Nonsingular matrices and transformations

This concerns *square* matrices and also transformations $T : V \to W$ where V and W have the *same* finite dimension.

Theorem. Let A be an $n \times n$ matrix with entries in a field F, with the corresponding matrix transformation $\tau_A : F^n \to F^n$. Also let $T : V \to W$ be a linear transformation between vector spaces over F, both of dimension n, such that T has matrix A with respect to particular bases of V and W. The following conditions are equivalent.

1. $\det A \neq 0$.
2. A row-reduces to the $n \times n$ identity matrix.
3. A has rank n ("full rank", meaning the maximum rank possible).
4. The rows of A are linearly independent.
5. The columns of A are linearly independent.
6. *Some* system of linear equations with coefficient matrix A has a unique solution.
7. *Every* system of linear equations with coefficient matrix A has a unique solution.
8. A has a right inverse, i.e., there is an $n \times n$ matrix B with $AB = I$.
9. A has a left inverse, i.e., there is an $n \times n$ matrix B with $BA = I$.
10. A has a two-sided inverse A^{-1}.
11. A has nullity 0; in other words, nullspace $\tau_A = \{\mathbf{0}\}$.
12. $Av = \mathbf{0} \Rightarrow v = \mathbf{0}$
13. 0 is not an eigenvalue of A, i.e., there is no $v \neq 0$ with $Av = 0v$.
14. τ_A is one-to-one.
15. τ_A is onto.
16. τ_A is an isomorphism of F^n with itself (an "automorphism" of F^n)
17. Nullspace$(T) = \{\mathbf{0}\}$.

18. T is one-to-one.
19. T is onto.
20. T is an isomorphism.
21. The matrix of T with respect to any bases of V and W is nonsingular.

Definition. When any (and so all) of these conditions is satisfied, then A is said to be *nonsingular.* Otherwise A is *singular.*

("Singular" means "special" or "unusual", not to be confused with "single". So a system of linear equations with *nonsingular* coefficient matrix has a *single* solution.)