Assignment #7

Quiz 7 in discussion section **Tuesday, November 13**:

Just be able to write down correctly the last example from lecture notes on 6-M showing how a linear transformation derived from a square matrix can have a simpler matrix with respect to a basis different from the standard basis, applied at both ends.

Specifically, we looked at the linear transformation $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$ with $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and found that its matrix with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (at both ends) is diagonal.

There will be different numbers instead of 3,2,2,3 but the basis will be the same. You will not be asked to mention or derive eigenvalues, even though this problem does illustrate them.

Assignment due nominally in lecture on Wednesday, November 14 but you can hand it in **Friday, November 16**.

where	Do but don't hand in	Hand in
p. 95	Ex. 3, 5	Ex. 2
V	V-4, V-5, V-6, V-9	V-7, V-13, V-14
	V10, V-11, V-12	V-15, V-16
W		W-1, W-2, W-3, W-4

Problem W-1. Count the number of invertible 3×3 matrices over $\mathrm{GF}(q)$. What is this number for $q = 2$?

Method: The rows should be a basis. For the first row any nonzero triple will do. The second row should not be in the span of the first row. The third row should not be in the span of the first two rows. How many choices at each step?

Problem W-2. For matrices over $\mathbb{R}$ or $\mathbb{C}$, it is possible to add up infinite series. Questions of convergence are similar to Math 33B, but we'll look only at examples that do converge.

(a) Find e^D for $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(Method: Take the usual series for e^x and put $x = D$. As you did for polynomials applied to matrices, the term 1 becomes I. Add up entries and simplify.)

(b) Find e^{tJ} for $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Here t is an unspecified scalar and J is the the same as a 90° rotation and also the same as the matrix used to represent $i = \sqrt{-1}$ in Problem G-6. You will need to compute the powers of J to see the pattern; then express each entry of e^{tJ} as a power series and see if it is familiar.

(c) Find e^{tN_4} (simplifying if possible), where N_4 is as in Problem V-5.

Problem W-3. Let V be the subspace of $\mathbb{R}^3$ that is the solution space of $x + y + z = 0$. Recall that early in the course we used V as an example of vector space that has no "best" basis.

The transformation τ_P on $\mathbb{R}^3$ given by $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ takes V to itself, so

if $T : V \to V$ is the "restriction" of τ_P to V then T is a linear transformation on V to itself. Notice that T is really a rotation on V by 120°.

What is the matrix of T with respect to the basis v_1, v_2 of V (used at each end of the transformation) if

(a) $v_1 = (1, -1, 0)$ and $v_2 = (1, 0, -1)$?

(b) $v_1 = (1, -1, 0)$ and $v_2 = (1, 1, -2)$?

Problem W-4. We have discussed the fact that one basis of a vector space V can always be mapped to another basis of V using a linear transformation[1]. In the case $V = F^n$, such a transformation will be τ_M for some matrix M. One easy case you know how to do is where the first basis is the standard basis and the second is possibly "nonstandard": Then the columns of M are the nonstandard basis.

(a) Working with $\mathbb{R}^2$, find M so that τ_M takes the nonstandard basis $(1, 1), (1, -1)$ to the standard basis e_1, e_2.

(Method: First solve the problem in the other direction, then invert. Remember that you need to rewrite the basis vectors as columns. Check your answer by seeing if τ_M does what it is supposed to.)

(b) Again working with $\mathbb{R}^2$, find M so that τ_M takes the nonstandard basis $(1, 1), (1, -1)$ to the nonstandard basis $(1, 2), (3, 4)$.

(Method: First solve the problem of taking the standard basis to the first nonstandard basis $(1, 1), (1, -1)$, getting a matrix A. Then solve the problem of taking the standard basis to the second nonstandard basis given, getting a matrix B. Finally, let $M = BA^{-1}$, since $\tau_B \circ \tau_A^{-1}$ takes the first basis to the standard basis and then to the second basis.)

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[1] Actually, a basis of n elements can be mapped to any list of n vectors.