Solutions to Assignment #5

p. 73, Ex. 4.

Yes, there is such a linear transformation: The two vectors $v_1 = (1, -1, 1)$ and $v_2 = (1, 1, 1)$ are not scalar multiples of each other and so are linearly independent. Therefore they can be extended to a basis v_1, v_2, v_3 of $\mathbb{R}^3$. By Theorem 1, given any three vectors w_1, w_2, w_3 in a second vector space, which can be $\mathbb{R}^2$, there is a linear transformation taking $v_i \mapsto w_i$ for each i. We can choose $w_1 = (1, 0)$, $w_2 = (0, 1)$ and $w_3 =$ anything we want, say $(0, 0)$.

In this problem we were not actually asked to find the transformation.

p. 73, ex. 5.

Notice that $\alpha_3 = -\alpha_1 - \alpha_2$. Since a linear transformation preserves linear relations, if such a T exists then we should have $\beta_3 = -\beta_1 - \beta_2$ also. But instead, $\beta_3 = \beta_1 + \beta_2 \neq -\beta_1 - \beta_2$. Therefore such a linear transformation does not exist.

p. 73, Ex. 6.

By ϵ_i the authors mean our standard basis vector e_i. We know $T = \tau_A$ for $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. To display the answer in a form like Example 1, calculate $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$, which says $T(x, y) = (ax + cy, bx + dy)$.

p. 73, Ex. 8.

This is a transformation on $\mathbb{R}^3 \to \mathbb{R}^3$, so we can use τ_A where A is a 3×3 matrix. The range of τ_A is the column space of A, so take A to have columns $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. For the third column of A we can take any vector already in the span of those two columns, say $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So we can choose

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix}.$$

For Problem O-1:

Advice: First write down some examples of functions $f : \{1,2,3\} \to \mathbb{R}$. You will see that you need a list of three numbers for each, so each function corresponds to a triple. Furthermore, functions add pointwise and triples coordinatewise, so that vector addition is compatible with the correspondence, and similarly for multiplication by scalars. That gives the following idea.

Given $f : \{1,\dots,n\} \to F$, define $\phi(f) = (f(1),\dots,f(n))$. Then ϕ : Functions$(\{1,\dots,n\} \to F) \to F^n$ is one-to-one and onto. Also, ϕ is a linear transformation, because the pointwise operations on functions correspond to the coordinatewise operations on n-tuples.

The importance of this problem is the perspective that it gives: Our first examples of vector spaces included some consisting of n-tuples and some consisting of functions. Now it is becoming apparent that even the n-tuples were really made of functions in disguise. How about examples such as Mat$(F, m \times n)$? This could be viewed as Functions(S, F), where S is the set $\{(i,j)|1 \le i \le m, 1 \le j \le n\}$ of pairs of indices, or more simply, $S = \{1,\dots,m\} \times \{1,\dots,n\}$.

For Problem O-2: (b) Just make sure that both of $T(e_1)$ and $T(e_2)$ are on the same line. One of them could be the zero vector.

A *shear* is a transformation that moves points parallel to some line. The line is the x-axis in the case of this program.

For Problem O-3:

(a) Yes: For closure under addition,, suppose $w_1, w_2 \in T(S)$. We must show that $w_1 + w_2 \in T(S)$. To say $w_i \in T(S)$ is the same as saying that $w_i = T(v_i)$ for some $v_i \in S$ $(i = 1,2)$. Since S is a subspace, $v_1 + v_2 \in S$, and since T preserves addition, $T(S)$ contains $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$. For closure under multiplication by scalars ... [continue with similar proof].

(b) Yes: For closure under addition, suppose $v_1, v_2 \in T^{-1}(U)$, which is the same as saying that $T(v_i) \in U$. We must show that $v_1 + v_2 \in T^{-1}(U)$, which is the same as saying that $T(v_1 + v_2) \in U$. Since T is linear, $T(v_1 + v_2) = T(v_1) + T(v_2)$, which *is* in U because U is closed under addition. Therefore $v_1 + v_2 \in T^{-1}(U)$ as required. For closure under multiplication by scalars ... [continue with similar proof].

For Problem O-4:

(ii) is false, while (i), (iii), (iv) are true. (iv) is proved in the handout.

For (i):

$y \in f(A_1 \cup A_2) \overset{(1)}{\Leftrightarrow} y = f(x)$ for some $x \in A_1 \cup A_2 \overset{(2)}{\Leftrightarrow} y = f(x)$ for some x with $x \in A_1$ or $x \in A_2 \overset{(3)}{\Leftrightarrow} y = f(x)$ for some $x \in A_1$ or some $x \in A_2 \overset{(4)}{\Leftrightarrow}$ $y \in f(A_1)$ or $y \in f(A_2) \overset{(5)}{\Leftrightarrow} x \in f(A_1) \cup f(A_2)$, where the reasons for the double implications are (1) definition of f(subset), (2) definition of $\cup$, (3) the way “for some” (really $\exists$) interacts with “or”, (4) definition of f(subset), (5) definition of $\cup$. Therefore $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,

For (ii):

The clearest kind of example is one where $A_1 \cap A_2 = \emptyset$ but $f(A_1) \cap f(A_2)$ is not empty. For instance let $X = \{1, 2\}$, let $Y = \{3\}$, define $f : X \to Y$ by $f(1) = f(2) = 3$, let $A_1 = \{1\}$, and let $A_2 = \{2\}$. Then $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$ but $f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\} \neq \emptyset$.

For (iii):

$x \in f^{-1}(B_1 \cup B_2) \overset{(1)}{\Leftrightarrow} f(x) \in B_1 \cup B_2 \overset{(2)}{\Leftrightarrow} f(x) \in B_1$ or $f(x) \in B_2 \overset{(3)}{\Leftrightarrow}$
$x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2) \overset{(4)}{\Leftrightarrow} x \in f^{-1}(B_1) \cup f^{-1}(B_2)$,

where the reasons for the double implications are (1) definition of f^{-1}, (2) definition of $\cup$, (3) definition of f^{-1}, (4) definition of $\cup$. Therefore
$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.