divides $a$" means "$a = kd$ for some integer $k$", these rules are not hard to prove officially.

(b) $221 = 2 \cdot 91 + 39$; $91 = 2 \cdot 39 + 13$; $39 = 3 \cdot 13 + 0$, so the gcd is 13.

(There was some secret algebra in inventing this problem: $(10+3)(10-3) = 100 - 9 = 91$ and $(15+2)(15-2) = 15^2 - 2^2 = 225 - 4 = 221$. Both of these involve 13.)

**For Problem Q-5:**

Some informal reasoning that could be made official:

Run the Euclidean algorithm backwards. The integers involved should build up as slowly as possible when the final result, the gcd, is 1 and $q$ is 1 in every step—except the last (or the first going backwards), where $q = 1$ won't work since the list of numbers used has to increase. So going backwards, we have

$2 = 2 \cdot 1 + 0$
$3 = 1 \cdot 2 + 1$
$5 = 1 \cdot 3 + 2$
$8 = 1 \cdot 5 + 3$
$13 = 1 \cdot 8 + 5$
$21 = 1 \cdot 13 + 8$
$34 = 1 \cdot 21 + 13$
$55 = 1 \cdot 34 + 21.$

*Note* (not part this course). The sequence $1, 2, 3, 5, 8, \ldots$, in which each number from the third on is the sum of the preceding two, consists of the **Fibonacci numbers** (fib-o-na-chi). Actually it's better to start with $0, 1$, getting $0, 1, 1, 2, 3, 5, 8, \ldots$, even though that doesn't quite fit the Euclidean algorithm application.

Fibonnaci numbers have a close connection to the **golden mean** (golden ratio) of the ancient Greeks, the number that is the positive solution of the equation $r^2 = r + 1$, specifically $r = (1 + \sqrt{5})/2 \approx 1.618$. In fact, the $n$-th Fibonacci number is the integer closest to $r^n/\sqrt{5}$ if we agree that 0 is the 0-th Fibonacci number. For example, $r^9/\sqrt{5} \approx 33.9941$ and $r^{10}/\sqrt{5} \approx 55.0036$. Fibonacci numbers and the golden mean turn up in unexpected places.

**For Problem Q-6:**

$$\begin{bmatrix} 91 & 1 & 0 \\ 221 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 91 & 1 & 0 \\ 39 & -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 13 & 5 & -2 \\ 39 & -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 13 & 5 & -2 \\ 0 & -17 & 7 \end{bmatrix}.$$

As a check notice that the right $2 \times 2$ determinant is still 1. Now we can see that $(13, 5, -2)$ has to be $5(91, 1, 0) - 2(221, 0, 1)$, so $13 = 5 \cdot 91 - 2 \cdot 221$. This checks.

**For Problem Q-7:**

As suggested, $\begin{bmatrix} 101 & 1 & 0 \\ 97 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 41 & -1 & \\ 97 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & -1 \\ 1 & -24 & 25 \end{bmatrix} \rightsquigarrow$ $\begin{bmatrix} 0 & 97 & 101 \\ 1 & -24 & 25 \end{bmatrix}$. Then $(1, -24, 25) = -24(101, 1, 0) + 25(97, 0, 1)$, so $1 = -24 \cdot 101 + 25 \cdot 97$ (which is true). Then in $\mathbb{Z}_{101}$, $1 = 25 \cdot 97$ and the multiplicative inverse of 97 is 25.

The moral is that finding inverses in fields $\mathbb{Z}_p$ is fast and easy on a computer, even if $p$ is a large prime. (For a computer, large can mean *very* large–even 200 digits!)

**For Problem R-1:**

A 90° rotation, as a transformation on $\mathbb{R}^2 \to \mathbb{R}^2$, has no eigenvectors. (Later we'll see that it does have eigenvectors if complex numbers are allowed.)

A reflection in $\mathbb{R}^2$ has two eigenspaces (line of eigenvectors), one along the mirror line and one perpendicular to the mirror line.

A shear has just one eigenspace, and it doesn't help to allow complex numbers.

**For Problem R-2:**

(a) $P(D) = \begin{bmatrix} p(1) & 0 \\ 0 & p(2) \end{bmatrix}$, so we want $p(1) = 0$ and $p(2) = 0$. Therefore we should use $p(x) = (x-1)(x-2) = x^2 - 3x + 2$.

(b) For the same reason, let $p(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$.

**For Problem R-3:**

To review: Writing $\epsilon$ for an elementary matrix, when we do successive elementary row operations starting from $A$ we are making $\epsilon_1 A$, $\epsilon_2\epsilon_1 A$, etc. So if $B$ is the end result after $k$ elementary row operations, then $B = MA$, where $M = \epsilon_k \ldots \epsilon_1$. We know $M$ is invertible since $M$ can be undone by the inverses of the elementary row operations: $M^{-1} = \epsilon_1^{-1} \ldots \epsilon_k^{-1}$.

(a) If $A$ is row-reduced to $B$ then $MA = B$ as just explained. The isomorphism is simply $\tau_M$, since $\tau_M$ is invertible (so is an isomorphism) and takes each column of $A$ to the corresponding column of $B$.

One of you suggested that this point of view is another good way to see why linear relations among columns are preserved by row reduction even though the column space is changed: The column space is changed isomorphically!

(b) Since the isomorphism in (a) takes columns of $A$ to corresponding columns of $B$, a selection of columns that makes a basis for the column space of $A$ corresponds to the same thing for $B$.