## Solutions to Assignments #8-#10, Part I

**For Problem U-2:** $\tau_A(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has been multiplied by 5. $\tau_A(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ has been multiplied by 3.

**For Problem U-3:** Just use the definition of the new matrix, carefully. Take $\mathbf{v}_1$, transform it, and write the result using the basis $\mathbf{v}_1, \mathbf{v}_2$, to get the first column of the answer: $\tau_A(\mathbf{v}_1) = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\mathbf{v}_1 = 5\mathbf{v}_1 + 0\mathbf{v}_2$, so the first column is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Similarly, $\tau_A(\mathbf{v}_2) = \cdots = 3\mathbf{v}_2 = 0\mathbf{v}_1 + 3\mathbf{v}_2$, so the second column of the answer is $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Together they give $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

**For Problem U-4:** Checking is easy: $A\mathbf{v}_1 = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\mathbf{v}_1$ and $A\mathbf{v}_2 = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}_2$, so the corresponding eigenvalues are 5 and 3.

**For Problem U-5:** $A\mathbf{v} = A(\mathbf{v}_1 + \mathbf{v}_2) = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$, which is not a scalar times $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

**For Problem U-6:** This rotation can't have any eigenvectors because the result of rotating a nonzero vector $\mathbf{v}$ by $90°$ is never a scalar times $\mathbf{v}$.

**For Problem U-8:** (a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (a+b) \\ (c+d) \end{bmatrix}$, the vector of row sums. So $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ if and only if the row sums are both 1.

(b) Let $A$ be a stochastic $2 \times 2$ matrix, meaning that the *column* sums are 1. Part (a) was about *row* sums being 1, so $A^t$ has the eigenvalue 1, i.e., 1 is a root of the characteristic polynomial of $A$. By Observation 2, $A$ itself has the same characteristic polynomial and so also has the eigenvalue 1.

(c) By (b) we just look for an eigenvector for the eigenvalue 1. $A - 1 \cdot I = \begin{bmatrix} -.8 & .7 \\ .8 & -.7 \end{bmatrix}$; $(A - I)\mathbf{v} = \mathbf{0}$ says $\begin{bmatrix} -.8 & .7 \\ .8 & -.7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; one solution is $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. But with stochastic matrices, usually it is best to have vectors whose entries sum to 1, so we can scale $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ to get $\begin{bmatrix} \frac{7}{15} \\ \frac{8}{15} \end{bmatrix}$.

**For Problem U-9:** $A\mathbf{v} = 0\mathbf{v}$ just says $A\mathbf{v} = \mathbf{0}$, so $\mathbf{v}$ is in the kernel of $\tau_A =$ the null space of $A$. In this case, $A$ is singular, since for a nonsingular matrix $A$ we would have $A\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0}$.

Important: Notice that an eigen**value** can be 0 but an eigen**vector** can't be $\mathbf{0}$.

**For Problem U-10:** In the handout with pictures of a house, #1 to #1 is the identity matrix, which is scalar; #1 to #2 is the shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which is defective; #1 to #3 is diagonal with distinct diagonal entries; #1 to #4 is a 90° rotation, so has no real eigenvalues; #1 to #5 is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, which has characteristic polynomial $\lambda^2 + 2\lambda + 2$ with roots $1 \pm i$ (from the quadratic formula), so no real eigenvalues—or notice that $\tau_A$ rotates each vector by 45° and lengthens it, so no real vector is an eigenvector; #1 to #6 is diagonal with distinct diagonal entries; #1 to #7 is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, symmetric but not scalar; #1 to #8 is $\begin{bmatrix} 3.5 & -1.5 \\ 3 & 0 \end{bmatrix}$, which has characteristic polynomial $\lambda^2 + 3.5\lambda + 4.5$, which by the quadratic formula has no real roots, so there are no real eigenvalues.

**For Problem U-12:** It was shown earlier that $A$ has eigenvalues $5, 3$ and corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $P^{-1}AP = D$ for $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

**For Problem U-15:** (a) $A = I$ (or any example with no eigenvalues equal to 5); (b) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (c) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (d) $A = 5I$.

**For Problem V-16:**

*All* of them are equivalence relations:

(v) is reflexive since $x - x = 0$, which is even; symmetric since $x - y$ is even implies $y - x$ is even; transitive since $x - y$ and $y - z$ both even imply $x - z = (x - y) + (y - z)$ is even.

(vi) is reflexive since $f(x) = f(x)$; symmetric since $f(x_1) = f(x_2)$ implies $f(x_2) = f(x_1)$; transitive since if $f$ has equal values at $x_1$ and $x_2$ and equal values at $x_2$ and $x_3$ then $f$ has equal values at $x_1$ and $x_3$.

(vii) is reflexive since $x - x = \mathbf{0} \in W$; symmetric since $x - y \in W$ implies $y - x = -(x - y) \in W$; transitive since $x - y \in W$ and $y - z \in W$ imply $x - z = (x - y) + (y - z) \in W$.

(viii) is reflexive since $A$ has the same row space as itself; symmetric since if $A$ and $B$ have the same row space then $B$ and $A$ have the same row space; transitive since if $A$ and $B$ have the same row space and $B$ and $C$ have the same row space then $A$ and $C$ have the same row space.

(ix) is reflexive since $A = I^{-1}AI$ so $A \sim A$; symmetric since if $A \sim B$ using $P^{-1} \dots P$ then $B \sim A$ using $P \dots P^{-1}$; transitive since $A \sim B$ and $B \sim C$ imply $P^{-1}AP = B$ and $Q^{-1}BQ = C$ for some invertible $P$ and $Q$, so that $Q^{-1}(P^{-1}AP)Q = C$, which is the same as $(PQ)^{-1}A(PQ) = C$, so $A \sim C$.

(x) is all three since any assertion that two elements are related is true, and the three properties consist of such assertions.

(xi) is reflexive since any element is in the same block as itself; symmetric since if $x$ and $y$ are in the same block then $y$ and $x$ are in the same block; transitive since if $x$ and $y$ are in the same block and $y$ and $z$ are in the same block then $x$ and $z$ are in the same block.

**For Problem V-17:** The "proof" gives the impression that it is talking about all possible $x \in X$, but what it actually says is that *if* $x$ is related to $y$ something is true. So what the proof actually proves is that *if* $x$ is related to *some* $y$ then $x$ is related to itself. It doesn't say anything about an $x$ that is related to no elements.

For a counterexample to the assertion, then, let $X = \mathbb{R}$ and say $x\rho y$ when $x$ and $y$ are both integers. This $\rho$ is symmetric and transitive but not symmetric. For a simpler counterexample, take any nonempty set $S$ and let $\rho$ be the "empty relation"—no two elements are related.

**For Problem V-18:**

For (i): Let's rephrase the three properties of equivalence relations using the definition of $B_x$: Reflexivity says that (1) $x \in B_x$; symmetry says that (2) $y \in B_x \Rightarrow x \in B_y$; transitivity says that (3) $y \in B_x$ and $z \in B_y$ imply $z \in B_x$.

Notice that we also have
(4) if $x \in B_y$ then $B_x = B_y$.

The reason: Suppose $x \in B_y$. To show $B_x \subseteq B_y$: If $z \in B_x$ then by $x \in B_y$ and (3) we have $z \in B_y$. To show $B_y \subseteq B_z$ notice that by (2) we can turn $x \in B_y$ around to get $y \in B_x$.

Now to show the distinct blocks make a partition: By (1), each $x$ is in its own block, so the union of all the blocks is all of $X$. To show that if two blocks are distinct then they are pairwise disjoint, let's do the contrapositive: If two blocks have an element $x$ in common, i.e., if $x \in B_z$ and $x \in B_z$ then $B_y = B_z$. But that's true by (4), since $B_z = B_x = B_y$.

For (ii): We are asked to start with $\rho$, make the partition described in (i), and then show that it gives $\rho$ back. In other words if the relation $\sigma$ is defined by $x\sigma y$ when $x$ and $y$ are in the same block, then $\sigma = \rho$. We can do this in two parts: First, does $x\sigma y \Rightarrow x\rho y$? Yes: If $x\sigma y$, so that $x$ and $y$ are in the same block, then since $x \in B_x$ that block must be $B_x$, so $y \in B_x$, which by definition says $x\rho y$. Next, does $x\rho y \Rightarrow x\sigma y$? Yes: If $x\rho y$ then by (4) $B_x = B_y$ and by (1) both $x$ and $y$ are in this block.

**For Problem V-19:**

Of (i)-(iv) only (iii) (equality) is an equivalence relation; it corresponds to (g) (singleton blocks).

All of (v)-(xi) are equivalence relations.

(v) corresponds to (e) in Q-2 (even/odd), which is the same as congruence blocks modulo 2.

(vi) corresponds to (h).

(vii) corresponds to (b)

(viii)-(xi) have no correspondence to examples in Q-2.

**For Problem V-20:** Congruence modulo $n$ is reflexive, since $n|0$ so $n|x-x$. The relation is symmetric, since $n|y-x$ means $y-x = kn$ for some $k$, so $x - y = (-k)n$, giving $n|x-y$. The relation is transitive since $n|y-x$ and $n|z-y$ say $y-x = kn$ and $z-y = mn$ for some $k, m$ and then $z - x = (z-y) + (y-x) = jn + kn = (m+k)n$ so $n|z-x$. Therefore congruence modulo $n$ is an equivalence relation.

Notice that Example (i) in Problem V-16 is congruence modulo 2.

The blocks of a congruence relation are called "congruence classes modulo $n$". Again, it would be better if people called these "congruence blocks" instead.

**For Problem Z-1:** $\mathbb{Z}$ satisfies $(\exists x)(\exists y > x)(\forall z)(x \geq y \text{ or } y \geq z)$.

**For Problem Z-2:** $(\exists x)(\exists \epsilon > 0)(\forall y)(|x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon)$.