Bases[1]

Definition. Let V be a vector space over a field F. A list of vectors $v_1, \dots, v_k \in V$ is a *basis* for V if $v_1, \dots, v_k$ are linearly independent and span V.

Theorem. The following conditions are equivalent, for $v_1, \dots, v_n$ in a vector space V.

(1) $v_1, \dots, v_n$ form a basis for V.

(2) Every element of V can be written uniquely as a linear combination of $v_1, \dots, v_n$.

(3) There is an isomorphism of F^n with V such that the standard basis vectors $e_1, \dots, e_n$ of F^n correspond to $v_1, \dots, v_n \in V$.

(4) $v_1, \dots, v_n$ are a minimal spanning set, in the sense that if any one of them is omitted then the remaining vectors do not span V.

(5) $v_1, \dots, v_n$ is a maximal linearly independent set, in the sense that if any vector in V is added to the list, the new list is linearly **de**pendent.

Lemma. If V has a basis with n elements, then any $n+1$ elements of V are linearly dependent.

Theorem. If V has a basis with n elements, then any other basis also has n elements.

Definition. If V has a basis with n elements, we say V has **dimension** n over F.

Notes. (a) By the Theorem, there is no ambiguity about what the dimension is. (b) Some vector spaces, such as the vector space $\mathcal{C}(\mathbb{R} \to \mathbb{R})$ of continuous functions, are infinite dimensional; there is no finite basis.

Useful observation: In a linearly dependent list of vectors, some vector is in the span of the preceding vectors in the list. Or to put it the other way around, if a list of vectors has the property that no member is in the span of the preceding ones, it is linearly independent.

(What does "the span of the preceding ones" mean in the case of the first vector in the list?)

[1] We say one basis, two bases ("baseeze")

Developing Intuition

If you develop intuition based on $\mathbb{R}^3$, it should be a good guide to what is true about finite-dimensional vector spaces in general (except for facts that depend on the dimension of the whole space being 3!)

Try these. Where one is false, think of a counterexample. Where one is true, see if you can prove it.

True or false?

1. A subspace of a finite-dimensional vector space is always finite-dimensional.

2. For three vectors v_1, v_2, v_3 in a vector space V, if each two are linearly independent, then the three are linearly independent.

3. If $v_1, \dots, v_n$ span V, then some subset of them is a basis for V.

4. If $v_1, \dots, v_n$ are a basis of V and W is a subspace of V, then some subset of $\{v_1, \dots, v_n\}$ is a basis for W.

5. If $w \in \text{Span}(v_1, .., v_n)$, then $\text{Span}(v_1, \dots, v_n, w) = \text{Span}(v_1, .., v_n)$.

6. If $v_1, \dots, v_k$ are linearly independent in a finite-dimensional vector space V, then this list can be extended to a basis.

7. Any linearly independent list of vectors is a basis for the subspace they span.

8. If W_1, W_2 are subspaces of a finite-dimensional vector space V, then $\dim W_1 + \dim W_2 \leq \dim V$.

9. If $\text{Span}(v_1) = \text{Span}(v_2)$ then $v_1 = rv_2$ for some scalar r.

10. For any two 2-dimensional subspaces of a vector space, their intersection contains a nonzero vector.