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HOMEWORK 1 - SOLUTIONS

Problem 1 (16.1.15). Use symmetry to evaluate $\int\int_{R} x^3 \, dA$ for $R = [-4, 4] \times [0, 5]$.

Solution. For each $y$, the $x$-integral goes from $-4$ to $4$, so is symmetric about the $x$-axis, while the function $x^3$ is odd, meaning that $(-x)^3 = -x^3$. This means that the negative and positive portions of the integral cancel, so each $x$-integral is 0, hence the entire area integral is 0.

Problem 2 (16.1.17). Use symmetry to evaluate $\int\int_{R} \sin x \, dA$ for $R = [0, 2\pi] \times [0, 2\pi]$.

Solution. For each $y$, the $x$-integral is the integral of $\sin x$ over a whole period, so is 0. Thus the entire area integral is 0.

Problem 3 (16.1.24). Evaluate $\int_{-1}^{1} \int_{0}^{\pi} x^2 \sin y \, dx \, dy$.

Solution.

\[
\int_{y=-1}^{y=1} \int_{x=0}^{x=\pi} x^2 \sin y \, dx \, dy = \int_{y=-1}^{y=1} \left[ \frac{x^3}{3} \sin y \right]_{x=0}^{x=\pi} dy \\
= \int_{y=-1}^{y=1} \frac{\pi^3}{3} \sin y \, dy \\
= \frac{\pi^3}{3} \left[ -\cos y \right]_{y=-1}^{y=1} \\
= \frac{\pi^3}{3} (\cos(-1) - \cos 1) \\
= 0.
\]

Problem 4 (16.1.31). Evaluate $\int_{1}^{2} \int_{0}^{4} \frac{dy \, dx}{x + y}$.

Solution.

\[
\int_{x=1}^{x=2} \int_{y=0}^{y=4} \frac{dy \, dx}{x + y} = \int_{x=1}^{x=2} \left[ \ln(x + y) \right]_{y=0}^{y=4} \, dx \\
= \int_{x=1}^{x=2} (\ln(x + 4) - \ln(x)) \, dx \\
= \left[ (x + 4) \ln(x + 4) - (x + 4) \right]_{x=1}^{x=2} - \left[ x \ln x - x \right]_{x=1}^{x=2} \\
= \left[ (6 \ln 6 - 6) - (5 \ln 5 - 5) \right] - \left[ (2 \ln 2 - 2) - (1 \ln 1 - 1) \right] \\
= 6 \ln 6 - 5 \ln 5 - 2 \ln 2.
\]
Problem 5 (16.1.34). Evaluate \( \int_0^2 \int_1^8 \frac{x \, dx \, dy}{\sqrt{x^2 + y}} \).

Solution.

\[
\begin{align*}
\int_{y=0}^{y=8} \int_{x=1}^{x=2} \frac{x \, dx \, dy}{\sqrt{x^2 + y}} &= \int_{u=1+y}^{u=8+y} \int_{y=0}^{y=8} \frac{(1/2) \, du \, dy}{\sqrt{u}} \\
&= \int_{y=0}^{y=8} \left[ \frac{2}{3} (4 + y)^{3/2} - \frac{2}{3} (1 + y)^{3/2} \right]_{y=0}^{y=8} \, dy \\
&= \frac{2}{3} \left[ 24\sqrt{3} - 34 \right] = -68 + 48\sqrt{3}.
\end{align*}
\]

Problem 6 (16.1.40). Evaluate \( \iint_{\mathcal{R}} \frac{y}{x+1} \, dA \) for \( \mathcal{R} = [0, 2] \times [0, 4] \).

Solution.

\[
\begin{align*}
\iint_{\mathcal{R}} \frac{y}{x+1} \, dA &= \int_{y=0}^{y=2} \int_{x=0}^{x=2} \frac{y}{x+1} \, dy \, dx \\
&= \int_{y=0}^{y=2} \left[ \frac{1}{2} \frac{y^2}{x+1} \right]_{x=0}^{x=2} \, dy \\
&= \int_{y=0}^{y=2} \frac{8}{x+1} \, dx \\
&= 8 \ln(x+1) \bigg|_{x=0}^{x=2} = 8 \ln 3.
\end{align*}
\]

Problem 7 (16.1.42). Evaluate \( \iint_{\mathcal{R}} e^{3x+4y} \, dA \) for \( \mathcal{R} = [0, 1] \times [1, 2] \).

Solution.

\[
\begin{align*}
\iint_{\mathcal{R}} e^{3x+4y} \, dA &= \int_{y=1}^{y=2} \int_{x=0}^{x=1} e^{3x+4y} \, dx \, dy \\
&= \int_{y=1}^{y=2} \left[ \frac{1}{3} e^{3x+4y} \right]_{x=0}^{x=1} \, dy \\
&= \frac{1}{3} e^{4y+3} - e^{4y} \bigg|_{y=1}^{y=2} \\
&= e^4 - \frac{1}{3} e^{3y} \bigg|_{y=1}^{y=2} = \frac{(e^3-1)(e^8-e^4)}{12}.
\end{align*}
\]
Problem 8 (16.1.47). Evaluate \( \int_0^1 \int_0^1 \frac{y}{1+xy} \, dy \, dx \).

Solution.

\[
\int_{x=0}^{x=1} \int_{y=0}^{y=1} \frac{y}{1+xy} \, dy \, dx = \int_{y=0}^{y=1} \int_{x=0}^{x=1} \frac{y}{1+xy} \, dx \, dy \quad \text{(Fubini to change order)}
\]

\[
= \int_{y=0}^{y=1} \int_{u=1+y}^{u=1} \frac{1}{u} \, du \, dy = \int_{y=0}^{y=1} \ln u \bigg|_{u=1+y}^{u=1} \, dy \quad (u = 1 + xy)
\]

\[
= \int_{y=0}^{y=1} \ln(1 + y) \, dy = [(1 + y) \ln(1 + y) - (1 + y)]_{y=0}^{y=1}
\]

\[
= [2 \ln 2 - 2] - [1 \ln 1 - 1] = 2 \ln 2 - 1.
\]

Problem 9 (16.1.49). Using Fubini’s theorem, argue that the solid in Figure 1 has volume \( AL \), where \( A \) is the area of the front face of the solid.

![Figure 1](image)

Solution. The surface bounding the solid from above is the graph of a positive function \( z = f(y) \) that does not depend on \( x \). (Here \( a \) is the largest value that \( y \) can take, which is not labeled in the diagram.) The volume of the solid is

\[
\iiint_{[0,L] \times [0,a]} f(y) \, dA = \int_{y=0}^{y=a} \int_{x=0}^{x=L} f(y) \, dx \, dy \quad \text{(Fubini)}
\]

\[
= \int_{y=0}^{y=a} [xf(y)]_{x=0}^{x=L} \, dy = \int_{y=0}^{y=a} L f(y) \, dy
\]

\[
= L \int_{y=0}^{y=a} f(y) \, dy = LA,
\]

where in the last step, we have used the fact that \( A \) is the area under the graph of \( z = f(y) \) in the \((y,z)\)-plane (which is equal to the area of the front face of the solid), which is then given by the integral of \( f \) over the interval \([0,a]\).
Problem 10 (16.2.10). Sketch the region $D$ between $y = x^2$ and $y = x(1 - x)$. Express $D$ as a simple region and calculate the integral of $f(x, y) = 2y$ over $D$.

Solution.

The region $D$ is both vertically simple and horizontally simple, but the bounds for $y$ in terms of $x$ are simpler than the bounds for $x$ in terms of $y$, so when we use Fubini’s theorem to evaluate the integral, we take the $y$-integral on the inside and the $x$-integral on the outside. This gives us

$$
\int\int_D f(x, y) \, dA = \int_{x=0}^{x=1/2} \int_{y=x^2}^{y=x(1-x)} 2y \, dy \, dx \\
= \int_{x=0}^{x=1/2} 2 \int_{y=x^2}^{y=x(1-x)} y \, dy \, dx \\
= \int_{x=0}^{x=1/2} 2 \left[ \frac{1}{2} y^2 \right]_{y=x^2}^{y=x(1-x)} \, dx \\
= \int_{x=0}^{x=1/2} 2 \left( x^2 - x^4 \right) \, dx \\
= \int_{x=0}^{x=1/2} 2x^2 \, dx - \int_{x=0}^{x=1/2} 2x^4 \, dx \\
= \left[ \frac{1}{3} x^3 - \frac{1}{2} x^4 \right]_{x=0}^{x=1/2} \\
= \frac{1}{24} - \frac{1}{32} = \frac{1}{96}.
$$

$\square$
Problem 11 (16.2.14). Integrate \( f(x,y) = (x + y + 1)^{-2} \) over the triangle with vertices (0,0), (4,0), and (0,8).

Solution.

\[
\int \int_D f(x,y) \, dA = \int_{x=0}^{x=4} \int_{y=0}^{y=-2x+8} (x + y + 1)^{-2} \, dy \, dx \quad \text{(Fubini)}
\]

\[
= \int_{x=0}^{x=4} \left[ -(x + y + 1)^{-1} \right]_{y=0}^{y=-2x+8} \, dx
\]

\[
= \int_{x=0}^{x=4} \left( (x + 1)^{-1} - (-x + 9)^{-1} \right) \, dx
\]

\[
= \int_{x=0}^{x=4} \left( (x + 1)^{-1} + (x - 9)^{-1} \right) \, dx
\]

\[
= \left[ \ln|x + 1| + \ln|x - 9| \right]_{x=0}^{x=4}
\]

\[
= \ln(5 + \ln 5) - (\ln 1 + \ln 9)
\]

\[
= \ln(25/9).
\]
Problem 12 (16.2.16). Integrate \( f(x, y) = x \) over the region bounded by \( y = x, \ y = 4x - x^2, \) and \( y = 0 \) in two ways: as a vertically simple region and as a horizontally simple region.

Solution.

If we regard this region as vertically simple, so the \( y \)-integral is inside, then we have to split the integral into two parts depending on which function bounds \( y \) from above. That is,

\[
\iint_D f(x, y) \, dA = \int_{x=0}^{x=4} \int_{y=0}^{y=y_{\text{max}}(x)} x \, dy \, dx \tag{Fubini}
\]

\[
= \int_{x=0}^{x=3} \int_{y=0}^{y=x} x \, dy \, dx + \int_{x=3}^{x=4} \int_{y=0}^{y=4x-x^2} x \, dy \, dx
\]

\[
= \int_{x=0}^{x=3} x^2 \, dx + \int_{x=3}^{x=4} x(4x - x^2) \, dx
\]

\[
= 9 + \left[ \frac{4}{3}x^3 - \frac{1}{4}x^4 \right]_{x=3}^{x=4} = \frac{175}{12}.
\]

If we regard this region as horizontally simple instead, so the \( x \)-integral is inside, then the left boundary is always given by \( x = y \) and the right boundary is always given by the larger value of \( x \) for which \( y = 4x - x^2 \). Using the quadratic formula for the equivalent equation \( x^2 - 4x + y = 0 \), this value is \( x = 2 + \sqrt{4-y} \), so we have

\[
\iint_D f(x, y) \, dA = \int_{y=0}^{y=3} \int_{x=x_{\text{min}}}^{x=x_{\text{max}}} x \, dx \, dy
\]

\[
= \frac{1}{2} \int_{y=0}^{y=3} (2 + \sqrt{4-y})^2 - y^2 \, dy
\]

\[
= \frac{175}{12}.
\]
Problem 13 (16.2.20). Compute the double integral of \( f(x, y) = \cos(2x + y) \) over the domain \( 1/2 \leq x \leq \pi/2 \) and \( 1 \leq y \leq 2x \).

Solution.

\[
\int_{x=\pi/2}^{x=\pi/2} \int_{y=2x}^{y=1} \cos(2x + y) \, dy \, dx = \int_{x=1/2}^{x=\pi/2} \sin(4x) - \sin(2x + 1) \, dx
\]

\[
= \left[ -\frac{1}{4} \cos(4x) + \frac{1}{2} \cos(2x + 1) \right]_{x=1/2}^{x=\pi/2}
\]

\[
= \left[ -\frac{1}{4} \cos(2\pi) + \frac{1}{2} \cos(\pi + 1) \right] - \left[ -\frac{1}{4} \cos 2 + \frac{1}{2} \cos 2 \right]
\]

\[
= -\frac{1}{4} - \frac{1}{2} \cos 1 - \frac{1}{4} \cos 2.
\]

\[\square\]

Problem 14 (16.2.21). Compute the double integral of \( f(x, y) = 2xy \) over the domain bounded by \( x = y \) and \( x = y^2 \).

Solution.

As a horizontally simple region, for each value of \( y \), the left boundary is given by \( x = y^2 \) and the right boundary is given by \( x = y \), so

\[
\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} 2xy \, dx \, dy = \int_{y=0}^{y=1} (y^3 - y^5) \, dy = \frac{1}{12}.
\]

\[\square\]
Problem 15 (16.2.28). For $\int_0^1 \int_{e^x}^e f(x,y) \, dy \, dx$, sketch the domain of integration and express as an iterated integral in the opposite order.

Solution. The domain of integration is the set of points $(x, y)$ for which $0 \leq x \leq 1$ and $e^x \leq y \leq e$, which produces the diagram below.

The iterated integral expresses the integral over the domain interpreted as a vertically simple region. It is also a horizontally simple region, with $1 \leq y \leq e$ and $0 \leq x \leq \ln y$, so

$$\int_{x=0}^{x=1} \int_{y=e^x}^{y=e} f(x,y) \, dy \, dx = \int_{y=1}^{y=e} \int_{x=0}^{x=\ln y} f(x,y) \, dx \, dy.$$
Problem 16 (16.2.35). For \( \int_{0}^{1} \int_{y=x}^{1} xe^{y^3} \, dy \, dx \), sketch the domain of integration. Then change the order of integration and compute. Explain the simplification achieved by changing the order.

Solution. The domain of integration is \( 0 \leq x \leq 1 \) and \( x \leq y \leq 1 \).

Changing the order, so the domain of integration is equivalently given by \( 0 \leq y \leq 1 \) and \( 0 \leq x \leq y \),

\[
\int_{x=0}^{x=1} \int_{y=x}^{y=1} xe^{y^3} \, dy \, dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} xe^{y^3} \, dx \, dy
\]

\[
= \frac{1}{2} \int_{y=0}^{y=1} y^2 e^{y^3} \, dy
\]

\[
= \frac{1}{2} \int_{u=0}^{u=1} e^{u^3} \, \frac{du}{3}
\]

\[
= \frac{e - 1}{6}.
\]

The function we were required to integrate can only be exactly integrated in one direction, so changing the order of integration to use this direction first gives us a chance to obtain something which can be (exactly) integrated a second time.
**Problem 17** (16.2.52). Calculate the average height above the $x$-axis of a point in the region $0 \leq x \leq 1$ and $0 \leq y \leq x^2$.

*Solution.* For a point in the plane, its height above the $y$-axis is its $y$-coordinate, so the average height in the given region is

$$\text{average height} = \frac{1}{\text{area of region}} \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} y \, dy \, dx.$$  

The area of a region is found by integrating 1 over the region, so

$$\text{average height} = \left( \int_{x=0}^{x=1} x^2 \, dx \right)^{-1} \left( \int_{x=0}^{x=1} x \, dx \right) = \left( \frac{1}{3} \right)^{-1} \left( \frac{1}{10} \right) = \frac{3}{10}.$$ 

**Problem 18** (16.2.55). What is the average value of the linear function

$$f(x, y) = mx + ny + p$$
on the ellipse $(x/a)^2 + (y/b)^2 \leq 1$? Argue by symmetry rather than calculation.

*Solution.* If $(x, y)$ is a point in the ellipse, then $(-x, -y)$ is a point in the ellipse as well, and if we pair these two points together, they average to

$$\frac{f(x, y) + f(-x, -y)}{2} = \frac{(mx + ny + p) + (-mx - ny + p)}{2} = p.$$  

Doing this for every pair of points, the average of $f(x, y)$ on the ellipse as a whole is $p$. 

**Problem 19** (16.2.59). Prove the inequality $\int\int_D \frac{dA}{4 + x^2 + y^2} \leq \pi$, where $D$ is the disc $x^2 + y^2 \leq 4$.

*Solution.* Since $x^2, y^2 \geq 0$ for all $(x, y)$, we have

$$\frac{1}{4 + x^2 + y^2} \leq \frac{1}{4}$$

for all $(x, y)$. Thus

$$\int\int_D \frac{dA}{4 + x^2 + y^2} \leq \int\int_D \frac{dA}{4} = \frac{1}{4} \cdot (\text{area of } D) = \frac{1}{4} \cdot 4\pi = \pi.$$
HOMEWORK 2 - SOLUTIONS

Problem 1 (16.3.3). Evaluate \( \iiint_B xe^{y-2z} \, dV \) for the box \( B = \{ 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1 \} \).

Solution.
\[
\iiint_B xe^{y-2z} \, dV = \int_{z=0}^{z=1} \int_{y=0}^{y=1} \int_{x=0}^{x=2} xe^{y}e^{-2z} \, dx \, dy \, dz
= \int_{z=0}^{z=1} e^{-2z} \, dz \int_{y=0}^{y=1} e^{y} \, dy \int_{x=0}^{x=2} x \, dx
= \frac{1}{2} (1 - e^{-2}) \cdot (e - 1) \cdot 2 = (e - 1)(1 - e^{-2}).
\]

\( \square \)

Problem 2 (16.3.10). Evaluate \( \iiint_W e^{x+y+z} \, dV \) over the region
\( W = \{ 0 \leq z \leq 1, 0 \leq y \leq x, 0 \leq x \leq 1 \} \).

Solution.
\[
\iiint_W e^{x+y+z} \, dV = \int_{z=0}^{z=1} \int_{y=0}^{y=x} \int_{x=0}^{x=1} e^{x+y+z} \, dy \, dx \, dz
= \int_{z=0}^{z=1} e^{z} \, dz \int_{y=0}^{y=x} e^{x} \, dy \int_{x=0}^{x=1} e^{x} \, dx
= (e - 1) \int_{u=e^{-1}}^{u=1} u \, du = \frac{1}{2} (e - 1)^3.
\]

\( \square \)

Problem 3 (16.3.15). Calculate the integral of \( f(x,y,z) = z \) over the region \( W \) below the hemisphere of radius 3 and lying over the triangle \( D \) in the \( xy \)-plane bounded by \( x = 1 \), \( y = 0 \), and \( x = y \).

Solution. We have \( D = \{ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x \} \), and for \( z \) to lie below the hemisphere of radius 3, we must have \( x^2 + y^2 + z^2 \leq 9 \), so
\( W = \{ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x \text{ and } 0 \leq z \leq \sqrt{9 - x^2 - y^2} \} \).

Then
\[
\iiint_W f(x,y,z) \, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \int_{z=0}^{z=\sqrt{9-x^2-y^2}} z \, dz \, dy \, dx
= \int_{x=0}^{x=1} \int_{y=0}^{y=x} 9 - x^2 - y^2 \, dy \, dx
= \int_{x=0}^{x=1} 9x - x^3 - \frac{1}{3} z^3 \, dx
= \frac{9}{2} - \frac{1}{4} - \frac{1}{12} = \frac{25}{6}.
\]

\( \square \)
Problem 4 (16.3.17). Integrate \( f(x, y, z) = x \) over the region in the first octant \((x, y, z \geq 0)\) above \( z = y^2 \) and below \( z = 8 - 2x^2 - y^2 \).

Solution. The bounds on \((x, y)\) are given by the intersection of the two surfaces,
\[
z = y^2 \quad \text{and} \quad z = 8 - 2x^2 - y^2 \implies x^2 + y^2 = 4.
\]
Hence the region of integration is
\[
W = \{0 \leq y \leq 2 \text{ and } 0 \leq x \leq \sqrt{4-y^2} \text{ and } y^2 \leq z \leq 8 - 2x^2 - y^2\},
\]
and
\[
\iiint_W f(x, y, z) \, dV = \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} \int_{z=y^2}^{z=8-2x^2-y^2} x \, dz \, dx \, dy
\]
\[
= \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} x(8-2x^2-2y^2) \, dx \, dy
\]
\[
= \int_{y=0}^{y=2} \left[ \frac{1}{8} (8-2y^2)^2 \right]_0^u \, du
\]
\[
= \int_{y=0}^{y=2} \frac{1}{8} u^2 \, du
\]
\[
= \int_{y=0}^{y=2} \frac{1}{8} (8-2y^2)^2 \, dy
\]
\[
= \int_{y=0}^{y=2} \frac{1}{2} (8-2y^2) \, dy
\]
\[
= \int_{y=0}^{y=2} \frac{1}{2} (8^2 - 4y^2 + 8) \, dy
\]
\[
= \left[ \frac{32}{10} - \frac{32}{3} + 16 \right] = 128/15.
\]

Problem 5 (16.3.20). Find the volume of the solid in \( \mathbb{R}^3 \) bounded by \( y = x^2, \, x = y^2, \, z = x+y+5, \) and \( z = 0. \)

Solution. Note that the solid is symmetric about the plane \( x = y \) (the set of bounding equations is unchanged upon swapping \( x \) and \( y \)). Therefore, it suffices to compute the volume of the solid bounded by \( y = x^2, \, x = y, \, z = x+y+5, \) and \( z = 0, \) then double it. Hence we compute
\[
2 \int_{x=0}^{x=1} \int_{y=x}^{y=x+5} \int_{z=0}^{z=x+y+5} 1 \, dz \, dy \, dx = 2 \int_{x=0}^{x=1} \int_{y=x}^{y=x+5} (x+y+5) \, dy \, dx
\]
\[
= 2 \int_{x=0}^{x=1} \frac{1}{2} (x+y+5)^2 - \frac{1}{2} (x+x^2+5)^2 \, dx
\]
\[
= 2 \int_{x=0}^{x=1} -\frac{1}{2} x^4 - x^3 - \frac{7}{2} x^2 + 5x \, dx
\]
\[
= 2 \left( -\frac{1}{10} - \frac{1}{4} - \frac{7}{6} + \frac{5}{2} \right) = 59/30.
\]
Problem 6 (16.3.25). Describe the domain of integration of the integral
\[
\int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1}^{\sqrt{5-x^2-z^2}} f(x, y, z) \, dy \, dx \, dz.
\]

Solution. The domain is
\[
D = \{ -2 \leq z \leq 2 \text{ and } -\sqrt{4-z^2} \leq x \leq \sqrt{4-z^2} \text{ and } 1 \leq y \leq \sqrt{5-x^2-z^2} \}
\]
\[
= \{ -2 \leq z \leq 2 \text{ and } x^2 + z^2 \leq 4 \text{ and } 1 \leq y \leq \sqrt{5-x^2-z^2} \}
\]
\[
= \{ x^2 + z^2 \leq 4 \text{ and } y \geq 1 \text{ and } x^2 + y^2 + z^2 \leq 5 \}
\]
where in the last step, we removed the first inequality since it follows from the other two. This is the smaller region between the sphere of radius \( \sqrt{5} \) centered at the origin and the plane \( y = 1 \).

Problem 7 (16.3.28). Let \( W \) be the region bounded by \( y + z = 2, 2x = y, x = 0, \) and \( z = 0 \). Express and evaluate the triple integral of \( f(x, y, z) = z \) by projecting \( W \) onto the
(a) \( xy \)-plane;
(b) \( yz \)-plane;
(c) \( xz \)-plane.

Solution. 1. The projection onto the \( xy \)-plane is the triangle \( \{ 0 \leq x \leq 1 \text{ and } 2x \leq y \leq 2 \} \), so
\[
\iiint_W f(x, y, z) \, dV = \int_{x=0}^{1} \int_{y=2x}^{2} \int_{z=0}^{2-y} z \, dz \, dy \, dx
\]
\[
= \int_{x=0}^{1} \int_{y=2x}^{2} \frac{1}{6} (2-y)^3 \, dy \, dx
\]
\[
= \int_{x=0}^{1} \frac{4}{3} (1-x)^3 \, dx = \frac{1}{3}.
\]

2. The projection onto the \( yz \)-plane is the triangle \( \{ 0 \leq y \leq 2 \text{ and } 0 \leq z \leq 2-y \} \), so
\[
\iiint_W f(x, y, z) \, dV = \int_{y=0}^{2} \int_{z=0}^{2-y} \int_{x=0}^{y/2} z \, dx \, dz \, dy = \frac{1}{3}.
\]

3. The projection onto the \( xz \)-plane is the triangle \( \{ 0 \leq x \leq 1 \text{ and } 0 \leq z \leq 2-2x \} \), so
\[
\iiint_W f(x, y, z) \, dV = \int_{x=0}^{1} \int_{z=2-2x}^{2-z} \int_{y=x/2}^{y=2-z} z \, dy \, dz \, dx = \frac{1}{3}.
\]
Problem 8 (16.3.29). Let
\[ W = \left\{ (x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1 \right\}. \]
Express \( \iiint_W f(x, y, z) \, dV \) as an iterated integral in the order \( dz \, dy \, dx \).

Solution. We must have \( x^2 + y^2 \leq 1 \), so \( -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2} \). Thus
\[
\iiint_W f(x, y, z) \, dV = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=1} f(x, y, z) \, dz \, dy \, dx.
\]

Problem 9 (16.3.30). Repeat the previous exercise for the order \( dx \, dy \, dz \).

Solution. Here we have \( 0 \leq z \leq 1 \), then \( -z \leq y \leq z \), then \( -\sqrt{z^2 - y^2} \leq x \leq \sqrt{z^2 - y^2} \), so
\[
\iiint_W f(x, y, z) \, dV = \int_{z=0}^{z=1} \int_{y=-z}^{y=z} \int_{x=\sqrt{z^2-y^2}}^{x=\sqrt{z^2-y^2}} f(x, y, z) \, dx \, dy \, dz.
\]

Problem 10 (16.3.35). Draw the region \( W \) bounded by the surfaces given by \( z = y^2 \), \( y = z^2 \), \( x = 0 \), and \( x+y+z = 4 \). Set up, but do not compute, a single triple integral that yields the volume of \( W \).

Solution. [drawing omitted for now]
The region is
\[ W = \{ 0 \leq z \leq 1 \text{ and } z^2 \leq y \leq \sqrt{z} \text{ and } 0 \leq x \leq 4 - y - z \}, \]
so the volume of \( W \) is
\[
\iiint_W \, dV = \int_{z=0}^{z=1} \int_{y=z^2}^{y=\sqrt{z}} \int_{x=0}^{x=4-y-z} \, dx \, dy \, dz.
\]

Problem 11 (12.3.13). Convert the equation \( r = 2 \sin \theta \) to rectangular coordinates.

Solution. Multiply by \( r \) to get \( r^2 = 2r \sin \theta \), which gives \( x^2 + y^2 = 2y \), or \( x^2 + (y-1)^2 = 1 \).

Problem 12 (12.3.16). Convert the equation \( r = 1/(2 - \cos \theta) \) to rectangular coordinates.

Solution. Clearing denominators, \( 2r - r \cos \theta = 1 \), or \( 2\sqrt{x^2+y^2} = 1 + r \cos \theta = 1 + x \). Squaring, \( 4x^2 + 4y^2 = 1 + 2x + x^2 \), or \( 3x^2 - 2x + 4y^2 = 1 \).
Problem 13 (12.3.18). Convert the equation \( x = 5 \) to a polar equation of the form \( r = f(\theta) \).

Solution. This becomes \( r \cos \theta = 5 \), so \( r = 5 \sec \theta \).

Problem 14 (12.3.19). Convert the equation \( y = x^2 \) to a polar equation of the form \( r = f(\theta) \).

Solution. This becomes \( r \sin \theta = r^2 \cos^2 \theta \), so \( r = \sin \theta / \cos^2 \theta = \sec \theta \tan \theta \).

Problem 15 (12.3.20). Convert the equation \( xy = 1 \) to a polar equation of the form \( r = f(\theta) \).

Solution. This becomes \( r^2 \sin \theta \cos \theta = 1 \), so \( r = \sqrt{\sec \theta \csc \theta} \).

Problem 16 (12.3.23). Match each equation with its description:

(a) \( r = 2 \)  
(b) \( \theta = 2 \)  
(c) \( r = 2 \sec \theta \)  
(d) \( r = 2 \csc \theta \)

(i) Vertical line  
(ii) Horizontal line  
(iii) Circle  
(iv) Line through origin

Solution. (a) This is the circle of radius 2 around the origin, which fits (iii).

(b) This is a ray from the origin outward in the direction corresponding to \( \theta = 2 \), which fits most closely (though not quite) with (iv).

(c) This equation becomes \( r \cos \theta = 2 \), or \( x = 2 \), so we have a vertical line, which fits (i).

(d) This equation becomes \( r \sin \theta = 2 \), or \( y = 2 \), so we have a horizontal line, which fits (ii).

Problem 17 (12.3.37). Show that \( r = a \cos \theta + b \sin \theta \) is the equation of a circle passing through the origin. Express the radius and center (in rectangular coordinates) in terms of \( a \) and \( b \), and write down the equation in rectangular coordinates.

Solution. Multiply through by \( r \) to get \( r^2 = ar \cos \theta + br \sin \theta \), or \( x^2 + y^2 = ax + by \). Completing the square gives us

\[
\left( x - \frac{1}{2}a \right)^2 + \left( y - \frac{1}{2}b \right)^2 = \left( \frac{a^2 + b^2}{4} \right),
\]

so the center is \((a/2, b/2)\) and the radius is \(\sqrt{a^2 + b^2}/2\).

Problem 18 (12.3.47). Show that every line that does not pass through the origin has a polar equation of the form

\[
r = \frac{b}{\sin \theta - a \cos \theta},
\]

where \( b \neq 0 \).

Solution. Any line which does not pass through the origin has rectangular equation \( y = ax + b \) for some \( a \) and \( b \neq 0 \). Then \( r \sin \theta = ar \cos \theta + b \), and solving for \( r \), we get the required form.
HOMEWORK 3 - SOLUTIONS

Problem 1 (16.4.4). Sketch $D = \{y \geq 0 \text{ and } x^2 + y^2 \leq 1\}$ and integrate $f(x, y) = y(x^2 + y^2)^3$ over $D$ using polar coordinates.

Solution.

From the diagram, we can see that $D = \{0 \leq \theta \leq \pi \text{ and } 0 \leq r \leq 1\}$, so

$$
\int_{D} f(x, y) \, dA = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} r \sin \theta (r^2)^3 \cdot r \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \sin \theta \, d\theta \int_{r=0}^{r=1} r^8 \, dr = 2 \cdot \frac{1}{9} = \frac{2}{9}.
$$

Problem 2 (16.4.10). For $\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{16-x^2}} \tan^{-1}\left(\frac{y}{x}\right) \, dy \, dx$, sketch the region of integration and evaluate by changing to polar coordinates.

Solution.

In this range, $\tan^{-1}(y/x) = \theta$, so

$$
\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{16-x^2}} \tan^{-1}\left(\frac{y}{x}\right) \, dy \, dx = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} \theta \cdot r \, dr \, d\theta
$$

$$
= \int_{\theta=0}^{\theta=\pi/2} \theta \, d\theta \int_{r=0}^{r=1} r \, dr

= \frac{(\pi/2)^2}{2} \cdot \frac{1}{2} = \frac{\pi^2}{16}.
$$
Problem 3 (16.4.21). Find the volume of the wedge-shaped region contained in the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$ and below by the $xy$-plane.

Solution. In cylindrical polar coordinates, the region is given by

$$0 \leq r \leq 3, \quad -\pi/2 \leq \theta \leq \pi/2, \quad 0 \leq z \leq r \cos \theta,$$

so the volume is

$$\int_{\theta = -\pi/2}^{\pi/2} \int_{r = 0}^{3} \int_{z = 0}^{r \cos \theta} r \, dz \, dr \, d\theta = \int_{\theta = -\pi/2}^{\pi/2} \int_{r = 0}^{3} r^2 \cos \theta \, dr \, d\theta = \int_{\theta = -\pi/2}^{\pi/2} 9 \cos \theta \, d\theta = 18.$$

Figure 1

Problem 4 (16.4.23). Evaluate $\int\int_{D} \sqrt{x^2 + y^2} \, dA$, where $D$ is the domain above.

Hint: Find the equation of the inner circle in polar coordinates and treat the right and left parts of the region separately.

Problem 5 (16.4.30). Use cylindrical coordinates to compute $\int\int\int_{W} z \sqrt{x^2 + y^2} \, dV$ for the region given by $x^2 + y^2 \leq z \leq 8 - (x^2 + y^2)$.

Problem 6 (16.4.35). Express $\int_{-1}^{1} \int_{y = \sqrt{1-x^2}}^{y=0} \int_{z=0}^{\sqrt{x^2+y^2}} f(x, y, z) \, dz \, dy \, dx$.

Problem 7 (16.4.40). Use cylindrical coordinates to find the volume of a sphere of radius $a$ from which a central cylinder of radius $b$ has been removed, where $0 < b < a$.

Problem 8 (16.4.49). Use spherical coordinates to calculate the triple integral of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the region $x^2 + y^2 + z^2 \leq 2z$.

Problem 9 (16.4.52). Find the volume of the region lying above the cone $\phi = \phi_0$ and below the sphere $\rho = R$. 

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**Problem 10** (16.4.56). Let \( \mathcal{W} \) be the region within the cylinder \( x^2 + y^2 = 2 \) between \( z = 0 \) and the cone \( z = \sqrt{x^2 + y^2} \). Calculate the integral of \( f(x, y, z) = x^2 + y^2 \) over \( \mathcal{W} \), using both spherical and cylindrical coordinates.

**Problem 11** (16.5.8). Compute the total mass of the plate in Figure 2 assuming a mass density of \( f(x, y) = x^2 / (x^2 + y^2) \text{ g cm}^{-2} \).

![Figure 2: 16.5.8](image)

**Problem 12** (16.5.13). Find the centroid of the quarter circle \( x^2 + y^2 \leq R^2 \) with \( x, y \geq 0 \) assuming the density \( \delta(x, y) = 1 \).

**Problem 13** (16.5.16). Show that the centroid of the sector in Figure 3a has \( y \)-coordinate

\[
\bar{y} = \left( \frac{2R}{3} \right) \left( \frac{\sin \alpha \alpha}{\alpha} \right).
\]

![Figure 3](image)

**Problem 14** (16.5.20). Show that the \( z \)-coordinate of the centroid of the tetrahedron bounded by the coordinate planes and the plane \( (x/a) + (y/b) + (z/c) = 1 \) in Figure 3b is \( \bar{z} = c/4 \). Conclude by symmetry that the centroid is \((a/4, b/4, c/4)\).
Problem 15 (16.5.21). Find the centroid of the region $W$ lying above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$ (Figure 4).

Problem 16 (16.5.24). Find the center of mass of the region bounded by $y^2 = x + 4$ and $x = 0$ with mass density $\delta(x, y) = |y|$.

Problem 17 (16.5.27). Find the $z$-coordinate of the center of mass of the first octant of the unit sphere with mass density $\delta(x, y, z) = y$ (Figure 5).