Notes on logical implication

1. If–then reasoning

A key concept is "if–then" reasoning. For example,

For any real number x, if $x > 0$ then $x^2 > 0$.

This is true. For short, we usually just say "if $x > 0$ then $x^2 > 0$". Another way to write it is $x > 0 \Rightarrow x^2 > 0$, which in words is "$x > 0$ implies $x^2 > 0$".

Is this statement true the other way around? In other words, is it true that

For any number x, $x^2 > 0 \Rightarrow x > 0$??

NO, since $x = -1$ is a *counterexample*: $(-1)^2 > 0$ is true but $-1 > 0$ is false.

2. Ways to say it

Imagine that P and Q are statements, maybe about x. Then we can ask whether $P \Rightarrow Q$ and whether $Q \Rightarrow P$.

Problem B-1. [1] Here are some possible relationships between P and Q. Some of them may mean the same as $P \Rightarrow Q$, while some might mean the same as $Q \Rightarrow P$. Decide which is which. Are there any like $Q \Rightarrow P$?

To do this, think about the example above. A simple test is this: $P \Rightarrow Q$ when you can't have P true about some x while Q is false.

(a) If P then Q.
(b) $P \Rightarrow Q$ (P implies Q).
(c) Q if P.
(d) Q is implied by P.
(e) Q follows from P.
(f) P is a sufficient condition for Q.
(g) Q is a necessary condition for P.
(h) Q is a consequence of P.
(i) P holds only if Q holds.
(j) not-$Q \Rightarrow$ not-P (the "contrapositive").

Problem B-2. Here are some statements about x. Write down all the true implications between them, for example, (1) $\Rightarrow$ (2) (really meaning, "for any x, (1) $\Rightarrow$ (2)"). For each two, consider both directions.

(1) $x > 0$
(2) $x^2 > 0$
(3) $x \neq 0$
(4) $x \neq 0$ and $\frac{1}{x} > 0$
(5) $x = y^2$ for some real number y with $y \neq 0$

[1] Problems in handouts are to be done as homework only if assigned.

Problem B-3. In Problem 2, find three “nonimplications” $P \not\Rightarrow Q$ and for each give a “counterexample”—in other words, a value of x for which P is true but not Q.

3. If and only if

Sometimes two statements are true or false together. We express this by a double arrow or by the words “if and only if”.

For any number x, $x \geq 0 \Leftrightarrow x^3 \geq 0$.

An additional way to say this is that $x \geq 0$ is a *necessary and sufficient* condition for $x^3 \geq 0$. Still another way is to say the two statements are *equivalent.*

The words “if and only if” are sometimes abbreviated “iff”.

Problem B-4. In Problem B-2, say which pairs of statements (if any) are equivalent.

4. “The following statements are equivalent”

Sometimes more than two statements are true or false together. This fact is often expressed as in this example:

The following statements are equivalent:

(1) $x \geq 0$
(2) $x^3 \geq 0$
(3) $x = |x|$

This would be the same as saying (1) $\Leftrightarrow$ (2), (1) $\Leftrightarrow$ (3), and (2) $\Leftrightarrow$ (3). Often we just say, “the following are equivalent”, or even just “TFAE”. To prove such an equivalence, it would be enough to prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Problem B-5. In Problem B-2, break the statements into groups so that all the statements within each group are equivalent and no two statements in different groups are equivalent. For each group, state the equivalence using the phrase “the following statements are equivalent”.

(Don’t prove anything. A group could conceivably have just one statement, but that shouldn’t happen in this example.)

Note. The moral of this handout is that math is simpler than English! There is really only one underlying mathematical concept here, $\Rightarrow$, but there are a number of ways to express it in English. In this course, be on the lookout for implications.