Contents

HOMEWORK 1 - SOLUTIONS

Problem 1 (16.1.15)**.** Use symmetry to evaluate $\iint_{\mathcal{R}} x^3\, dA$ for $\mathcal{R} = [-4, 4] \times [0, 5]$.

Solution. For each y, the x-integral goes from -4 to 4, so is symmetric about the x-axis, while the function x^3 is odd, meaning that $(-x)^3 = -x^3$. This means that the negative and positive portions of the integral cancel, so each x-integral is 0, hence the entire area integral is 0. □

Problem 2 (16.1.17)**.** Use symmetry to evaluate $\iint_{\mathcal{R}} \sin x\, dA$ for $\mathcal{R} = [0, 2\pi] \times [0, 2\pi]$.

Solution. For each y, the x-integral is the integral of $\sin x$ over a whole period, so is 0. Thus the entire area integral is 0. □

Problem 3 (16.1.24)**.** Evaluate $\int_{-1}^{1} \int_{0}^{\pi} x^2 \sin y\, dx\, dy$.

Solution.

$$\begin{aligned}
\int_{y=-1}^{y=1} \int_{x=0}^{x=\pi} x^2 \sin y\, dx\, dy &= \int_{y=-1}^{y=1} \left[\frac{x^3}{3} \sin y \right]_{x=0}^{x=\pi} dy \\
&= \int_{y=-1}^{y=1} \frac{\pi^3}{3} \sin y\, dy \\
&= \frac{\pi^3}{3} [-\cos y]_{y=-1}^{y=1} \\
&= \frac{\pi^3}{3} (\cos(-1) - \cos 1) \\
&= 0.
\end{aligned}$$

□

Problem 4 (16.1.31)**.** Evaluate $\int_{1}^{2} \int_{0}^{4} \frac{dy\, dx}{x+y}$.

Solution.

$$\begin{aligned}
\int_{x=1}^{x=2} \int_{y=0}^{y=4} \frac{dy\, dx}{x+y} &= \int_{x=1}^{x=2} [\ln(x+y)]_{y=0}^{y=4}\, dx \\
&= \int_{x=1}^{x=2} (\ln(x+4) - \ln(x))\, dx \\
&= [(x+4)\ln(x+4) - (x+4)]_{x=1}^{x=2} - [x \ln x - x]_{x=1}^{x=2} \\
&= [(6 \ln 6 - 6) - (5 \ln 5 - 5)] - [(2 \ln 2 - 2) - (1 \ln 1 - 1)] \\
&= 6 \ln 6 - 5 \ln 5 - 2 \ln 2.
\end{aligned}$$

□

Problem 5 (16.1.34)**.** Evaluate $\displaystyle\int_0^8 \int_1^2 \frac{x\,dx\,dy}{\sqrt{x^2+y}}$.

Solution.

$$\begin{aligned}
\int_{y=0}^{y=8}\int_{x=1}^{x=2} \frac{x\,dx\,dy}{\sqrt{x^2+y}} &= \int_{y=0}^{y=8}\int_{u=1+y}^{u=4+y} \frac{(1/2)\,du\,dy}{\sqrt{u}} && (u = x^2+y)\\
&= \int_{y=0}^{y=8} \left[\sqrt{u}\right]_{u=1+y}^{u=4+y} dy = \int_{y=0}^{y=8} \left(\sqrt{4+y}-\sqrt{1+y}\right) dy\\
&= \left[\frac{2}{3}(4+y)^{3/2} - \frac{2}{3}(1+y)^{3/2}\right]_{y=0}^{y=8} = \frac{2}{3}\left[\left(12^{3/2}-9^{3/2}\right) - \left(4^{3/2}-1^{3/2}\right)\right]\\
&= \frac{2}{3}\left[24\sqrt{3}-34\right] = \frac{-68+48\sqrt{3}}{3}.
\end{aligned}$$

□

Problem 6 (16.1.40)**.** Evaluate $\displaystyle\iint_{\mathcal{R}} \frac{y}{x+1}\,dA$ for $\mathcal{R} = [0,2]\times[0,4]$.

Solution.

$$\begin{aligned}
\iint_{\mathcal{R}} \frac{y}{x+1}\,dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=4} \frac{y}{x+1}\,dy\,dx && \text{(Fubini)}\\
&= \int_{x=0}^{x=2} \left[\frac{1}{2}\frac{y^2}{x+1}\right]_{y=0}^{y=4} dx = \int_{x=0}^{x=2} \frac{8}{x+1}\,dx\\
&= 8\left[\ln(x+1)\right]_{x=0}^{x=2} = 8\ln 3.
\end{aligned}$$

□

Problem 7 (16.1.42)**.** Evaluate $\displaystyle\iint_{\mathcal{R}} e^{3x+4y}\,dA$ for $\mathcal{R} = [0,1]\times[1,2]$.

Solution.

$$\begin{aligned}
\iint_{\mathcal{R}} e^{3x+4y}\,dA &= \int_{y=1}^{y=2}\int_{x=0}^{x=1} e^{3x+4y}\,dx\,dy && \text{(Fubini)}\\
&= \int_{y=1}^{y=2} \left[\frac{1}{3}e^{3x+4y}\right]_{x=0}^{x=1} dy = \frac{1}{3}\int_{y=1}^{y=2} e^{4y+3} - e^{4y}\,dy\\
&= \frac{e^3-1}{3}\int_{y=1}^{y=2} e^{4y}\,dy = \frac{e^3-1}{3}\left[\frac{1}{4}e^{4y}\right]_{y=1}^{y=2} = \frac{(e^3-1)(e^8-e^4)}{12}.
\end{aligned}$$

□

Problem 8 (16.1.47)**.** Evaluate $\displaystyle\int_0^1 \int_0^1 \frac{y}{1+xy}\,dy\,dx$.

Solution.

$$\begin{aligned}
\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{y}{1+xy}\,dy\,dx &= \int_{y=0}^{y=1}\int_{x=0}^{x=1} \frac{y}{1+xy}\,dx\,dy && \text{(Fubini to change order)}\\
&= \int_{y=0}^{y=1}\int_{u=1}^{u=1+y} \frac{1}{u}\,du\,dy = \int_{y=0}^{y=1} [\ln u]_{u=1}^{u=1+y}\,dy && (u = 1+xy)\\
&= \int_{y=0}^{y=1} \ln(1+y)\,dy = [(1+y)\ln(1+y) - (1+y)]_{y=0}^{y=1}\\
&= [2\ln 2 - 2] - [1\ln 1 - 1] = 2\ln 2 - 1.
\end{aligned}$$

□

Problem 9 (16.1.49)**.** Using Fubini's theorem, argue that the solid in Figure 1 has volume AL, where A is the area of the front face of the solid.

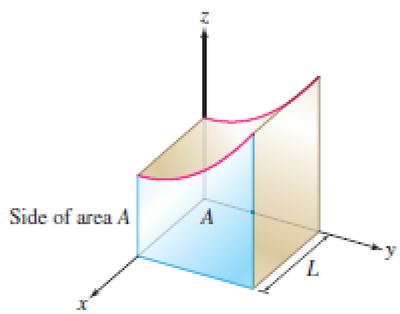


Figure 1

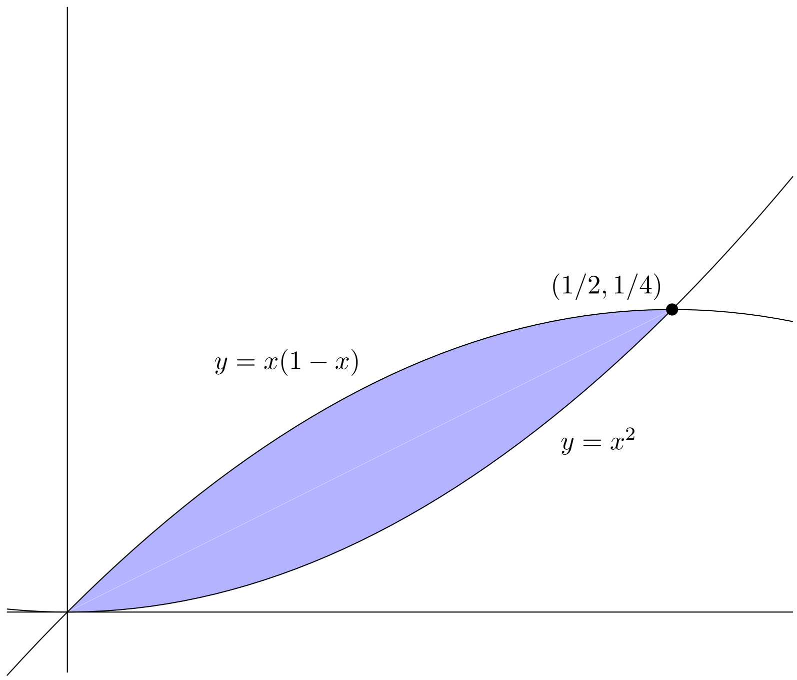
Solution. The surface bounding the solid from above is the graph of a positive function $z = f(y)$ that does not depend on x. (Here a is the largest value that y can take, which is not labeled in the diagram.) The volume of the solid is

$$\begin{aligned}
\iint_{[0,L]\times[0,a]} f(y)\,dA &= \int_{y=0}^{y=a}\int_{x=0}^{x=L} f(y)\,dx\,dy && \text{(Fubini)}\\
&= \int_{y=0}^{y=a} [xf(y)]_{x=0}^{x=L}\,dy = \int_{y=0}^{y=a} Lf(y)\,dy\\
&= L\int_{y=0}^{y=a} f(y)\,dy = LA,
\end{aligned}$$

where in the last step, we have used the fact that A is the area under the graph of $z = f(y)$ in the (y,z)-plane (which is equal to the area of the front face of the solid), which is then given by the integral of f over the interval $[0,a]$. □

Problem 10 (16.2.10)**.** Sketch the region $\mathcal{D}$ between $y = x^2$ and $y = x(1-x)$. Express $\mathcal{D}$ as a simple region and calculate the integral of $f(x,y) = 2y$ over $\mathcal{D}$.

Solution.



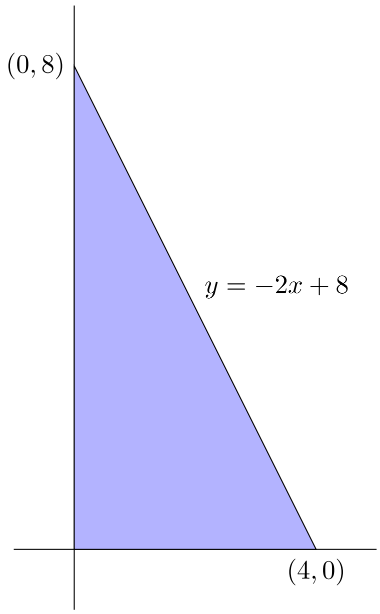
The region $\mathcal{D}$ is both vertically simple and horizontally simple, but the bounds for y in terms of x are simpler than the bounds for x in terms of y, so when we use Fubini's theorem to evaluate the integral, we take the y-integral on the inside and the x-integral on the outside. This gives us

$$\begin{aligned}
\iint_{\mathcal{D}} f(x,y)\,dA &= \int_{x=0}^{x=1/2} \int_{y=x^2}^{y=x(1-x)} 2y\,dy\,dx && \text{(Fubini)} \\
&= \int_{x=0}^{x=1/2} [y^2]_{y=x^2}^{y=x(1-x)}\,dx \\
&= \int_{x=0}^{x=1/2} x^2(1-x)^2 - x^4\,dx \\
&= \int_{x=0}^{x=1/2} x^2(1-2x)\,dx \\
&= \left[\frac{1}{3}x^3 - \frac{1}{2}x^4\right]_{x=0}^{x=1/2} \\
&= \frac{1}{24} - \frac{1}{32} = \frac{1}{96}.
\end{aligned}$$

□

Problem 11 (16.2.14)**.** Integrate $f(x,y) = (x+y+1)^{-2}$ over the triangle with vertices $(0,0)$, $(4,0)$, and $(0,8)$.

Solution.



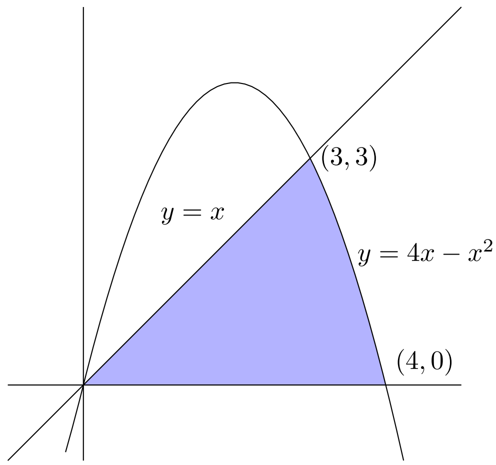
The bounds for y in terms of x are simpler than the bounds for x in terms of y (in particular, no fractions), so we take the y-integral on the inside and the x-integral on the outside. We have

$$\begin{aligned}
\iint_{\mathcal{D}} f(x,y)\,dA &= \int_{x=0}^{x=4} \int_{y=0}^{y=-2x+8} (x+y+1)^{-2}\,dy\,dx && \text{(Fubini)} \\
&= \int_{x=0}^{x=4} \left[-(x+y+1)^{-1}\right]_{y=0}^{y=-2x+8}\,dx \\
&= \int_{x=0}^{x=4} \left((x+1)^{-1} - (-x+9)^{-1}\right)\,dx \\
&= \int_{x=0}^{x=4} \left((x+1)^{-1} + (x-9)^{-1}\right)\,dx \\
&= \left[\ln|x+1| + \ln|x-9|\right]_{x=0}^{x=4} \\
&= (\ln 5 + \ln 5) - (\ln 1 + \ln 9) \\
&= \ln(25/9).
\end{aligned}$$

□

Problem 12 (16.2.16)**.** Integrate $f(x, y) = x$ over the region bounded by $y = x$, $y = 4x - x^2$, and $y = 0$ in two ways: as a vertically simple region and as a horizontally simple region.

Solution.



If we regard this region as vertically simple, so the y-integral is inside, then we have to split the integral into two parts depending on which function bounds y from above. That is,

$$\begin{aligned}
\iint_{\mathcal{D}} f(x,y)\,dA &= \int_{x=0}^{x=4}\int_{y=0}^{y=y_{\max}(x)} x\,dy\,dx && \text{(Fubini)}\\
&= \int_{x=0}^{x=3}\int_{y=0}^{y=x} x\,dy\,dx + \int_{x=3}^{x=4}\int_{y=0}^{y=4x-x^2} x\,dy\,dx\\
&= \int_{x=0}^{x=3} x^2\,dx + \int_{x=3}^{x=4}\int x(4x-x^2)\,dx\\
&= 9 + \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_{x=3}^{x=4} = \frac{175}{12}.
\end{aligned}$$

If we regard this region as horizontally simple instead, so the x-integral is inside, then the left boundary is always given by $x = y$ and the right boundary is always given by the larger value of x for which $y = 4x - x^2$. Using the quadratic formula for the equivalent equation $x^2 - 4x + y = 0$, this value is $x = 2 + \sqrt{4 - y}$, so we have

$$\begin{aligned}
\iint_{\mathcal{D}} f(x,y)\,dA &= \int_{y=0}^{y=3}\int_{x=y}^{x=2+\sqrt{4-y}} x\,dx\,dy\\
&= \frac{1}{2}\int_{y=0}^{y=3} (2+\sqrt{4-y})^2 - y^2\,dy\\
&= \frac{175}{12}.
\end{aligned}$$

□

Problem 13 (16.2.20)**.** Compute the double integral of $f(x, y) = \cos(2x + y)$ over the domain $1/2 \le x \le \pi/2$ and $1 \le y \le 2x$.

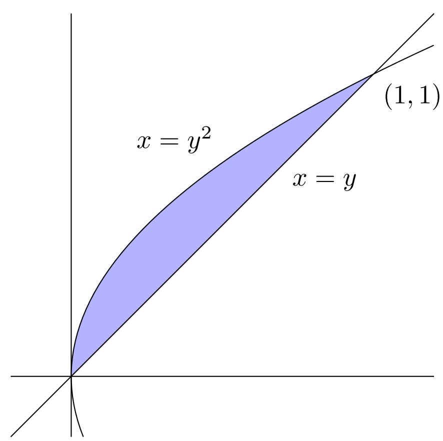
Solution.

$$\begin{aligned}
\int_{x=1/2}^{x=\pi/2} \int_{y=1}^{y=2x} \cos(2x+y)\,dy\,dx &= \int_{x=1/2}^{x=\pi/2} \sin(4x) - \sin(2x+1)\,dx \\
&= \left[-\frac{1}{4}\cos(4x) + \frac{1}{2}\cos(2x+1)\right]_{x=1/2}^{x=\pi/2} \\
&= \left[-\frac{1}{4}\cos(2\pi) + \frac{1}{2}\cos(\pi+1)\right] - \left[-\frac{1}{4}\cos 2 + \frac{1}{2}\cos 2\right] \\
&= -\frac{1}{4} - \frac{1}{2}\cos 1 - \frac{1}{4}\cos 2.
\end{aligned}$$

□

Problem 14 (16.2.21)**.** Compute the double integral of $f(x, y) = 2xy$ over the domain bounded by $x = y$ and $x = y^2$.

Solution.



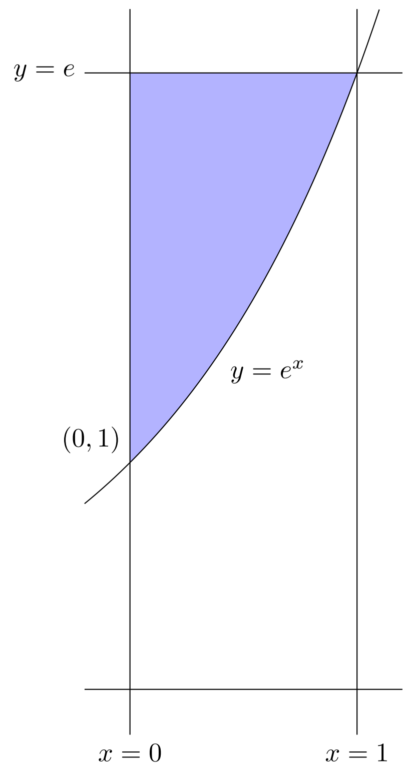
As a horizontally simple region, for each value of y, the left boundary is given by $x = y^2$ and the right boundary is given by $x = y$, so

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} 2xy\,dx\,dy = \int_{y=0}^{y=1} (y^3 - y^5)\,dy = \frac{1}{12}.$$

□

Problem 15 (16.2.28)**.** For $\int_0^1 \int_{e^x}^e f(x,y)\,dy\,dx$, sketch the domain of integration and express as an iterated integral in the opposite order.

Solution. The domain of integration is the set of points (x, y) for which $0 \le x \le 1$ and $e^x \le y \le e$, which produces the diagram below.



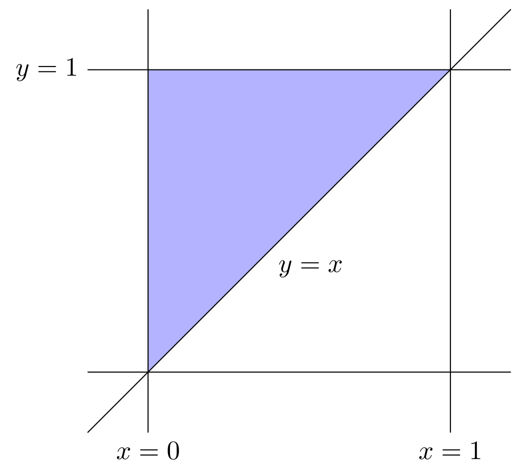
The iterated integral expresses the integral over the domain interpreted as a vertically simple region. It is also a horizontally simple region, with $1 \le y \le e$ and $0 \le x \le \ln y$, so

$$\int_{x=0}^{x=1} \int_{y=e^x}^{y=e} f(x,y)\,dy\,dx = \int_{y=1}^{y=e} \int_{x=0}^{x=\ln y} f(x,y)\,dx\,dy.$$

□

Problem 16 (16.2.35)**.** For $\int_0^1 \int_{y=x}^1 xe^{y^3}\, dy\, dx$, sketch the domain of integration. Then change the order of integration and compute. Explain the simplification achieved by changing the order.

Solution. The domain of integration is $0 \le x \le 1$ and $x \le y \le 1$.



Changing the order, so the domain of integration is equivalently given by $0 \le y \le 1$ and $0 \le x \le y$,

$$\begin{aligned}\int_{x=0}^{x=1} \int_{y=x}^{y=1} xe^{y^3}\, dy\, dx &= \int_{y=0}^{y=1} \int_{x=0}^{x=y} xe^{y^3}\, dx\, dy \\ &= \frac{1}{2} \int_{y=0}^{y=1} y^2 e^{y^3}\, dy \\ &= \frac{1}{2} \int_{u=0}^{u=1} \frac{e^u}{3}\, du \qquad (u = y^3) \\ &= \frac{e-1}{6}.\end{aligned}$$

The function we were required to integrate can only be exactly integrated in one direction, so changing the order of integration to use this direction first gives us a chance to obtain something which can be (exactly) integrated a second time. ☐

Problem 17 (16.2.52)**.** Calculate the average height above the x-axis of a point in the region $0 \le x \le 1$ and $0 \le y \le x^2$.

Solution. For a point in the plane, its height above the y-axis is its y-coordinate, so the average height in the given region is

$$\text{average height} = \frac{1}{\text{area of region}} \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} y\,dy\,dx.$$

The area of a region is found by integrating 1 over the region, so

$$\begin{aligned}
\text{average height} &= \left(\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} 1\,dy\,dx \right)^{-1} \left(\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} y\,dy\,dx \right) \\
&= \left(\int_{x=0}^{x=1} x^2\,dx \right)^{-1} \left(\int_{x=0}^{x=1} \frac{x^4}{2}\,dx \right) \\
&= \left(\frac{1}{3} \right)^{-1} \left(\frac{1}{10} \right) = \frac{3}{10}.
\end{aligned}$$

□

Problem 18 (16.2.55)**.** What is the average value of the linear function

$$f(x, y) = mx + ny + p$$

on the ellipse $(x/a)^2 + (y/b)^2 \le 1$? Argue by symmetry rather than calculation.

Solution. If (x, y) is a point in the ellipse, then $(-x, -y)$ is a point in the ellipse as well, and if we pair these two points together, they average to

$$\frac{f(x, y) + f(-x, -y)}{2} = \frac{(mx + ny + p) + (-mx - ny + p)}{2} = p.$$

Doing this for every pair of points, the average of $f(x, y)$ on the ellipse as a whole is p. □

Problem 19 (16.2.59)**.** Prove the inequality $\iint_{\mathcal{D}} \frac{dA}{4 + x^2 + y^2} \le \pi$, where $\mathcal{D}$ is the disc $x^2 + y^2 \le 4$.

Solution. Since $x^2, y^2 \ge 0$ for all (x, y), we have

$$\frac{1}{4 + x^2 + y^2} \le \frac{1}{4}$$

for all (x, y). Thus

$$\iint_{\mathcal{D}} \frac{dA}{4 + x^2 + y^2} \le \iint_{\mathcal{D}} \frac{dA}{4} = \frac{1}{4} \cdot (\text{area of } \mathcal{D}) = \frac{1}{4} \cdot 4\pi = \pi.$$

□

HOMEWORK 2 - SOLUTIONS

Problem 1 (16.3.3)**.** Evaluate $\iiint_{\mathcal{B}} xe^{y-2z}\,dV$ for the box $\mathcal{B} = \{0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 1\}$.

Solution.

$$\begin{aligned}\iiint_{\mathcal{B}} xe^{y-2z}\,dV &= \int_{z=0}^{z=1}\int_{y=0}^{y=1}\int_{x=0}^{x=2} xe^{y}e^{-2z}\,dx\,dy\,dz \\ &= \int_{z=0}^{z=1} e^{-2z}\,dz \int_{y=0}^{y=1} e^{y}\,dy \int_{x=0}^{x=2} x\,dx \\ &= \frac{1}{2}(1-e^{-2})\cdot(e-1)\cdot 2 = (e-1)(1-e^{-2}).\end{aligned}$$

□

Problem 2 (16.3.10)**.** Evaluate $\iiint_{\mathcal{W}} e^{x+y+z}\,dV$ over the region

$$\mathcal{W} = \{0 \le z \le 1, 0 \le y \le x, 0 \le x \le 1\}.$$

Solution.

$$\begin{aligned}\iiint_{\mathcal{W}} e^{x+y+z}\,dV &= \int_{z=0}^{z=1}\int_{x=0}^{x=1}\int_{y=0}^{y=x} e^{x}e^{y}e^{z}\,dy\,dx\,dz \\ &= \int_{z=0}^{z=1} e^{z}\,dz \int_{x=0}^{x=1}\int_{y=0}^{y=x} e^{x}e^{y}\,dy\,dx = (e-1)\int_{x=0}^{x=1} e^{x}(e^{x}-1)\,dx \\ &= (e-1)\int_{u=0}^{u=e-1} u\,du = \frac{1}{2}(e-1)^3. \qquad (u = e^x - 1)\end{aligned}$$

□

Problem 3 (16.3.15)**.** Calculate the integral of $f(x,y,z) = z$ over the region $\mathcal{W}$ below the hemisphere of radius 3 and lying over the triangle $\mathcal{D}$ in the xy-plane bounded by $x = 1$, $y = 0$, and $x = y$.

Solution. We have $\mathcal{D} = \{0 \le x \le 1 \text{ and } 0 \le y \le x\}$, and for z to lie below the hemisphere of radius 3, we must have $x^2 + y^2 + z^2 \le 9$, so

$$\mathcal{W} = \{0 \le x \le 1 \text{ and } 0 \le y \le x \text{ and } 0 \le z \le \sqrt{9 - x^2 - y^2}\}.$$

Then

$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,dV &= \int_{x=0}^{x=1}\int_{y=0}^{y=x}\int_{z=0}^{z=\sqrt{9-x^2-y^2}} z\,dz\,dy\,dx \\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=x} 9 - x^2 - y^2\,dy\,dx = \int_{x=0}^{x=1} 9x - x^3 - \frac{1}{3}x^3\,dx \\ &= \frac{9}{2} - \frac{1}{4} - \frac{1}{12} = \frac{25}{6}.\end{aligned}$$

□

Problem 4 (16.3.17)**.** Integrate $f(x, y, z) = x$ over the region in the first octant $(x, y, z \geq 0)$ above $z = y^2$ and below $z = 8 - 2x^2 - y^2$.

Solution. The bounds on (x, y) are given by the intersection of the two surfaces,

$$z = y^2 \qquad \text{and} \qquad z = 8 - 2x^2 - y^2 \implies x^2 + y^2 = 4.$$

Hence the region of interation is

$$\mathcal{W} = \{0 \leq y \leq 2 \text{ and } 0 \leq x \leq \sqrt{4 - y^2} \text{ and } y^2 \leq z \leq 8 - 2x^2 - y^2\},$$

and

$$\begin{aligned}
\iiint_{\mathcal{W}} f(x, y, z)\, dV &= \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} \int_{z=y^2}^{z=8-2x^2-y^2} x\, dz\, dx\, dy \\
&= \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} x(8 - 2x^2 - 2y^2)\, dx\, dy \\
&= \int_{y=0}^{y=2} \int_{u=8-2y^2}^{u=0} -\frac{1}{4} u\, du\, dy \qquad (u = 8 - 2x^2 - 2y^2) \\
&= \int_{y=0}^{y=2} \frac{1}{8}(8 - 2y^2)^2\, dy \\
&= \int_{y=0}^{y=2} \frac{1}{2} y^4 - 4y^2 + 8\, dy \\
&= \frac{32}{10} - \frac{32}{3} + 16 = \frac{128}{15}.
\end{aligned}$$

□

Problem 5 (16.3.20)**.** Find the volume of the solid in $\mathbb{R}^3$ bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

Solution. Note that the solid is symmetric about the plane $x = y$ (the set of bounding equations is unchanged upon swapping x and y). Therefore, it suffices to compute the volume of the solid bounded by $y = x^2$, $y = x$, $z = x + y + 5$, and $z = 0$, then double it. Hence we compute

$$\begin{aligned}
2 \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} \int_{z=0}^{z=x+y+5} 1\, dz\, dy\, dx &= 2 \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} x + y + 5\, dy\, dx \\
&= 2 \int_{x=0}^{x=1} \frac{1}{2}(x + x + 5)^2 - \frac{1}{2}(x + x^2 + 5)^2\, dx \\
&= 2 \int_{x=0}^{x=1} -\frac{1}{2} x^4 - x^3 - \frac{7}{2} x^2 + 5x\, dx \\
&= 2 \left(-\frac{1}{10} - \frac{1}{4} - \frac{7}{6} + \frac{5}{2} \right) = \frac{59}{30}.
\end{aligned}$$

□

Problem 6 (16.3.25)**.** Describe the domain of integration of the integral

$$\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{1}^{\sqrt{5-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz.$$

Solution. The domain is

$$\begin{aligned}\mathcal{D} &= \{-2 \le z \le 2 \text{ and } -\sqrt{4-z^2} \le x \le \sqrt{4-z^2} \text{ and } 1 \le y \le \sqrt{5-x^2-z^2}\} \\ &= \{-2 \le z \le 2 \text{ and } x^2+z^2 \le 4 \text{ and } 1 \le y \le \sqrt{5-x^2-z^2}\} \\ &= \{x^2+z^2 \le 4 \text{ and } y \ge 1 \text{ and } x^2+y^2+z^2 \le 5\} \\ &= \{y \ge 1 \text{ and } x^2+y^2+z^2 \le 5\},\end{aligned}$$

where in the last step, we removed the first inequality since it follows from the other two. This is the smaller region between the sphere of radius $\sqrt{5}$ centered at the origin and the plane $y = 1$. □

Problem 7 (16.3.28)**.** Let $\mathcal{W}$ be the region bounded by $y + z = 2$, $2x = y$, $x = 0$, and $z = 0$. Express and evaluate the triple integral of $f(x, y, z) = z$ by projecting $\mathcal{W}$ onto the

(a) xy-plane;

(b) yz-plane;

(c) xz-plane.

Solution.

1. The projection onto the xy-plane is the triangle $\{0 \le x \le 1 \text{ and } 2x \le y \le 2\}$, so

$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,dV &= \int_{x=0}^{x=1}\int_{y=2x}^{x=1}\int_{z=0}^{z=2-y} z\,dz\,dy\,dx \\ &= \int_{x=0}^{x=1}\int_{y=2x}^{y=2} \frac{1}{2}(2-y)^2\,dy\,dx \\ &= \int_{x=0}^{x=1} \frac{1}{6}(2-2x)^3\,dx \\ &= \int_{x=0}^{x=1} \frac{4}{3}(1-x)^3\,dx = \frac{1}{3}.\end{aligned}$$

2. The projection onto the yz-plane is the triangle $\{0 \le y \le 2 \text{ and } 0 \le z \le 2-y\}$, so

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \int_{y=0}^{y=2}\int_{z=0}^{z=2-y}\int_{x=0}^{x=y/2} z\,dx\,dz\,dy = \frac{1}{3}.$$

3. The projection onto the $xz-$plane is the triangle $\{0 \le x \le 1 \text{ and } 0 \le z \le 2-2x\}$, so

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \int_{x=0}^{x=1}\int_{z=0}^{z=2-2x}\int_{y=x/2}^{y=2-z} z\,dy\,dz\,dx = \frac{1}{3}.$$

□

Problem 8 (16.3.29)**.** Let

$$\mathcal{W} = \left\{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\right\}.$$

Express $\iiint_{\mathcal{W}} f(x, y, z)\, dV$ as an iterated integral in the order $dz\, dy\, dx$.

Solution. We must have $x^2 + y^2 \le 1$, so $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. Thus

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=1} f(x, y, z)\, dz\, dy\, dx.$$

□

Problem 9 (16.3.30)**.** Repeat the previous exercise for the order $dx\, dy\, dz$.

Solution. Here we have $0 \le z \le 1$, then $-z \le y \le z$, then $-\sqrt{z^2 - y^2} \le x \le \sqrt{z^2 - y^2}$, so

$$\iiint_{\mathcal{W}} f(x, y, z)\, dV = \int_{z=0}^{z=1} \int_{y=-z}^{y=z} \int_{x=-\sqrt{z^2-y^2}}^{x=\sqrt{z^2-y^2}} f(x, y, z)\, dx\, dy\, dz.$$

□

Problem 10 (16.3.35)**.** Draw the region $\mathcal{W}$ bounded by the surfaces given by $z = y^2$, $y = z^2$, $x = 0$, and $x + y + z = 4$. Set up, but do not compute, a single triple integral that yields the volume of $\mathcal{W}$.

Solution. [drawing omitted for now]
The region is

$$\mathcal{W} = \{0 \le z \le 1 \text{ and } z^2 \le y \le \sqrt{z} \text{ and } 0 \le x \le 4 - y - z\},$$

so the volume of $\mathcal{W}$ is

$$\iiint_{\mathcal{W}} dV = \int_{z=0}^{z=1} \int_{y=z^2}^{y=\sqrt{z}} \int_{x=0}^{x=4-y-z} dx\, dy\, dz.$$

□

Problem 11 (12.3.13)**.** Convert the equation $r = 2\sin\theta$ to rectangular coordinates.

Solution. Multiply by r to get $r^2 = 2r\sin\theta$, which gives $x^2 + y^2 = 2y$, or $x^2 + (y - 1)^2 = 1$. □

Problem 12 (12.3.16)**.** Convert the equation $r = 1/(2 - \cos\theta)$ to rectangular coordinates.

Solution. Clearing denominators, $2r - r\cos\theta = 1$, or $2\sqrt{x^2 + y^2} = 1 + r\cos\theta = 1 + x$. Squaring, $4x^2 + 4y^2 = 1 + 2x + x^2$, or $3x^2 - 2x + 4y^2 = 1$. □

Problem 13 (12.3.18)**.** Convert the equation $x = 5$ to a polar equation of the form $r = f(\theta)$.

Solution. This becomes $r\cos\theta = 5$, so $r = 5\sec\theta$. □

Problem 14 (12.3.19)**.** Convert the equation $y = x^2$ to a polar equation of the form $r = f(\theta)$.

Solution. This becomes $r\sin\theta = r^2\cos^2\theta$, so $r = \sin\theta/\cos^2\theta = \sec\theta\tan\theta$. □

Problem 15 (12.3.20)**.** Convert the equation $xy = 1$ to a polar equation of the form $r = f(\theta)$.

Solution. This becomes $r^2\sin\theta\cos\theta = 1$, so $r = \sqrt{\sec\theta\csc\theta}$. □

Problem 16 (12.3.23)**.** Match each equation with its description:

(a) $r = 2$	(i) Vertical line
(b) $\theta = 2$	(ii) Horizontal line
(c) $r = 2\sec\theta$	(iii) Circle
(d) $r = 2\csc\theta$	(iv) Line through origin

Solution. (a) This is the circle of radius 2 around the origin, which fits (iii).

(b) This is a ray from the origin outward in the direction corresponding to $\theta = 2$, which fits most closely (though not quite) with (iv).

(c) This equation becomes $r\cos\theta = 2$, or $x = 2$, so we have a vertical line, which fits (i).

(d) This equation becomes $r\sin\theta = 2$, or $y = 2$, so we have a horizontal line, which fits (ii).

□

Problem 17 (12.3.37)**.** Show that

$$r = a\cos\theta + b\sin\theta$$

is the equation of a circle passing through the origin. Express the radius and center (in rectangular coordinates) in terms of a and b, and write down the equation in rectangular coordinates.

Solution. Multiply through by r to get $r^2 = ar\cos\theta + br\sin\theta$, or $x^2 + y^2 = ax + by$. Completing the square gives us

$$\left(x - \frac{1}{2}a\right)^2 + \left(y - \frac{1}{2}b\right)^2 = \frac{a^2 + b^2}{4},$$

so the center is $(a/2, b/2)$ and the radius is $\sqrt{a^2 + b^2}/2$. □

Problem 18 (12.3.47)**.** Show that every line that does not pass through the origin has a polar equation of the form

$$r = \frac{b}{\sin\theta - a\cos\theta},$$

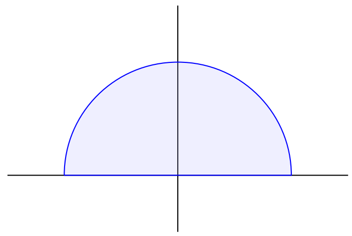
where $b \neq 0$.

Solution. Any line which does not pass through the origin has rectangular equation $y = ax + b$ for some a and $b \neq 0$. Then $r\sin\theta = ar\cos\theta + b$, and solving for r, we get the required form. □

HOMEWORK 3 - SOLUTIONS

Problem 1 (16.4.4)**.** Sketch $\mathcal{D} = \{y \geq 0 \text{ and } x^2 + y^2 \leq 1\}$ and integrate $f(x, y) = y(x^2 + y^2)^3$ over $\mathcal{D}$ using polar coordinates.

Solution.



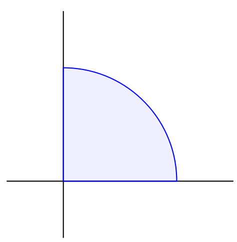
From the diagram, we can see that $\mathcal{D} = \{0 \leq \theta \leq \pi \text{ and } 0 \leq r \leq 1\}$, so

$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} r \sin\theta (r^2)^3 \cdot r\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \int_{r=0}^{r=1} r^8\, dr = 2 \cdot \frac{1}{9} = \frac{2}{9}.$$

□

Problem 2 (16.4.10)**.** For $\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{16-x^2}} \tan^{-1}\left(\frac{y}{x}\right) dy\, dx$, sketch the region of integration and evaluate by changing to polar coordinates.

Solution.



In this range, $\tan^{-1}(y/x) = \theta$, so

$$\begin{aligned}\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{16-x^2}} \tan^{-1}\left(\frac{y}{x}\right) dy\, dx &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} \theta \cdot r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} \theta\, d\theta \int_{r=0}^{r=1} r\, dr \\ &= \frac{(\pi/2)^2}{2} \cdot \frac{1}{2} = \frac{\pi^2}{16}.\end{aligned}$$

□

Problem 3 (16.4.21)**.** Find the volume of the wedge-shaped region contained in the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$ and below by the xy-plane.

Solution. In cylindrical polar coordinates, the region is given by

$$0 \le r \le 3, \qquad -\pi/2 \le \theta \le \pi/2, \qquad 0 \le z \le r\cos\theta,$$

so the volume is

$$\begin{aligned}\int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=3}\int_{z=0}^{z=r\cos\theta} r\,dz\,dr\,d\theta &= \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=3} r^2\cos\theta\,dr\,d\theta \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2} 9\cos\theta\,d\theta = 18.\end{aligned}$$

□

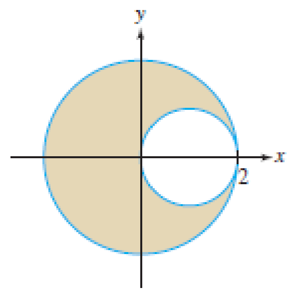


Figure 1

Problem 4 (16.4.23)**.** Evaluate $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\,dA$, where $\mathcal{D}$ is the domain above.

Hint: Find the equation of the inner circle in polar coordinates and treat the right and left parts of the region separately.

Problem 5 (16.4.30)**.** Use cylindrical coordinates to compute $\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\,dV$ for the region given by $x^2+y^2 \le z \le 8-(x^2+y^2)$.

Problem 6 (16.4.35)**.** Express $\int_{-1}^{1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=0}^{x^2+y^2} f(x,y,z)\,dz\,dy\,dx$.

Problem 7 (16.4.40)**.** Use cylindrical coordinates to find the volume of a sphere of radius a from which a central cylinder of radius b has been removed, where $0 < b < a$.

Problem 8 (16.4.49)**.** Use spherical coordinates to calculate the triple integral of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ over the region $x^2+y^2+z^2 \le 2z$.

Problem 9 (16.4.52)**.** Find the volume of the region lying above the cone $\phi = \phi_0$ and below the sphere $\rho = R$.

Problem 10 (16.4.56)**.** Let $\mathcal{W}$ be the region within the cylinder $x^2 + y^2 = 2$ between $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$. Calculate the integral of $f(x, y, z) = x^2 + y^2$ over $\mathcal{W}$, using both spherical and cylindrical coordinates.

Problem 11 (16.5.8)**.** Compute the total mass of the plate in Figure 2 assuming a mass density of $f(x, y) = x^2/(x^2 + y^2)$ $\mathsf{g} \cdot \mathsf{cm}^{-2}$.

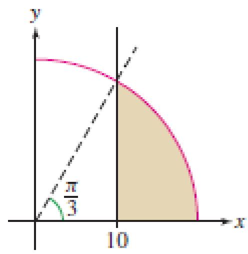
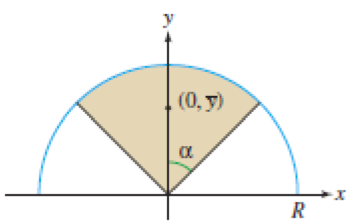


Figure 2: 16.5.8

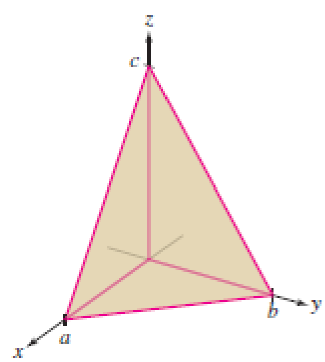
Problem 12 (16.5.13)**.** Find the centroid of the quarter circle $x^2 + y^2 \leq R^2$ with $x, y \geq 0$ assuming the density $\delta(x, y) = 1$.

Problem 13 (16.5.16)**.** Show that the centroid of the sector in Figure 3a has y-coordinate

$$\overline{y} = \left(\frac{2R}{3}\right)\left(\frac{\sin\alpha}{\alpha}\right).$$



(a) 16.5.16



(b) 16.5.20

Figure 3

Problem 14 (16.5.20)**.** Show that the z-coordinate of the centroid of the tetrahedron bounded by the coordinate planes and the plane $(x/a) + (y/b) + (z/c) = 1$ in Figure 3b is $\overline{z} = c/4$. Conclude by symmetry that the centroid is $(a/4, b/4, c/4)$.

Problem 15 (16.5.21)**.** Find the centroid of the region $\mathcal{W}$ lying above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$ (Figure 4).

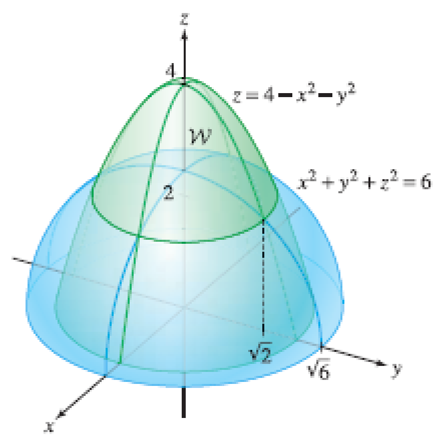


Figure 4: 16.5.21

Problem 16 (16.5.24)**.** Find the center of mass of the region bounded by $y^2 = x + 4$ and $x = 0$ with mass density $\delta(x, y) = |y|$.

Problem 17 (16.5.27)**.** Find the z-coordinate of the center of mass of the first octant of the unit sphere with mass density $\delta(x, y, z) = y$ (Figure 5).

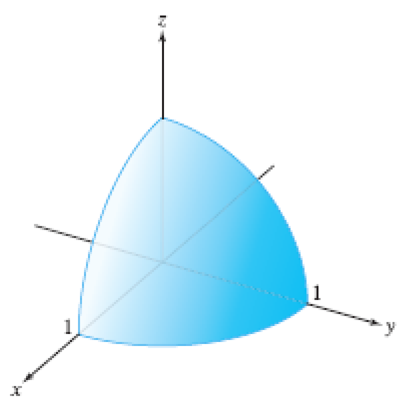


Figure 5: 16.5.27