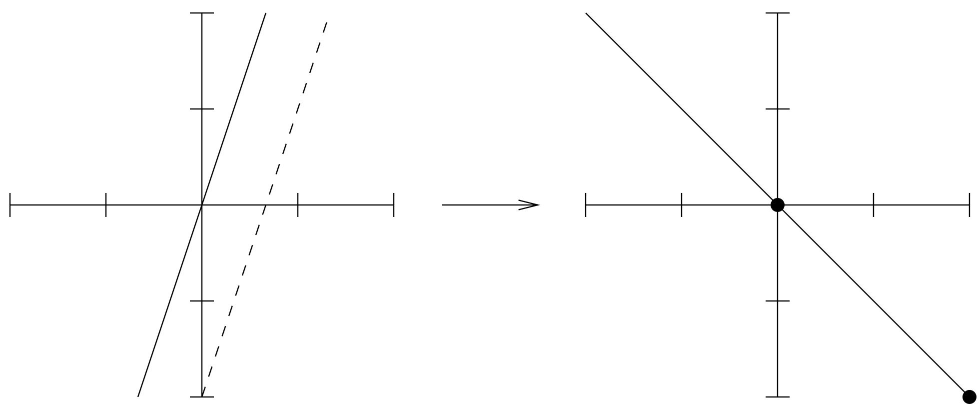
Solutions to Assignments #8-#10, Part II

For Problem CC-1:

The transformation is the same as $T(x,y) = (3x-y, -3x+y)$. The range has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The null space has basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The inverse image of $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is the graph of $3x - y = 2$. Graphs are these:



For Problem CC-2: $\begin{aligned} T(\cos 2x) &= -2\sin 2x &= 0\cos 2x - 2\sin 2x \\ T(\sin 2x) &= 2\cos 2x &= 2\cos 2x + 0\sin 2x \end{aligned}$ so the matrix is $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Notice that this is a scaledrotation: $-2R_{90^\circ}$.

For Problem CC-3:

(a) $p_A(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$, so eigenvalues are $\lambda = 4$ and $\lambda = 1$. $A - 4I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. We could write out equations, but to save effort, notice that we're just looking for a nonzero vector in the null space, or equivalently, a vector perpendicular to both rows, so we can use $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as the eigenvector. $A - I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and a vector perpendicular to both rows is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore $P^{-1}AP = D$ with $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Using the same P as in (a) we get $P^{-1}A^2P = D^2 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$. It is not even necessary to calculate the entries of A^2.

For Problem CC-4:

(a) $P_A(\lambda) = \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1)$, so eigenvalues are $\lambda = 8$, $\lambda = -1$. We get $A - 8I = \begin{bmatrix} -6 & 3 \\ 6 & -3 \end{bmatrix}$ and (as in the solution to CC-3) a vector perpendicular to both rows is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We get $A - (-1)I = A + I = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$, and a vector perpendicular to both rows is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore $P^{-1}AP = D$ with $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$. Later we'll need P^{-1}, which is $\frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.

(b) We want $B^3 = A$, so if we let $E = P^{-1}BP$ then we'll have $E^3 = D$. We don't know B or E yet, but we can get them by letting E be the cube root of D, specifically, $E = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and then setting $B = PEP^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. This answer can be checked by cubing it.

Incidentally, do you see connections between this problem and Problem CC-3?

For Problem CC-5: $(AB)_{ij} = \sum_k A_{ik}B_{kj}$ so $\text{trace}(AB) = \sum_i \sum_k A_{ik}B_{ki}$, while $(BA)_{ij} = \sum_k B_{ik}A_{kj}$ so $\text{trace}(BA) = \sum_i \sum_k B_{ik}A_{ki}$. If we switch the order of summation in this second equation and then switch the letters i and k, neither of which changes the value, we get the first equation.

For Problem CC-6:

For (a): Say A is invertible. Then $A^{-1}(AB)A = BA$. The case where B is invertible is the same with the letters switched.

(b) It is possible to have $AB = \mathcal{O}$ and $BA \neq \mathcal{O}$ (mathcal O meaning zero matrices here): Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The only matrix similar to a zero matrix is the zero matrix itself, so AB and BA can't be similar.

For Problem CC-7:

(a) The characteristic polynomial of A is $\lambda^2 - t\lambda + \Delta$, where $t = \text{trace } A$ and $\Delta = \det A$. But t is also the sum of the eigenvalues and Δ is also the product of the eigenvalues, so $t = 1$ and $\Delta = -1$, and $p_A(\lambda) = \lambda^2 - \lambda - 1$. From here there are two approaches:

(i) a diagonal matrix as the solution: Solving with the quadratic equation, we find that the eigenvalues are $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$, so an answer is $A = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1-\sqrt{5}) \end{bmatrix}$.

(ii) (easier) just invent a matrix with the proper trace and determinant. For diagonal entries take $1, 0$. For off-diagonal entries to make the determinant come out right, take $1, 1$. So an answer is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Note: In a problem of this kind, you can always take one diagonal entry to be 0 and still be sure of finding other entries that will work.

(b) One way: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We know $\sin\theta = 0$ for $\theta = 0$ and $\theta = \pi$. The first of these is excluded, and the other gives $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

A better way: A rotation matrix is characterized by having orthonormal columns and determinant 1. In a diagonal matrix the columns are already perpendicular; to be orthonormal the diagonal entries should be ± 1. To get determinant 1, they should both be 1 or both be -1. The first is excluded, so the only possible answer is $-I$.

(c) This problem mentions Cayley's Theorem, which we'll discuss in class: Theorem: For any $n \times n$ matrix A, $p_A(A)$ is the $n \times n$ zero matrix. Working as in (a)-(ii), let the diagonal entries be $1, 0$ and the off-diagonal entries be $1, -1$, so the matrix is $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. It can be checked that A works. (The method (a)-(i) is also OK, but this time the eigenvalues are complex.)

Incidentally, recalling the polynomial factorization $x^3 - 1 = (x-1)(x^2 - x + 1)$, we see that for this A we have $A^3 - I = (A - I)(A^2 - A + I) = 0$ (the zero matrix), so $A^3 = I$. For the same reason, the eigenvalues of A must be complex cube roots of 1, which are studied in Math 132.

(d) The simplest examples would be to take A to be the 2×2 identity matrix and $\lambda = 1$, or the 2×2 zero matrix and $\lambda = 0$; the eigenspace is the whole space either way. If the problem had asked for a 3×3 example, it's still simplest to use a diagonal matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\lambda = 1$, or $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\lambda = 0$; the eigenspace is the x, y-plane in either case.

For Problem CC-8:

(a) We might as well try for the n-cycle $\begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}$, so $1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1$. Let's start with (1 2). This takes 1 to 2, which is good, but 2 to 1, which is bad. To get 2 to go to 3, follow with (1 3), on the left since we are composing functions. So far we have (1 3)(1 2). This takes 1 to 2 and 2 to 3 but 3 to 1, so follow with (1 4), and so on. Therefore one answer is $(1\ n)(1\ (n-1))\dots(1\ 3)(1\ 2)$. More generally, if symbols are $a_1, \dots, a_n$, we have $(a_1\ a_n)(a_1\ a_{n-1})\dots(a_1\ a_2) = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$.

Incidentally, when we say "and so on", we are really using informal induction. That's appropriate for small proofs like this one, as long as we know how to make it formal if required. The induction step here would involve checking that $\begin{pmatrix} 1 & n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & (n-1) \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}$. In this course there wasn't much practice with induction, so you will not be expected to use it in unfamiliar contexts like this one.

For another answer, first start with a naive attempt: $(1\ 2)(2\ 3)\dots((n-1)\ n)$. This has two drawbacks: First, it involves thinking from left to right, instead of right to left, and second, it gives the undesired answer $\begin{pmatrix} 1 & n & \dots & 2 \end{pmatrix}$. But that *is* an n-cycle, so in a sense it has solved the problem, even if awkwardly. We can fix up this idea by looking at it and reversing the appearances of $n, \dots, 2$, while leaving the appearance of 1 alone. We get

$(1\ n)((n-1)\ (n-2))\dots(3\ 2) = \begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}$ as desired. More generally, $(a_1\ a_n)(a_{n-1}\ a_{n-2})\dots(a_3\ a_2) = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$.

(b) As stated, every permutation is a product of disjoint cycles, and by (a) every cycle is a product of transpositions, so every permutation is a product of transpositions.

It is also possible to prove (b) directly. Take a permutation σ. Imagine doing σ to a deck of cards numbered $1, 2, \dots, n$, so they're all out of order. Can you put them back in order using transpositions? Yes: Wherever card 1 is, switch it to the top. Then switch card 2 to the second position from the top, etc. When you're done the deck will be in order. In other words, you have accomplished σ^{-1} by doing a product of $n-1$ transpositions, say $\sigma^{-1} = \tau_{n-1}\dots\tau_1$, where the τ_i are transpositions. Then $\sigma = (\tau_{n-1}\dots\tau_1)^{-1} = \tau_1^{-1}\dots\tau_{n-1}^{-1}$ (reversing just like inverting a product of matrices), and since each transposition is its own inverse this shows $\sigma = \tau_1\dots\tau_{n-1}$.

(c) In S_3, **1** and the two 3-cycles are even; the three transpositions are odd. As stated it can be checked that even times even or odd times odd is even, while even times odd or odd times odd is odd.

For Problem CC-9:

(a) Since these are separate DE's, $x(t) = x(0)e^{4t} = e^{4t}$ and $y(t) = y(0)e^{-7t} = 2e^{-7t}$.

(b) $\mathbf{x}' = D\mathbf{x}$ with $D = \begin{bmatrix} 4 & 0 \\ 0 & -7 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) $\mathbf{x}(t) = e^{Dt}\mathbf{x}(0)$ says $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-7t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, the same as (a).

(d) If $\mathbf{x}(t) = e^{Dt}\mathbf{x}(0)$, differentiating and assuming that ordinary rules apply, we get $\mathbf{x}'(t) = De^{Dt}\mathbf{x}(0) = D\mathbf{x}(t)$, and clearly setting $t = 0$ we do get $\mathbf{x}(0)$ as the value.

Note: Since e^{Dt} involves terms that are all powers of D, it commutes with D, so it doesn't matter whether we write De^{Dt} or $e^{Dt}D$, but the first of these fits better here.

For Problem CC-10:

(a) To summarize: $P^{-1}AP = D$ with $D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Substituting $\mathbf{x} = P\mathbf{z}$ we get $(P\mathbf{z})' = AP\mathbf{z}$, which is the same as $P\mathbf{z}' = AP\mathbf{z}$ (since the derivative of a vector means the vector of derivatives, and since we are multiplying by the constant matrix P). Putting P on the right we get $\mathbf{z}' = P^{-1}AP\mathbf{z}$, or $\mathbf{z}' = D\mathbf{z}$, which has the solution $z(t) = z(0)e^{5t}$, $w(t) = w(0)e^{t}$. But what are these initial values? We had $\mathbf{x} = P\mathbf{z}$, so $\mathbf{z} = P^{-1}\mathbf{x}$, $\begin{bmatrix} z(0) \\ w(0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$. The solution, then, from $\mathbf{x} = P\mathbf{z}$, is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{3}{2}e^{5t} \\ -\frac{1}{2}e^{t} \end{bmatrix}$, or $x = \frac{3}{2}e^{5t} - \frac{1}{2}e^{t}$, $y = \frac{3}{2}e^{5t} + \frac{1}{2}e^{t}$. This checks in the original DE.

In matrix form the answer is $\mathbf{x} = Pe^{Dt}P^{-1}\mathbf{x}(0)$.

(b) The matrix power answer ought to be $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$. Inventing reasonable rules, the derivative of the right-hand side is $Ae^{At}\mathbf{x}(0) = A\mathbf{x}(t)$, so that checks.

Note: This answer and the previous answer are consistent: Because e^{Dt} is an infinite sum of scaled powers of Dt, and because similarity by a fixed P preserves matrix multiplication and addition, we get $Pe^{Dt}P^{-1} = e^{PDtP^{-1}} = e^{At}$.

For Problem DD-1: Assume (3). For (1): All matrices have real entries. Since D has real eigenvalues (the diagonal entries), since $A \sim D$, and since similar matrices have the same eigenvalues, A has real eigenvalues.

For (2): Notice that it's OK if eigenvalues are repeated (i.e., have multiplicity greater than 1). For the diagonal matrix D and a given λ, the eigenspace of λ is the span of all e_i with $D_{ii} = \lambda$. Since the standard basis vectors are perpendicular (orthogonal) to each other, the eigenspaces are perpendicular. What is the relationship between the eigenvectors of A and those of D? $Dv = \lambda v$ says $(P^{-1}AP)v = \lambda v$, or $APv = \lambda Pv$. Therefore the eigenspace of A for λ is the eigenspace of D for λ rotated by τ_R. Since R is a rotation matrix, τ_R is rigid and preserves angles, so the eigenspaces of A are also perpendicular to one another.

Note. Maybe that was the hard way to show (2), since it's easy to prove (2) directly, as on p. DD 5.

Also notice that for an eigenspace E_λ, λ doesn't have to be an an eigenvalue, but if it's not, then $E_\lambda = \{\mathbf{0}\}$.

For Problem DD-2:

$p_A(\lambda) = \lambda^2 - 15\lambda + 50 = (\lambda - 10)(\lambda - 5)$, so eigenvalues are 10 and 5. For $\lambda = 10$ we have $A - 10I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ and an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which scales to $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ of length 1. For $\lambda = 5$ we have $A - 5I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ so an eigenvector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which scales to $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$. However, the matrix $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$ has determinant -1, so let's negate the second column to get the answer $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$. Now $P^{-1}AP = D$ for $D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$.

For Problem DD-3: Suppose $v_1, \ldots, v_n$ are orthonormal. Suppose $r_1v_1 + \cdots + r_nv_2 = \mathbf{0}$. We must show that $r_1 = \cdots = r_n = 0$. Take the dot product of both sides with v_1. Since the dot product is linear in each of its two arguments (with the other held fixed), we get $r_1v_1 \cdot v_1 + r_2v_1 \cdot v_2 + \cdots + r_nv_1 \cdot v_n = 0$. Since $v_i \cdot v_j = \delta_{ij}$, this says $r_1 = 0$. Similarly, dotting both sides with each v_i in turn, we get $r_i = 0$ for all i, so the vectors are linearly independent.

For Problem DD-4:

In a matrix product AB, the entries are the dot products of the rows of A with the columns of B. In P^tP, the rows of P^t are the same as the columns of P, transposed. If the columns of P are $v_1, \ldots, v_n$, then, the entries of P^tP are the numbers $v_i \cdot v_j$. Therefore $P_tP = I$ when $v_i \cdot v_j = I_{ij} = \delta_{ij}$ (the Kronecker delta symbol), which says the columns of P are orthonormal.

For Problem DD-5: It's an orthogonal matrix (orthonormal columns), so just take the transpose.

For Problem DD-6:

$P^tP = I$ implies $\det(P^tP) = \det I = 1$. Since determinants are compatible with multiplication, $\det P^t \det P =$. Since $\det(P^t) = \det P$, we get $(\det P)^2 = 1$. The solutions of $x^2 = 1$ are $x = \pm 1$ so $\det P = \pm 1$.

For Problem DD-7:

One end of the semiminor axis is the point where $r = \frac{1}{2}$, $s = 0$; here $\mathbf{x} = R\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = [v_1|v_2]\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \frac{1}{2}v_1 = \frac{1}{2}\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}$. It can be checked that this point is on the curve.

Note: Sometimes "semimajor axis" means the length of the line segment from the origin and sometimes it means the line segment itself. We can tell by the context.

For Problem DD-8:

(a) As in the earlier explanation, when we start with $\mathbf{x}^tA\mathbf{x} = 1$ and substitute $\mathbf{x} = R\mathbf{r}$ we get $\mathbf{r}^tR^tAR\mathbf{r} = 1$, or $\mathbf{r}^tD\mathbf{r} = 1$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Written out, $\mathbf{r}^tD\mathbf{r} = 1$ becomes $\lambda_1 r^2 + \lambda_2 s^2 = 1$.

(b) If λ_1 and λ_2 are both positive, we get an ellipse (or a circle, if they are equal). If one is positive and the other negative, we get a hyperbola. If one is positive and the other is 0, we get two parallel lines. In all other cases there are no points.

Note: It is interesting to consider taking λ_1 to be a fixed positive number and gradually changing λ_2, starting from λ_1 and then going negative. We'd see an ellipse that expands along its major axis until it becomes two parallel lines, which then bend back and becomes a hyperbola.

For Problem DD-9:

(a) Here $A = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$, as in Problem DD-2. In that problem we found $\lambda_1 = 10$, $\lambda_2 = 5$, so the transformed equation is $10r^2 + 5s^2 = 1$, giving semiaxes $\frac{1}{\sqrt{10}}$ and $\frac{1}{\sqrt{5}}$. Since the first is smaller, it's the semiminor axis and the second is the semimajor axis.

(b) In the earlier problem we saw $R = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ (called P there). With

$\mathbf{x} = R\mathbf{r}$, for $r = 1$, $s = 0$ on the r-axis we have $\mathbf{x} = R \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. The slope of the r axis is the y-value over the x-value, so is 2.

For Problem DD-10:

(a) If $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, we can write $p_A(\lambda) = \lambda^2 - t\lambda + \Delta$, where $t = a + d$ and $\Delta = ad - b^2$. The quadratic formula says the roots are $\lambda = \frac{1}{2}(t \pm \sqrt{t^2 - 4\Delta})$. Simplifying the discriminant (the part inside the square root), we get $t^2 - 4\Delta = (a+d)^2 - 4(ad - b^2) = a^2 + 2ad + d^2 - 4ad + 4b^2 = (a-d)^2 + (2b)^2$.

(b) Since the discriminant is the sum of two squares and so is ≥ 0, the square root is real. Therefore the eigenvalues are real and are $\frac{1}{2}(a + d \pm \sqrt{(a+d)^2 + (2b)^2})$.

(c) As suggested, the roots are equal when $(a-d)^2 + (2b)^2 = 0$. That happens when $a = d$ and $b = 0$, which is when $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, a scalar matrix.

For Problem DD-11:

$\langle \mathbf{u}, A\mathbf{v} \rangle = \mathbf{u}^t A\mathbf{v}$ and $\langle A\mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u})^t\mathbf{v} = \mathbf{u}^t A^t \mathbf{v} = \mathbf{u}^t A\mathbf{v}$ (since A is symmetric), $= \langle \mathbf{u}, A\mathbf{v} \rangle$.