Comments/solutions for Assignments #1 and #2

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Ex. 5. Comment: This describes addition and multiplication in $\mathrm{GF}(2) = \mathbb{Z}_2$. Unary minus (negation) is $-x = x$. The intended solution to the problem is to check each law. After checking commutativity of addition and multiplication, you can save some effort in checking associativity.

Ex. 7: Suppose F is a subfield of $\mathbb{R}$. To be a field, F must contain 0 and 1 (the same 0 and 1 as in $\mathbb{R}$, since those are the only neutral elements for addition and multiplication). Then F also contains $1+1, 1+1+1, ..$ and their negatives, so F contains at least $\mathbb{Z}$. But to be a subfield F must also contain the multiplicative inverse of each of its nonzero elements, so F contains $\frac{1}{2}, \frac{1}{3}$, etc. Multiplying these in F by integers, we get all of $\mathbb{Q}$. In other words, $\mathbb{Q} \subseteq F$.

Ex. 8: Let F be any field of characteristic 0.

Informal version: In F there is an element 1. $1 + 1$ is like 2, so call it $\bar{2}$. Similarly, we can make $\bar{3} = 1 + 1 + 1$, etc. Their negations are $-\bar{2}$, etc. If we fill out by writing $\bar{0} = 0$ and $\bar{1} = 1$, we have a copy of of $\mathbb{Z}$, sort of pseuo-integers. Their ratios make a copy of $\mathbb{Q}$.

There are a couple of issues here, though: First, where did we use the fact that that characteristic is 0? The answer is that to be sure about the copy of $\mathbb{Z}$ we need to know that the sums of 1's and their negations are all *distinct* (i.e., different from each other). For example, could $\bar{5} = \bar{2}$? Since we can cancel additively in a field, $1+1+1+1+1 = 1+1$ would imply $1+1+1 = 0$, which can't happen since the characteristic is 0. In other words, by additive cancellation, if $\bar{m} = \bar{n}$, then $\bar{0} = \overline{m - n}$, which for characteristic 0 implies $m = n$.

Another issue is whether the sums of 1's have operations like those of integers. For example, do we have $\bar{2} + \bar{3} = \bar{5}$ and $\bar{2}\bar{3} = \bar{6}$? Yes, the first left-hand side is the sum of five 1's and $\bar{2}\bar{3} = (1+1)(1+1+1) = 1+1+1+1+1+1 = \bar{6}$, by the distributive law in F.

Since the problem is early in the book, most likely the authors expected only an informal solution, but it's possible to be more formal, as follows. A "copy" of $\mathbb{Q}$ means a subfield isomorphic to $\mathbb{Q}$ as a field, so we try to construct a one-to-one function $\phi : \mathbb{Q} \to F$, whose "image" will be a subfield of F. Let $\phi(0) = 0$. For an integer $n > 0$ we define $\phi(n) = \bar{n}$. (Here $\phi(-2) = \overline{-2}$ means

$-\bar{2}$, etc.) As in the informal version we can check that ϕ so far is one-to-one and preserves addition and multiplication of integers.

Now for any fraction $\frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ we define $\phi(\frac{a}{b})$ to be $\dfrac{\phi(a)}{\phi(b)} = \dfrac{\bar{a}}{\bar{b}}$, meaning $\bar{a}\bar{b}^{-1}$. But there is also an issue here: Since different fractions can represent the same rational number, e.g., $\dfrac{6}{4} = \dfrac{15}{10}$, could we conceivably get inconsistent values from our definition? Is $\dfrac{\bar{6}}{\bar{4}} = \dfrac{\bar{15}}{\bar{10}}$, necessarily? In other words, we must check that if $\frac{a}{b} = \frac{c}{d}$ then $\frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$ in F. But it's OK: $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc \Rightarrow \bar{a}\bar{d} = \bar{b}\bar{c} \Rightarrow \frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$, as hoped. We say that "ϕ is well defined".

To finish the proof we need to check that ϕ preserves addition and multiplication for rational numbers and is one-to-one. So far we know this on $\mathbb{Z}$ only. For $\frac{a}{b}, \frac{c}{b} \in \mathbb{Q}$ we have $\phi(\frac{a}{b} + \frac{c}{d}) = \phi(\dfrac{ad+bc}{bd}) = \dfrac{\phi(ad+bc)}{\phi(bd)} = \dfrac{\bar{a}\bar{d}+\bar{b}\bar{c}}{\bar{b}\bar{d}} = \frac{\bar{a}}{\bar{b}} + \frac{\bar{c}}{\bar{d}} = \phi(\frac{a}{b}) + \phi(\frac{c}{b})$, which is the correct result. Multiplication is checked similarly.

For being one-to-one, let's see if $\phi(\frac{a}{b}) = \phi(\frac{c}{d})$ implies $\frac{a}{b} = \frac{c}{b}$: $\phi(\frac{a}{b}) = \phi(\frac{c}{d})$ says $\frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$, so $\bar{a}\bar{d} = \bar{b}\bar{c}$, or $\phi(ad) = \phi(bc)$. Since we know ϕ is one-to-one for integers, $ad = bc$, so $\frac{a}{b} = \frac{c}{d}$ as hoped.

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Ex. 1 (outline). The various properties come from similar properties of F (which you can quote) and from the fact that operations in F^n are defined coordinatewise.

Ex. 5. $\oplus$ is not commutative or associative. We do have $x \oplus \mathbf{0} = x$, though, and for each x, $x \oplus x = \mathbf{0}$, so x is its own additive inverse. Also, $1 \cdot v = -v$ rather than v, so (4a) fails. But (4b), (4c), (4d) hold, since in each case both sides have the same number of minuses when rewritten using ordinary multiplication by scalars.

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Ex. 1. Just (b) is a subspace. (a) is not closed under multiplication by negative scalars; (c) is not closed under addition nor under multiplication by scalars; (d) is not closed under addition; (e) is not closed under multiplication by scalars, since irrational scalars don't work.

Ex. 2. (b), (d), and (e) are subspaces. This may seem surprising, but they work because operations on functions are defined pointwise. Saying $f(-1) = 0$, for example, is somewhat like saying $a_2 = 0$ for $(a_1, a_2, a_3) \in \mathbb{R}^3$.

For (e), we need to quote the theorems that the sum of two continuous functions is continuous and that a continuous function times a constant is continuous.

(a) is not a subspace because the constant function 1 is in the set but twice it (the constant function 2) is not. (c) is not a subspace because the 0 function (constant 0) is not in it.

Ex. 3. Method #1: Outline: Write the first vector as a linear combination of the others with unknown coefficients, make linear equations, and solve.

Method #2: Make the vectors the columns of a matrix M, with the "target" vector as the last column, and row reduce to a matrix E in row-reduced echelon form. Since the linear relations between the columns don't change, you can check in E to see whether the last column is in the span of the preceding ones (and if so, with what coefficients), and the same will hold for columns of M. Notice, though, that M and E are exactly the same as what you get in Method #1.

Ex. 4. This is asking for a basis of the solution space of the homogeneous equations, which is the same thing as a basis of the null space of the matrix of coefficients. Row reduce, etc.

Ex. 5. Just (c) is a subspace. The set of matrices in (c) contains the zero matrix and is closed under multiplication by scalars. It is also closed under addition, since if A_1, A_2 are in the set, we have $(A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2)$.

Notice that matrix multiplication is used in defining the subspace but is not used for operations on the vector space of matrices involved.

For (a), the zero matrix is not in the set. For (b), the sum of two noninvertible matrices could be invertible, e.g, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (Here it's important that $n \geq 2$.) For (d), the set is not closed under multiplication by 2.

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Ex. 1. If v and w are linearly dependent, there is a nontrivial linear relation $rv + sw = \mathbf{0}$. Nontrivial means that $r \neq 0$ or $s \neq 0$ or both. If $r \neq 0$ we can solve to get $v = (-\frac{s}{r})w$, so that v is a scalar multiple of w. If $s \neq 0$, a similar proof applies the other way around.

Ex. 2. Method: Make a matrix with these as either the rows or the columns and row reduce. If you get the identity matrix, so the rank is 4, then they are linearly independent; otherwise not.

Ex. 3. Method #1: Make a matrix with the four vectors as rows and row-reduce. The nonzero rows of the row-reduced echelon form are a basis for the row space.

Method #2: Make a matrix with the four vectors as columns and row-reduce. See which columns are pivot columns. The same-numbered columns of the original matrix are a basis for the column space.

Ex. 4. If you were asked only to show that the vectors form a basis, you could do this like Ex. 2 above. Since you are asked also to express the standard basis vectors relative to this basis, you might as well do that first and then you will have shown that these vectors form a basis.

Naive method: Try $e_1 = r\alpha_1 + s\alpha_2 + t\alpha_3$, which results in three equations in the three unknowns, which you can solve by row reduction. Then do the same for e_2 and e_3, each time getting new values for the coefficients.

Organized method: Make a matrix with these three vectors as the first three columns and e_1, e_2, e_3 as the last three columns, so the matrix looks like $[A|I]$. Row reduce. You should get a matrix that looks like $[I|B]$. Since linear relations between columns are preserved, looking at each column of the B part as a linear combination of the first three columns tells you how to represent each of e_1, e_2, e_3 as a linear combination of columns of A. (Note. Actually $B = A^{-1}$ in this situation.)

Ex. 5. Take three vectors in a plane, such as $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$.

Ex. 6. Thinking of the 2×2 matrices as $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$, W_1 can be described by $s = -r$ and W_2 can be described by $t = -r$.

(a) Both of these equations are preserved under addition of matrices and multiplying matrices by scalars. Also both contain the zero matrix. So W_1, W_2 are subspaces of V.

(b) Intuitively, each of W_1 and W_2 involves choosing three parameters freely, so their dimensions ought to be 3. More officially, a basis for W_1 consists of the matrices you get by choosing one parameter to be 1 and the others 0: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Similarly, a basis for W_2 consists of $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Since each of W_1, W_2 consists of linear combinations of the respective basis matrices, $W_1 + W_2$ consists of linear combinations of all six, or really five since one is repeated. But the five are dependent—obviously, since we are in the space V of dimension 4. One way to determine the actual dimension is to use an isomorphism of V with $\mathbb{R}^4$, let's say $(r, s, t, u) \mapsto \begin{bmatrix} r & s \\ t & u \end{bmatrix}$. We are then asking about five 4-tuples. Make a matrix with the five as its rows or columns, row-reduce, and see what the rank is. You should get 4. Another way is that since W_1 has dimension 3 in a space V of dimension 4, if you just notice that the first basis vector of W_2 is not in W_1, their sum must be larger than W_2, so all of V.

For $W_1 \cap W_2$, go back to the equations in r, s, t, u. We have $s = -r = t$, which u varies freely, so the general form of a matrix in this subspace is $\begin{bmatrix} r & -r \\ -r & u \end{bmatrix}$.

A basis is $\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so the space has dimension 2.

Another way would be to use the equation $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$ and solve $3 + 3 = 4 + x$, getting $x = 2$.

Ex. 9. See the solution of D-1 below. With the isomorphism method you get vectors $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ in $\mathbb{R}^3$. Put them as the rows of a matrix and row reduce; you get the identity matrix, so the rank is 3 and the original rows were linearly independent.

Ex. 12. A basis consists of the mn matrices each with one entry of 1 and the rest 0's. One way to see this is that if you use an isomorphism with $\mathbb{R}^{mn}$, these matrices correspond to the standard basis.

For Problem B-1: All are like $P \Rightarrow Q$. For example, (g) says Q is a necessary condition for P. This means that if P is true, then Q, being necessary for P, also has to be true. So $P \Rightarrow Q$.

For Problem B-2: Each of (1), (4), (5) implies the others (and itself, for that matter). Each of (2), (3) implies the other (and itself). Each of (1), (4), (5) implies each of (2), (3).

For Problem B-3: Neither of (2), (3) implies any of (1), (4), (5); a counterexample in each case is $x = -1$.

For Problem B-4: (1) $\Leftrightarrow$ (4), (4) $\Leftrightarrow$ (5), (1) $\Leftrightarrow$ (5), and (2) $\Leftrightarrow$ (3).

For Problem B-5:

The following statements are equivalent:

(1) $x > 0$
(2) $x \neq 0$ and $\frac{1}{x} > 0$
(3) $x = y^2$ for some real number y with $y \neq 0$

Also, the following statements are equivalent:

(1) $x^2 > 0$
(2) $x \neq 0$

For Problem C-1: Our list starts with $\{1,2\}$ and $\{1,3\}$. Their complements are $\{3,4\}$ and $\{2,4\}$. Intersecting two of these four sets at a time we get the empty set and the singletons $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$. Taking unions of two or more singletons gives all other subsets of S, including S itself. (S can also be obtained as the union of $\{1,2\}$ and its complement.)

For Problem C-2: If the unknown weights are r, s, t, u, we get equations $12r + 24s + 25t + 25u = 23$, etc., from the first four rows. The first four rows are the augmented matrix of this system. Row-reducing over $\mathrm{GF}(2) = \mathbb{Z}_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{3}{20} \\ 0 & 1 & 0 & 0 & \frac{1}{20} \\ 0 & 0 & 1 & 0 & \frac{7}{25} \\ 0 & 0 & 0 & 1 & \frac{13}{25} \end{bmatrix}$$ so the weights are $r = 15\%, s = 5\%, t = 28\%, u = 52\%$.

Using these weights in the fifth row gives the weighted average 93.

For Problem C-3: Suppose that instead $\sqrt{p} \in \mathbb{Q}$, say $\sqrt{p} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, in lowest terms (meaning that a and b have no positive integer divisors in common except 1). Then $p = \frac{a^2}{b^2}$, $a^2 = pb^2$. Since p is a prime factor of a^2, p must be a prime factor of a, so $a = kp$ for some integer k. Then $a^2 = pb^2$ becomes $k^2p^2 = pb^2$. Cancel to get $k^2p = b^2$. Then p is a prime factor of b^2, and by the same reasoning as before, p must be a prime factor of b, as it was of a. But then $\frac{a}{b}$ was not in lowest terms, a contradiction. We conclude that $\sqrt{p} \notin \mathbb{Q}$. (We used the fact that every positive integer can be expressed uniquely as a product of primes, so a^2 involves the same primes as a and b^2 involves the same primes as b.)

For Problem C-4:

As suggested, let $V = \mathbb{R}^2$ as a set and use the usual $+$ and $-$, but make a fudged product by scalars in which a scalar times a vector is always $\mathbf{0}$. Since every defining law of vector spaces except $1v = v$ involves a product by scalars on either both sides or neither side, all those laws are still true. However, the law $1v = v$ fails for any choice of a nonzero v, since the left-hand side is $\mathbf{0}$ and right-hand side is not.

For Problem C-5:

(a) The subspace spanned is the whole space, because for any $(a, b, c) \in \mathbb{R}^3$, the equation $r(1, 0, 0) + s(2, 3, 0) + t(4, 5, 6) = (a, b, c)$ gives $r + 2s + 4t = a$, $3s + 5t = b$, $6t = c$, which can always be solved by finding t, then s, then r.

(b) The subspace spanned is again the whole space, because for any $a + bx + cx^2 \in \text{Pols}(\mathbb{R}, 2)$, the equation $r + s(2 + 3x) + t(4 + 5x + 6x^2) = a + bx + cx^2$ gives the same equations as in (a) and so can be solved.

(c) The subspace spanned is again the whole space, because for any $ax^2+bx+c \in \text{Pols}(\mathbb{R}, 2)$, the equation $rx^2+s(2x+3)+t(4x^2+5x+6) = ax^2+bx+c$ gives the same equations as in (a) and so can be solved. [If you write polynomials in some other order you get different equations but they are solved similarly.]

For Problem D-1:

(a) If $r \neq 0$, then since scalars are in a field we can multiply the equation $rv = \mathbf{0}$ on both sides by r^{-1} on the left, obtaining $r^{-1}(rv) = r^{-1}\mathbf{0}$. By laws (f) and (e) and the lhs (left-hand-side) is $(r^{-1}r)v = 1v = v$, while the rhs is $\mathbf{0}$ by (i). Therefore we get $v = \mathbf{0}$. In other words, if r isn't the zero scalar then v is the zero vector, so one of them is zero, as stated.

(b) The outline is

$$r(v - w) = rv - rw = \mathbf{0} \Rightarrow r(v - w) = \mathbf{0} \Rightarrow v - w = \mathbf{0} \Rightarrow v = w,$$

where r is canceled using part (a) (not law (a)).

For a full proof we need to justify our use of familiar algebra for binary minus, since binary minus and laws for it are not in our definition of a vector space. Instead, binary minus is defined outside the definition of a vector space by $v - w = v + (-w)$. We do have laws such as $r(-w) = -(rw)$ because by laws (k) and (f) we have $r(-w) = r((-1)w) = (r(-1))w = ((-1)r)w = (-1)(rw)$.

The first equation in the outline above comes from $r(v - w) = r(v + (-w)) = rv + r(-w) = rv + (-(rw)) = rv - rw$. The last implication in the outline comes from adding w to both sides: $v - w = \mathbf{0} \Rightarrow (v + (-w)) + w = \mathbf{0} + w \Rightarrow v + (-w + w) = w \Rightarrow v = w$, where various defining laws have been used.

(c) In outline,

$$rv = sv \Rightarrow rv - sv = \mathbf{0} \Rightarrow (r - s)v = \mathbf{0} \Rightarrow r - s = 0 \Rightarrow r = s.$$

The justification of using familiar laws for binary minus is as in part (b).

Notice that in the original handout there was a misprint in law (h), which should say $(r + s)v = rv + sv$.

For Problem D-2:

We can define $-v$ to mean $(-1)v$. Then $v + (-v) = 1v + (-1)v$ by this definition and by (e), which equals $(1 + (-1))v$ by (h), which equals $0v$ by the fact that $1 + (-1) = 0$ in a field, which equals $\mathbf{0}$ by (j). We have shown $v + (-v) = \ldots = \mathbf{0}$, which is (d).

For Problem E-1:

Method #1: If $ru+s(u+v)+t(u+v+w) = \mathbf{0}$, expanding and then regrouping we get $(r+s+t)u+(s+t)v+tw = \mathbf{0}$. Since u, v, w are linearly independent, these coefficients must be 0, so $r + s + t = 0, s + t = 0, t = 0$. Working backwards from $t = 0$ we get $s = 0$, $r = 0$. Therefore $ru + s(u + v) + t(u + v + w) = \mathbf{0}$ implies $r = s = t = 0$, and so $u, u + v, u + v + w$ are linearly independent.

Method #2: The subspace W of V generated by u, v, w is isomorphic to F^3 with u, v, w corresponding to e_1, e_2, e_3, since u, v, w span it and are linearly independent. Under this isomorphism, $u, u + v, u + v + w$ correspond to $(1, 0, 0), (1, 1, 0), (1, 1, 1)$. Making these the rows of a matrix and row-reducing we get the identity matrix, which has rank 3, so they must be linearly independent. Using the isomorphism, $u, u + v, u + v + w$ must be linearly independent.

For Problem E-2:

Using the augmented matrix and row reducing,

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

from which we see there is the unique solution $x = 1, y = 0, z = 1, w = 1$. This checks.

For Problem E-3:

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 & -2 \\ 1 & 2 & 2 & 3 & 0 \\ 1 & 2 & 3 & 3 & 2 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow E = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The two nonzero rows of E are a basis for the row space.

For the null space, we write the general solution in variables, say, $x_1, \ldots, x_5$, getting $\begin{cases} x_1 &= -2x_2 & -3x_4 & +x_5 \\ x_2 &= x_2 & & \\ x_3 &= & & -2x_5 \\ x_4 &= & x_4 & \\ x_5 &= & & x_5 \end{cases}$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$. These three column vectors are then the basis.

For the column space, we look at E and see that the pivot columns are columns 1 and 3, so a basis for the column space is columns 1 and 3 of M. (Alternatively, we could transpose M, row reduce, and take the resulting nonzero rows.)

For Problem E-4:

(a) one-to-one and onto; (b) neither; (c) onto; (d) neither; (e) one-to-one.

For Problem E-5:

If $t_1 \neq t_2$ in T, is $g(t_1) \neq g(t_2)$? Since f is onto we know $t_1 = f(s_1)$, $t_2 = f(s_2)$, for some $s_1, s_2 \in S$; we must have $s_1 \neq s_2$ or else we would have $t_1 eq t_2$. $g(t_1), g(t_2)$ are defined to be s_1, s_2, so the answer is yes. Therefore g is one-to-one.

If $s \in S$ is there some $t \in T$ with $g(t) = s$? Yes, $t = f(s)$ works. So g is onto.

For Problem E-6:

First, some notation: If we have a function like $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and we want to describe it without naming it, we can just write $x \mapsto x^2$, which we say as "x maps to x^2". The different-looking arrow is a reminder that we're talking about elements rather than whole spaces. But sometimes it's handy to use this kind of notation with the name of a function; for example, instead of "Let $t = f(s)$" we can say "Let $s \overset{f}{\mapsto} t$", read as "s maps to t under f" or "via f".

In this notation, if $f : S \to T$ is a one-to-one correspondence then $s \overset{f}{\mapsto} t$ is the same as $t \overset{f^{-1}}{\mapsto} s$.

The same notation can be used to to express the idea that a linear transformation $T : V \to W$ preserves addition of vectors:

Instead of $T(v_1+v_2) = T(v_1)+T(v_2)$ we can say
"If $v_1 \overset{T}{\mapsto} w_1$ and $v_2 \overset{T}{\mapsto} w_2$ then $v_1+v_2 \overset{T}{\mapsto} w_1+w_2$."

To say that T preserves multiplication by scalars, we can say
"If $v \overset{T}{\mapsto} w$ then $rv \overset{T}{\mapsto} rw$, for any $r \in F$".

Now to do the problem: To check that T^{-1} preserves addition, we must show that for any $w_1, w_2 \in W$ if $w_1 \overset{T^{-1}}{\mapsto} v_1$ and $w_2 \overset{T^{-1}}{\mapsto} v_2$ then $w_1+w_2 \overset{T^{-1}}{\mapsto} v_1+v_2$. But this is exactly the same as saying that if $v_1 \overset{T}{\mapsto} w_1$ and $v_2 \overset{T}{\mapsto} w_2$ then $v_1+v_2 \overset{T}{\mapsto} w_1+w_2$, which is true since T preserves addition. Therefore T^{-1} preserves addition.

Similarly, to check multiplication by scalars, if $w \overset{T^{-1}}{\mapsto} v$ then we want $rw \overset{T^{-1}}{\mapsto} rv$. But this is the same as $rv \overset{T}{\mapsto} rw$, which is true since T preserves multiplication by scalars.

Therefore T^{-1} is an isomorphism.

For Problem E-7:

In (b), with the basis $1, x, x^2$ for Pols$(\mathbb{R}, 2)$ there is an isomorphism $\mathbb{R}^3 \equiv$ Pols$(\mathbb{R}, 2)$ taking e_1, e_2, e_3 to $1, x, x^2$. Under this isomorphism, part (b) becomes part (a), so the span is the whole space.

(c) In (c), using the basis $x^2, x, 1$ for Pols$(\mathbb{R}, 2)$, the resulting isomorphism with $\mathbb{R}^3$ turns part (c) into part (a), so the span is the whole space.

For Problem E-8:

For (a):

(ii) says the pivot columns of A are columns 1, 3, and 5.

(iii) expresses the other columns as linear combinations of the pivot columns.

Therefore columns 2, 4, and 6 are $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$.

Putting these together says that the mystery matrix is

$$M = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

For (b): Knowing the linear relations means we know which columns are linear combinations of preceding columns, and what the coefficients are. So just as for the mystery matrix, if we are given any matrix in row-reduced echelon form, the pivot columns are those which are not the in the span of the preceding columns; these columns look like some columns of the identity matrix, in order. Each remaining column is a linear combination of the preceding pivot columns, and the coefficients tell its entries.

For (c): Suppose A is row-reduced by two people to matrices E_1 and E_2 in row-reduced echelon form. Both E_1 and E_2 have the same linear relations between columns as A does and so have the same linear relations between columns as each other. But by (b) we know these linear relations uniquely determine the entries of E_1 and E_2, so that $E_1 = E_2$. Therefore the row-reduced echelon form is unique.