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REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the Journal in 1936 until the end of 1999. The Association moved the reviews to this Bulletin, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Matthew Foreman, Gerhard Jäger, Penelope Maddy, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the Journal, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this Bulletin, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the Journal, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the Journal, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the Journal, that is, to Tarski's *Truth and proof*.

References such as 186***4*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the Journal, vol. 1, pp. 121–218).

Petr Hájek. ***Metamathematics of fuzzy logic.*** Trends in logic, vol. 4. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1998, viii + 297 pp.

'Fuzzy logic' means different things to different people. For some it is a philosophy of life—"a way to break the stranglehold that the black-and-white thinking of the Western tradition has upon us." For others it is a more accurate way of describing our ordinary language (and thought) "in which we do not think that everything is either true or false but where we recognize the shades of grey that populate our thought and linguistic communication." Some see fuzzy logic as an ontological device that "more accurately describes reality than claiming that, for every existent item a and for every property F, either a manifests F or a manifests *not-F*." Some theorists view the fuzzy–classical debate as being a modern analogue–digital debate; and in various engineering fields, fuzzy logic is viewed as the background theoretical framework in which varying a dial to degree n results in a greater change somewhere else in the factory (or machine) than had the dial not been varied quite to degree n.

Although it is undeniable that the engineering feats accomplished by clever use of this last technique have led to wonderful advances in camera and refrigerator technology, many people have questioned whether it really is "applied fuzzy logic" or whether it even presupposes anything more than classical two-valued logic. (Does it really require fuzzy logic to model a dial that maps minutes-on-barbecue into degree-of-doneness?) And needless to say, the claims made by fuzzy-theorists in the other areas cited above have not been met with universal acceptance. In fact, it is not much of an exaggeration to say that the phrase 'fuzzy logic' divides the world into two: those who advocate it with missionary zeal and those who think

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that it is logical pornography. (It is intriguing and exciting at first sight, but quickly is seen as shallow and just a little disgusting.)

Hájek's book is a welcome item in this stewpot full of differing frameworks. His first maneuver is to distinguish "fuzzy logic in the wide sense" from "fuzzy logic in the narrow sense"; and he claims that the aim of the present book is to investigate the narrow notion, which he sees as one with the study of formal calculi of many-valued logic. Nonetheless, despite this concern in the main part of the book with the formal properties of fuzzy logic, Hájek spends some opening pages discussing such topics as whether fuzzy logic actually has any applications, or whether it can be used as a model of vagueness and uncertainty. (Hájek is adamant that it does not mirror probability theory.) And there is a short but interesting discussion concerning "where the truth values come from" and "how the values can be interpreted."

Hájek claims that his readers are "assumed to be able to follow mathematical arguments and to have at least a basic mathematical knowledge." They further need "at least a partial experience with the classical (Boolean) propositional logic [and] some knowledge of predicate calculus." He claims that he "do[es] not assume any deep knowledge of mathematical logic or algebra." This reviewer's experience is that Hájek overestimates how much can be easily comprehended by a reader with only this much background.

In "the narrow sense," fuzzy logic is infinitely-many valued logic. Despite the belief of many "popular fuzzy logicians" that Lotfi Zadeh invented infinite-valued logic in 1965 (see JSL XXXVIII 656), it was in fact investigated in the 1920's and 1930's by Łukasiewicz and Tarski and there was a very strong resurgence of interest in the 1950's and early 1960's. Hájek cites some of this historically relevant material in a very short concluding chapter to the book (only two pages of it to cover the period before Zadeh's 1965 paper). This reviewer would have preferred to see the author develop fuzzy logic as a development of the historical quest for infinite-valued logics; but then to each his or her own. Hájek has a clear idea of how to investigate the topic of infinitely-many valued logics, and he does a nice job within his own terms, and indeed, a nice job even with no qualifications.

Every other presentation of fuzzy logic seen by this reviewer starts with a claim like this: "A propositional letter takes a value in the real interval [0, 1]. A conjunction of sentences is the minimum value of its conjuncts, while a disjunction of sentences is the maximum value of its disjuncts. A negation is interpreted as 1 minus the value of the unnegated component." Hájek instead starts his semantic account by describing a very general notion of a conjunctive truth function, $*$. It must agree with classical conjunction on the values 0 and 1; it must be commutative and associative; it must be non-decreasing with respect to both arguments; $(1*x) = x$ and $(0*x) = 0$; and it must be a continuous function on $[0,1]^2$ into $[0,1]$. Hájek's strategy is to develop a semantic theory for a basic logic (BL) that has just this "weak" type of conjunction, and then to add on various restrictions to the conjunctive operator which will generate semantics for various different fuzzy logics. Different ways to add restrictions to this operator give rise to distinct "t-norms," as Hájek calls them. Three different restrictions are considered:

Łukasiewicz:	$(x * y) = \max(0, x + y - 1)$
Gödel:	$(x * y) = \min(x, y)$
Goguen:	$(x * y) = xy$ (product of reals)

(It should be noted that Hájek sometimes uses 'Goguen' for this last, and sometimes uses 'product'.) On these definitions of conjunction in a fuzzy logic, only the Gödel logic resembles fuzzy logic as traditionally understood.

Given any one of these t-norms, one can define a (semantic) conditional, $\Rightarrow$, as its *residuum* in the following way:

$$(x \Rightarrow y) = \text{the maximal } z \text{ satisfying } (x * z) \leqq y$$

Corresponding to the three different definitions of $*$, we have three different conditionals. Each conditional yields the same value, namely 1, whenever the antecedent has the same or smaller truth-value than the consequent, but they differ in the cases where the antecedent is "truer" than the consequent:

$$\begin{array}{ll} \multicolumn{2}{l}{\text{If } x \leqq y, \text{ then } (x \Rightarrow y) = 1; \text{ otherwise:}} \\ \quad\text{Łukasiewicz:} & (x \Rightarrow y) = 1 - x + y \\ \quad\text{Gödel:} & (x \Rightarrow y) = y \\ \quad\text{Goguen:} & (x \Rightarrow y) = y/x \ \text{ (division of reals)} \end{array}$$

There has long been (and still is) disagreement in the fuzzy logic community about the correct definition of the conditional, and we see here how three of the ones used turn out to be consequences of the choice of conjunction. (There are other common suggestions in the literature for implication that are not generated like this; some of these are even simply 2-valued.)

Hájek uses $\Rightarrow$ to define $(-)$, a semantic negation operator: $(-)x =_{\text{df}} (x \Rightarrow 0)$. For our three different types of $\Rightarrow$ this yields two negations:

$$\begin{array}{ll} \text{Łukasiewicz:} & (-)x = 1 - x \\ \text{Gödel:} & (-)x = 1, \text{ if } x = 0; = 0 \text{ otherwise} \\ \text{Goguen:} & (-)x = 1, \text{ if } x = 0; = 0 \text{ otherwise} \end{array}$$

It will be noted that only the Łukasiewicz negation corresponds to the traditional fuzzy logic definition of negation, and that the other negation is essentially a 2-valued operator that says whether the unnegated sentence has the value 0 or not.

Nonetheless, given any one of these t-norms, using & to be the connective in the language that represents $*$ and using $\rightarrow$ as the connective to represent $\Rightarrow$, we can define the traditional fuzzy $\wedge$ and $\vee$ (min and max operators):

$$\begin{array}{lcl} A \wedge B & \text{is} & A \,\&\, (A \rightarrow B) \\ A \vee B & \text{is} & ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A) \end{array}$$

(Notice that $\wedge$ rather than & is the conjunctive operator used in defining $\vee$.) Given the constant $\mathbf{0}$ in the language, then negation and equivalence are defined:

$$\begin{array}{rcl} \neg A & \text{is} & A \rightarrow \mathbf{0} \\ A \equiv B & \text{is} & (A \rightarrow B) \,\&\, (B \rightarrow A) \end{array}$$

(Note that & rather than $\wedge$ is the conjunctive operator used in defining $\equiv$.)

The reader might note that many of the standard properties of the different connectives are preserved in BL, such as the usual distributivity laws, commutativity laws, the De Morgan laws (for $\wedge$, $\vee$, and $\neg$), and the like. Note that $(p \rightarrow \neg\neg p)$ is a theorem but that $(\neg\neg p \rightarrow p)$ is not (and indeed it fails under the Gödelian-Goguenian interpretation). We note also that the deduction theorem does not hold in full generality in BL. What we have instead is

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff there is an } n \text{ such that } \Gamma \vdash (\varphi^n \rightarrow \psi)$$

where φ^n is $(\varphi \,\&\, \varphi \,\&\, \cdots \,\&\, \varphi)$, n times. (The unqualified deduction theorem only holds for the Gödel logic.) One might also wonder about semantic compactness: Is it true that

$$\Gamma \models \varphi \text{ iff } \Delta \models \varphi \text{ for some } \Delta \text{ which is a finite subset of } \Gamma?$$

Given the resources of fuzzy logic, it seems possible to construct an infinite contradictory set Γ that has no contradictory finite subsets. But then $\Gamma \models \varphi$ would be valid but $\Delta \models \varphi$ would not be, for any finite Δ. This would have consequences for completeness, for then there could be a valid argument $\Gamma \models \varphi$ for which there was no derivation (because derivations are finite).

As can be seen, many truth functions can be defined from these t-norms. But a simple cardinality argument shows that fuzzy logic is expressively incomplete, for there are uncountably many n-ary truth functions from $[0, 1]^n$ into $[0, 1]$. In a very interesting theorem, Hájek shows that every *continuous* t-norm (every reasonable definition of conjunction) is a combination of the three t-norms he has singled out.

Hájek spends two chapters on the three defined fuzzy logics, axiomatizing them and showing the cases in which they are inter-interpretable. At the end of Chapter 3 (on Łukasiewicz fuzzy logic) he discusses how this logic would deal with the sorites paradox. At the end of this discussion he claims that the example will be further dealt with in the following chapters on Gödel and Goguen logic as well as in the fuzzy predicate logic. The reviewer thinks it unfortunate that this very interesting discussion was not continued.

In Chapter 5 Hájek extends his analysis to predicate logic. Once again, we start with BL and consider the logic generated by adding predicates, constants, variables, and the quantifiers $\forall$ and $\exists$. (The logic is called BL$\forall$.) Of some interest are the theorems

$$\exists x\, \varphi \to \neg\forall x\neg\varphi \quad \text{and} \quad \neg\exists x\, \varphi \equiv \forall x\neg\varphi$$

and the fact that the converse of the first of these is not a theorem of BL$\forall$. Semantically the idea is that constants denote (precisely!) elements of a domain, while n-place predicates designate a "fuzzy n-ary relation" (which gives the truth value of that predicate for any n-tuple chosen from the domain). $\forall x\, \varphi$ has the truth value of the "least true instance of φ" while $\exists x\, \varphi$ has the truth value of the "most true instance of φ." If there are no such instances then the value of the formula is undefined. (This seems a departure from traditional fuzzy logic where such formulas are assigned the limit-value of the various φ's.) This restriction is carried out by having only "safe" models be relevant to considerations of completeness; completeness is attained with respect to the class of linearly ordered $\mathcal{C}$-algebras that are safe. Hájek thus avoids the result that fuzzy logic is not recursively axiomatizable by enlarging the class of truth-value algebras. A traditional fuzzy logician ought to say that this is not "real" completeness, but only completeness with respect to a limited set of models. Both the traditional fuzzy logician and the doubter of fuzzy logic ought to say that the importance or relevance of this new class of models needs to be made plausible before we can responsibly say that fuzzy logic has been given a complete axiomatization that is reasonable.

After developing BL$\forall$, Hájek moves on to the task of extending the Łukasiewicz, Gödel, and Goguen logics with predicates, names, and quantifiers. Hájek shows that tautologies of the Gödel predicate logic coincide with the tautologies over "the standard G-algebra $[0,1]_{\mathrm{G}}$ of truth functions [and] thus G$\forall$ is recursively axiomatized." The interested reader might note that this is done with respect to the *countable* subinterval of $[0,1]$. Hájek then shows that no similar result is possible for either the Łukasiewicz or Goguen logics: no recursive axiomatization for either of these logics is possible. (This follows from a result of Scarpellini's, JSL XXIX 145.) Many observers might think this should sound the death knell for fuzzy logic. This reviewer thinks Hájek might have spent much more time on this issue.

There are many further very interesting topics covered in this book, such as the treatment of equality (similarity?) in fuzzy logic, issues of computational complexity and undecidability, a whole chapter on "approximate inference," generalized quantifiers, modalities, an exposition of other fuzzy logics, a discussion of the liar paradox in fuzzy logic ("This sentence is at least a little false"), and various other topics. There are many hidden gems in these discussions; the reviewer found Hájek's lengthy discussion in Chapter 7 of Zadeh's notion of a "compositional rule of inference" and the possibility of other "fuzzy rules of inference" to be especially interesting and of especial importance to those theorists who are not logicians but who are interested in developing accounts of reasoning or accounts of fuzzy controllers.

The reviewer found enough (inconsequential) typographical errors to believe that there are probably quite a few in the book. Some of the proofs seem like verbatim transcriptions of a proof-sketch off of Hájek's blackboard: the abbreviations and informal conventions employed are sometimes difficult to unravel.

Anyone interested in fuzzy logic—whether they are "true believers" or simply "logical pornographers"—should have this book and learn from it. It is not much of a compliment to say that it is the best book there is on the formal properties of fuzzy logic, since there are no others. But even if there were others, it is still fuzzily-valid to infer that this one would still be the best. (Thanks to Alasdair Urquhart for discussion.)

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PETR HÁJEK, LLUIS GODO, and FRANCESC ESTEVA. *A complete many-valued logic with product-conjunction*. ***Archive for mathematical logic***, vol. 35 (1996), pp. 191–208.

A beautiful result essentially due to P. S. Mostert and A. I. Shields (*On the structure of semigroups on a compact manifold with boundary*, ***Annals of mathematics***, vol. 65 (1957), pp. 117–143; cf. also D. H. Ling, *Representation of associative functions*, ***Publicationes mathematicae*** (Debrecen), vol. 12 (1965), pp. 189–212), says that every continuous t-norm on $[0,1]$ is in some sense a combination of three fundamental t-norms: Łukasiewicz t-norm $x \cdot y = \max\{x + y - 1, 0\}$, Gödel's t-norm $x \wedge y = \min\{x, y\}$, and product t-norm $\odot$ (ordinary product).

In his book ***Metamathematics of fuzzy logic*** (see the preceding review), P. Hájek emphasizes the relationships between fuzzy logics and continuous t-norms with their residua. According to this (quite reasonable) point of view, four logics emerge: Łukasiewicz logic, Gödel logic, product logic, and basic logic. As shown recently by Cignoli, Esteva, Godo, and Torrens in *Basic fuzzy logic is the logic of continuous t-norms and their residua* (***Soft computing***, vol. 4 no. 2 (2000), pp. 106–112), basic logic is the logic of arbitrary continuous t-norms and their residua.

Whereas Łukasiewicz logic and Gödel logic have been investigated for many years (completeness proofs for these logics have been known since the fifties), product logic and basic logic are rather recent subjects.

The present paper contains an axiomatization and a completeness proof for product logic. For the axiomatization, the idea is that the structure $\langle (0,1), \odot, \to, 0, 1\rangle$ (with 0 excluded), where $\odot$ is product and $\to$ is its residuum, is a residuated cancellative commutative monoid, and that $0 \odot x = x \odot 0 = 0$, $0 \to x = 1$, and for $x \neq 0$, $x \to 0 = 0$. The crucial axiom expressing that the cancellation property holds with the exception of 0 is

$$\neg\neg\varphi \to ((\varphi \odot \psi) \to (\varphi \odot \gamma)) \to (\psi \to \gamma),$$

where $\neg\varphi$ is short for $\varphi \to 0$. Furthermore, the behavior of $\neg$ is described by the axiom $\neg(\varphi \wedge \neg\varphi)$ (where $\varphi \wedge \psi$ is short for $\varphi \odot (\varphi \to \psi)$).

The completeness proof is in three steps. First, the authors prove some syntactic results that establish some basic properties of product logic, and allow for the introduction of a complete algebraic semantics, based on product algebras.

As a second step, it is proved that every product algebra is isomorphic to a subdirect product of a family of linearly ordered product algebras. From this, it follows that product logic is complete with respect to evaluations on linearly ordered product algebras.

Finally, it is shown that any finite partial subalgebra of a linearly ordered product algebra can be embedded into the standard product algebra on the real interval $[0,1]$. The completeness of product logic with respect to evaluations on the standard product algebra is an easy consequence of the results mentioned above.

Although Łukasiewicz logic is probably deeper, and Gödel logic has a nicer proof theory, product logic is also very interesting. First of all, product is a quite natural t-norm. Moreover, the algebraic structure of product algebras is far from trivial: thinking of product algebras as negative cones of lattice-ordered Abelian groups with an added bottom element is a simplification that works only in some particular cases. Finally, Łukasiewicz logic is interpretable in product logic (with an added constant c and with the axiom $\neg\neg c$), whereas interpreting product logic in Łukasiewicz logic is more problematic (Łukasiewicz connectives are continuous, whereas product implication is discontinuous). Thus, a paper like the present one, which proves a basic result about product logic, constitutes a significant contribution to fuzzy logic in general.

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Patrick Grim, Gary Mar, and Paul St. Denis. ***The philosophical computer. Exploratory essays in philosophical computer modeling.*** With the Group for Logic and Formal Semantics. The MIT Press, Cambridge, Mass., and London, 1998, viii + 323 pp. + CD-ROM.

This is an unusual, inspiring, and innovative book. The aim of the book is to present examples of using computer models in philosophical research. Visual models (computer graphics) are emphasized. Theoretical apparatus used includes (elementary) fuzzy logic, fractals, chaos theory, cellular automata, and game theory. In this review I concentrate mainly on *logical* aspects and disregard philosophical conclusions made, but I note with satisfaction that the authors are cautious enough to warn the reader not to identify computer models with solutions of philosophical (or ethical) problems (e.g. p. 156 f). The book consists of six chapters, whose preliminary forms were partially published in journals. I shall comment on each chapter separately.

Chapter 1: Chaos, fractals, and the semantics of paradox. Several self-referring sentences and pairs (or triples) of self-referring sentences are studied, beginning naturally with the liar's sentence. The sentences are presented in a verbal formulation and then conditions for their truth values are studied. The authors decide (very properly) to base their analysis on a simple fuzzy propositional logic (Łukasiewicz logic with truth values from the real unit interval, possibly extended by hedges, such as "very" and the like). A truth predicate Vpx (saying that the truth value of p is x) is also used.

Self-referential sentences lead to axioms of the form $p \equiv A(p)$ where $A(p)$ is a formula built from the propositional variable p; e.g. the liar's sentence "I am false" gives $p \equiv \neg p$. This gives a condition on the truth value $|p|$ of p: $|p| = f(|p|)$ where f is the truth function of A. (For the liar: $|p| = 1 - |p|$.) Looking for fixed point(s) of f is the *static* point of view; the authors suggest a *dynamic* approach. They take for each $x \in [0, 1]$ the sequence $x, f(x), f(f(x)), \dots, f^{(n)}(x), \dots$ and investigate its properties. The properties are often stated without proofs but visualized by figures. Surprisingly perhaps, some simple formulas $A(p)$ (rather than the corresponding self-referential axioms $p \equiv A(p)$) lead to functions that are chaotic in the sense of chaos theory; in particular, for each pair I, J of open intervals in $[0, 1]$ there is an $x \in I$ and a natural number n such that $f^{(n)}(x) \in J$.

I will briefly present some self-referential sentences the authors investigate. *Half-sayer:* This sentence is as true as half of its estimated value ($\varphi \equiv (\varphi \equiv \frac{1}{2}\varphi)$). (Here $\frac{1}{2}$ is a hedge: $|\frac{1}{2}\varphi| = \frac{1}{2}|\varphi|$. Square and square root are handled similarly.) *Modest liar:* This sentence is fairly false ($\varphi \equiv \sqrt{\neg\varphi}$). *Chaotic liar:* This sentence is as true as it is estimated to be false ($\varphi \equiv (\varphi \equiv \neg\varphi)$). *Logistic liar:* It is very false that this sentence is as true as it is estimated to be false ($\varphi \equiv (\neg(\varphi \equiv \neg\varphi))^2$). Here, for example, the "chaotic" formula gives the function $f(x) = 1 - |1 - 2x|$ with fixed points 0 and $\frac{2}{3}$; this function is chaotic on $[0, 1]$ (and is visualized by a progressive web diagram).

Chaotic pair: $\varphi \equiv (\varphi \equiv \psi)$, $\psi \equiv (\psi \equiv \neg\varphi)$. This gives sequences $x_{n+1} = 1 - |x_n - y_n|$, $y_{n+1} = 1 - |1 - x_n - y_n|$; thus for each point (x_0, y_0) in the unit square we have a sequence of points starting from it. This may be used to define a fractal, which is very nice looking (but not fully described). This is the first connection between logical formulas and fractals:

simple formulas of fuzzy propositional logic lead to surprisingly nice and complex fractals. Three-dimensioned fractals given by triples of formulas are also investigated.

The last part of Chapter 1 contains two theorems on undefinability of chaos which need some comment. The first theorem says that if Γ is the set of Gödel numbers of expressions that determine functions chaotic in the interval $[0, 1]$ and T is a consistent system of real arithmetic adequate for number theory, then Γ is not representable in T. This is true and the use of a diagonal lemma for such theories can be shown to be correct: Assuming that Robinson's arithmetic Q is relatively interpretable in T and letting Y, Z be variables for reals and x a variable for natural numbers, given a formula $\psi(Y, Z, x)$, there is a formula φ such that $T \vdash \varphi(Y, Z) \equiv \psi(Y, Z, \overline{\varphi(Y, Z)})$. (One can easily check that the usual proof of the diagonal lemma works.) But in fact the result does not say much since the same proof could be used to prove non-representability of Γ' where Γ' is the set of all functions f such that $f(0) = 0$. (The authors point out the similarity with Rice's theorem of recursion theory.) The second theorem works with an (undefined) notion of a primitive recursive function of real arguments.

I find Chapter 1 most interesting from the logical point of view. Let me mention, in passing, that a related paper by P. Hájek, J. Paris, and J. Shepherdson, *The liar paradox and fuzzy logic* (***The journal of symbolic logic***, vol. 65 (2000), pp. 339–346), was written in ignorance of the book under review. It deals with the question whether crisp arithmetic can be extended by a fuzzy truth predicate satisfying dequotation ($\varphi \equiv Tr(\overline{\varphi})$) and allowing self-reference.

Chapter 2: Notes on epistemic dynamics. Here numbers from the real unit interval are treated as degrees of *accuracy* rather than degrees of truth; in addition, a notion of *reliability* of information source is introduced. Self-referential sentences concerning both notions are studied in the "dynamic" way as above; one has two sequences—one of degrees of accuracy given by $x_{n+1} = f(x_n)$ for an f, and one of degrees of reliability given by a fixed expression $R_{n+1} = (2R_n + x_n)/3$. Also pairs of self-referring sentences on accuracy are studied, leading again to sequences (x_n, y_n) satisfying $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_{n+1}, y_n)$. Different choices of initial values (x_0, R_0) and (x_0, y_0) lead to different sequences with various chaotic properties and yield fractals called "epistemic hazard maps." (For example: "This message is half as accurate as the next one received from Agent 002. This message is as accurate as the last message from Agent 001." These are interpreted as $x_{n+1} = 1 - |\frac{1}{2}y_n - x_n|$ and $y_{n+1} = 1 - |x_{n+1} - y_n|$.)

Chapter 3: Fractal images of formal systems. Here the authors offer several kinds of visual presentations of simple formal systems. One example: enumerate all formulas formed from a single propositional letter and the Sheffer stroke; think of them placed in a planar system of unit squares, naturally indexed by pairs of natural numbers by using a pairing function. Color tautologies black, non-tautologies white; if done properly you get a nice picture. Several other examples are presented, including graphical representations of truth functions of fuzzy connectives. Finally, patterns in the system of squares are related to cellular automata; an automaton starts from an initial configuration of black and white squares and develops in steps; the color of a square in the next step is given by the present color of the square and colors of its neighbors. I quote the last sentences of the chapter: "Number-theoretical analysis of logical systems forms a familiar and powerful part of the work of Gödel and others. Analysis in terms of geometry and fractal geometry, we want to suggest, may be a promising further step."

Chapter 4: The evolution of generosity in a Hobbesian model. Here games are heavily used. After a presentation of the "prisoner's dilemma" the authors discuss one-step games of two players, each having two possibilities (cooperate and defect) and playing simultaneously; the payoff is given by a 2×2 matrix. Then sequences of such games are studied along with strategies for the players, saying which possibility to play depending on the previous choice of the opponent. Empirical results comparing different strategies are presented. A probabilistic variant (of Nowak and Sigmund) is mentioned and a strategy "partially tolerant to defection" is discussed. The whole situation is then generalized to a system of players sitting in squares in a quadrant and playing against their neighbors. This is called the "spatialized prisoner's dilemma."

Chapter 5: Real-valued game theory: real life, cooperative chaos, and discrimination. The preceding is generalized by fuzzifying the two possibilities (cooperate, defect) to a continuum of degrees of cooperation. Various strategies are discussed, with emphasis on those that are "generous" in a sense. This chapter contains several small errors: Page 203, first equation: replace $2 + c_t$ by 2. Page 207, line 5 from below: replace 4 by 4.5. Page 211, middle: replace $(R-T)\cdot x+(T-P)+P$ by $(R-T)\cdot x+(T-P)\cdot y+P$ and replace $(S-P)+(R-S)+P$ by $(S-P)\cdot x+(R-S)\cdot y+P$. Page 212, line 10 from below: replace $(0, c)$ by $(1, c)$. Finally, the authors repeatedly define a function by an expression E and then say, "the function is assumed to have a maximum of 1 and a minimum of 0." This must be understood as saying that the function is defined by E iff $0 \leqq E \leqq 1$; if $E \geqq 1$ then the function has the value 1 and if $E \leqq 0$ then f takes the value 0.

Chapter 6: Computation and undecidability in the spatialized prisoner's dilemma. This chapter examines the problem: given an initial finite configuration of strategies (and a simple rule for how the players change their strategies during the run), will one strategy eventually dominate? This problem is proved to be undecidable, by reduction of a standard unsolvable problem to it. There is a detailed exposition on Turing and Minsky machines. (I have not checked details.)

Conclusion. The authors conclude that they have presented a sample rather than a survey of methods of philosophical computer modeling. They expect that the philosophical computer will expand the ancient tradition of conceptual modeling in unpredictable ways.

The book is complemented by a CD-ROM containing several examples in color, some of them dynamic, and containing the text of the whole book.

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Pablo Domínguez Prieto. ***Indeterminación y verdad. La polivalencia lógica en la escuela de Lvóv-Varsovia.*** Nossa y Jara Editores, Madrid 1995, 289 pp.

Łukasiewicz introduced his systems of many-valued logics in order to investigate modal propositions and the notions of possibility and necessity closely related to such propositions.

In 1910 Łukasiewicz published a book, in Polish (186***2***), and a paper, in German (186***1***), about the principle of non-contradiction in Aristotle's work. In both works he argued conclusively that this principle is not so evident and fundamental as it is considered. In 1918, before leaving Poland for two years, he announced that he had solved the problem concerning the possible existence of non-Aristotelian logics. In fact, in 1920 he presented his three-valued logic L_3 (186***4***), which does not contradict the principle of non-contradiction, but contradicts the logical principle of bivalence (and the Aristotelian principle of the excluded middle).

Łukasiewicz introduced, for every natural number $n \geqq 2$, an n-valued propositional calculus L_n, in which n distinct truth-values can be attributed to the propositions; and an infinite-valued propositional calculus $L_{\aleph_0}$, in which the propositions can have as truth-values any real number between zero and one. The bi-valued calculus coincides with the Aristotelian classical calculus.

Post, working independently of Łukasiewicz and motivated by certain formal properties of systems of propositions, introduced in 1921 his n-valued propositional calculi, known as Post logics (280*1*). So, Łukasiewicz and Post, working independently and motivated by distinct questions, are undoubtedly the creators of many-valued logic.

In this book—***Indeterminación y verdad, La polivalencia lógica en la escuela de Lvóv-Varsovia***—Łukasiewicz's positions are considered inseparable from his relations with other authors, among them Twardowski, Leśniewski, Kotarbiński, and Ajdukiewicz. These relations are very important to the understanding of the distinct positions of the Lvóv–Warsaw school of logic relative to many-valuedness, at the same time recognizing the unity of this school.

This work presents a double perspective: the historical perspective, by presenting the authors, their relationship, and several works important to the history of many-valued logic; and a theoretical perspective, by analyzing the position of these authors relative to the notion and interpretation of many-valuedness and of several questions related to it.

The object of study of the book is the conception of logical many-valuedness. The aim of the work is to elucidate the notion of indetermination in many-valued logic as it was used by the Lvóv–Warsaw school. It is understood that many-valuedness appears, in the thinkers of that school, as a probabilistic unfolding of bi-valued logic.

The first chapter of the book studies the question of logical many-valuedness, beginning from a discussion of the classical principle of bivalence. Many-valued systems are correctly presented as special kinds of non-classical logics, and the motivations for the appearance of these systems and the meaning of their non-classical truth-values are discussed. The role played by the Lvóv–Warsaw school of logic in the history of many-valued logic is analyzed.

In the next chapters the study of several authors of the Lvóv–Warsaw school begins with some biographical notes. In Twardowski the notion of truth and the analysis of its absolute and atemporal character is studied. According to Kotarbiński and Czeżowski, these subjects strongly influenced Łukasiewicz's metaphysical conception, from which his many-valued positions are derived.

Leśniewski's intuitive formalism is studied. The book's author discusses Leśniewski's conception of the "symbolic function" of formalized systems, his mechanisms of assignment of truth-values to the judgments of logic, and his analysis of the eternal character of the veritable values of the judgments of logic. Leśniewski studied some questions related to the notion of logical truth presented by Twardowski, and offered a possible logical solution—opposite to Łukasiewicz's solution—to some open metaphysical problems proposed by Twardowski.

In Kotarbiński, three subjects are studied: his analysis of the propositions on contingent future events, his conception of possibility, and his criticisms of the pre-eternal objectivity of the contingencies.

Ajdukiewicz's conception of language in the interpretation of the judgments of logic is analyzed through his three philosophical stages: radical conventionalism, moderated empirism, and radical empirism.

In the book, there are important references to the second generation of the Lvóv–Warsaw school, trying to show that the question of logical many-valuedness continues to be ever present. Among several authors, Jaśkowski and Zawirski are especially considered: besides several results related to many-valuedness, Jaśkowski's discussive system D_2 is correctly mentioned as the first formal analysis of contradictions, constituting the first known paraconsistent system in the literature.

Łukasiewicz is carefully studied. The interpretation of his many-valued logics is considered crucial, yet difficult. Some difficulties are presented for the analysis of Łukasiewicz logics: the evolution of Łukasiewicz's positions, from his metaphysical writings (1902–1915) until his logical writings (1915–1935–1956), and the multiple questions that have arisen from the interpretation of his complex conception of logic. The solutions introduced by

Łukasiewicz through his later many-valued positions are interpreted from his former metaphysical positions, and from this standpoint the distinct motivations of Łukasiewicz and Post are considered.

Łukasiewicz's three-valued system L_3 is studied, with two aims: to give a philosophical interpretation of such a system and to understand the meaning of the intermediate indeterminate truth-value $\frac{1}{2}$. Łukasiewicz's three-valued logic is interpreted as a probabilistic version of the classical bi-valued logic.

The relations among logical many-valuedness, temporal dimension and language constitute one of the fundamental questions of this study.

The author claims that his interpretation of Łukasiewicz's third truth-value, the indeterminate truth-value $\frac{1}{2}$, was possible only after having studied the positions of several authors of the Lvóv–Warsaw school. And from this interpretation he could try to answer some questions: Why did many-valued logic appear? How should these systems be interpreted? Is there an unique correct system of logic?

One of the most important aims of this work is to show that the apparently distinct positions of the authors studied, relative to many-valuedness and including their non-acceptance, can be integrated into the same interpretative scheme.

Finally, it is suggested that this genesis of many-valuedness in the Lvóv–Warsaw school shows that the divergence of logical systems does not correspond to a pluralism of rival logics and, on the other hand, that a strictly monistic vision of logic is not acceptable.

This book, with a *Foreword* by Joseph M. Życiński, corresponds to the doctoral dissertation of its author, Pablo Domínguez Prieto.

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STEPHEN BELLANTONI and STEPHEN COOK. *A new recursion-theoretic characterization of the polytime functions*. ***Computational complexity***, vol. 2 (1992), pp. 97–110.

ARNOLD BECKMANN and ANDREAS WEIERMANN. *A term rewriting characterization of the polytime functions and related complexity classes*. ***Archive for mathematical logic***, vol. 36 (1996), pp. 11–30.

In their paper under review, Bellantoni and Cook characterize the functions computable in polynomial time by a function algebra that controls the growth rate of functions without introducing explicit bounds in the defining expressions. The paper is well written and self-contained; the underlying idea of "implicit control" proved fruitful in characterizing many other complexity classes.

In order to give a machine-independent, purely functional characterization of the class FPTIME of number-theoretical functions computable on a Turing machine in time polynomial in the binary length $|x|$ of the inputs, one first simulates binary representations of natural numbers by the initial functions 0 and the *binary successors* s_0, s_1 satisfying $s_0(x) = 2 \cdot x$ and $s_1(x) = 2 \cdot x + 1$. This "data structure" gives rise to a canonical recursion scheme, that is, f is defined by *recursion on notation* from g, h_0, h_1 if $f(0, \vec{y}) = g(\vec{y})$ and $f(s_i(x), \vec{y}) = h_i(x, \vec{y}, f(x, \vec{y}))$ where $s_i(x) \neq 0$ is required in the case $i = 0$.

Recursion on notation is as strong as primitive recursion. For example, the function e satisfying $e(x) = 2^x - 1$ can be defined by recursion on notation from string concatenation $\oplus$ satisfying $x \oplus y = x \cdot 2^{|y|} + y$ by $e(0) := 0$, $e(s_0(x)) := e(x) \oplus e(x)$, and $e(s_1(x)) := s_1(e(x) \oplus e(x))$, where $\oplus$ is defined by $x \oplus 0 := x$ and $x \oplus s_i(y) := s_i(x \oplus y)$. Hence, in order to characterize FPTIME, we must restrict the use of recursion.

The first such restriction is due to Cobham (JSL XXXIV 657), who showed FPTIME = FP where FP is the smallest class containing the initial functions $0, s_0, s_1$, all projections π_i^n satisfying $\pi_i^n(\vec{x}) = x_i$, and smash where $\text{smash}(x, y) = 2^{|x| \cdot |y|}$, and closed under composition and *bounded* recursion on notation. The idea is that recursion on notation can be used as long as the resulting function is bounded by a function already defined in the class. To make this idea work, the "principal function" smash is added to the initial functions.

Bellantoni and Cook in their paper under review observe that the "explicit" reference to the growth rate of functions in Cobham's approach can be made "implicit" by ensuring that *computed values in recursions must not control other recursions*. To exemplify this central idea, consider string multiplication $\otimes$ satisfying $x \otimes y = x \cdot \sum_{i<|y|} 2^{i \cdot |x|}$ defined by $x \otimes 0 := 0$ and $x \otimes s_i(y) := (x \otimes y) \oplus x$. Although both functions e and $\otimes$ are defined by one recursion from $\oplus$, the former is not in FPTIME whereas the latter is. The reason is simply that when computing $e(s_i(x))$, the *previous value* $e(x)$ is fed into both input positions of $\oplus$, thus each previous value induces the unfolding of (in that way "controls") the recursion in $\oplus$. In contrast to e, no previous value in the recursion for $\otimes$ controls another recursion.

The idea that *computed values in recursions must not control other recursions* led to the class BC where each function is written in the form $f(\vec{x}; \vec{y})$ serving as a kind of bookkeeping of those input positions $\vec{x}$ (or $\vec{y}$) on which some (or no) recursion has been performed. Thus, recursion in BC is of the form

$$f(0, \vec{x}; \vec{y}) = g(\vec{x}; \vec{y}) \qquad f(s_i(x), \vec{x}; \vec{y}) = h_i(x, \vec{x}; \vec{y}, f(x, \vec{x}; \vec{y})).$$

Accordingly, composition is restricted to the form $f(\vec{x}; \vec{y}) = g(\vec{g}(\vec{x};); \vec{h}(\vec{x}; \vec{y}))$ and the initial functions come as 0, $s_0(; y)$, $s_1(; y)$, $\pi_i^{n,m}(\vec{x}; \vec{y})$, the *binary predecessor* $p(; y)$ satisfying $p(; 0) = 0$, $p(; s_i(y)) = y$, and the *binary conditional* $c(; x, y, z)$ satisfying $c(; s_0(x), y, z) = y$ and $c(; s_1(x), y, z) = z$.

An easy induction on the structure of $f(\vec{x}; \vec{y})$ shows that each f in BC is *length-bounded by a polynomial q_f in the lengths of $\vec{x}$*, i.e., $|f(\vec{x}; \vec{y})| \leqq q_f(|\vec{x}|) + \max(|\vec{y}|)$ for all $\vec{x}, \vec{y}$. Using this "bounding lemma," one can turn every recursion in BC into a bounded recursion in Cobham's class FP, implying BC $\subseteq$ FP. The converse holds by simulating the functions of FP in BC.

The paper by Beckmann and Weiermann under review investigates the natural question of what computational complexity of algorithms is obtained when the function algebra BC of the preceding paper is turned into a rewrite system. The major findings are that the length of any reduction sequence is still exponentially bounded in the binary lengths of the inputs, but if one requires that those arguments on which no recursions have been performed are to be evaluated first, then the length of every such reduction sequence is bounded by a polynomial in the binary lengths of the inputs.

The paper is hard going in parts, but quite rewarding if the reader perseveres. All relevant notions are defined and proofs are presented in full detail; familiarity with the function algebra BC, however, is presupposed. The latter is crucial for the understanding of the findings above and the way they are established.

The bounding lemma above is the pivot of the development. To set the stage, the function algebra BC is turned into a rewrite system over BC-*terms*, that is, terms built from variables and BC-*function symbols*. The latter are built from symbols for the initial functions by "name operations" corresponding to recursion and composition in BC. Reading the equations $f(\vec{r}; \vec{s}) = t$ in BC as *conversion rules* $f(\vec{r}; \vec{s}) \to t$, one obtains a rewrite system over BC-terms. Obviously, each *reduction sequence* for a BC-term t, that is, a maximal sequence $t \to t_1 \to t_2 \to \cdots$, is finite and results in a unique term $\mathrm{nf}(t)$. In particular, if s is *ground* (i.e., t contains no variables), then $\mathrm{nf}(t)$ is a "binary numeral" denoting the value of t.

In order to estimate the length of any reduction sequence for ground BC-terms t, Beckmann and Weiermann cleverly combine standard methods with the length-bounding polynomials q_f above. The standard methods ask for an interpretation I of BC-terms such that $I(t) > I(t')$ whenever $t \to t'$. Hence the length of any reduction sequence for $f(\vec{m}; \vec{n})$ is bounded by $I(f(\vec{m}; \vec{n}))$, where f is a BC-function symbol and $\vec{m}; \vec{n}$ is any list of binary numerals.

In the general case, the interpretation I is defined such that $I(f(\vec{x}, \vec{y})) = 2^{I_0(f)(|\vec{x}|)} \cdot q(|\vec{x}, \vec{y}|)$ where $I_0(f)$ is a polynomial defined along the structure of q_f, and q is a polynomial of degree 1. Thus, the length of any reduction sequence for $f(\vec{m}; \vec{n})$ is bounded by an exponential of a polynomial in the lengths of the inputs $\vec{m}, \vec{n}$. It is shown that there is no polynomial bound. The example given, however, is not convincing in so far as essentially projections $\pi_i^{n,m}(\vec{x}; \vec{y})$ providing many "redundant" input positions $\vec{y}$ are used to double reductions.

This leads, however, to restricted conversion rules $\to'$ requiring that $f(\vec{r}; \vec{s}) \to' t$ for compound BC-function symbols f is admissible only if $\vec{s}$ are already binary numerals $\vec{n}$. In other words, one requires that those arguments on which no recursions have been performed have to be evaluated first. In this case, an interpretation I can be defined such that $I(f(\vec{x}, \vec{y})) = I_0(f)(|\vec{x}|) \cdot q(|\vec{x}, \vec{y}|)$ where $I_0(f)$ is a polynomial mimicking essentially the structure of q_f above, and q is a polynomial of degree 1. Thus, the length of any $\to'$ reduction sequence for $f(\vec{m}; \vec{n})$ is bounded by a polynomial in the lengths of the inputs $\vec{m}, \vec{n}$.

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YURI GUREVICH and SAHARON SHELAH. *On finite rigid structures*. ***The journal of symbolic logic***, vol. 61 (1996), pp. 549–562.

This is one of the finest seminal papers to appear in the field of finite model theory within the last decade. On the face of it, the paper establishes a negative result by refuting Dawar's conjecture: a linear order is definable in the least fixed point logic over any finitely axiomatizable class of rigid finite structures (Anuj Dawar, *Feasible computation through model theory*, Ph.D. dissertation, University of Pennsylvania, 1993).

To appreciate fully the result as well as the technique developed in the paper, one needs to reflect on the role that is played in finite model theory and database theory (1) by linear order and its definability and (2) by least fixed point logic.

Finite model theory really blossomed when it was realized that querying relational database states could be thought of as defining classes of finite models by sentences of logical languages. This viewpoint led to addressing the PTIME-completeness problem for relational query languages formally, as follows: *Find a logical language that defines exactly those classes of finite models that are* PTIME-*computable*. (Other complexity classes have similarly been studied.)

In general, no logical language has been found (and, as the feelings of many go, is not going to be found) that captures exactly PTIME in this sense. If only linearly ordered models are considered, however, then least fixed point logic (LFP) captures exactly PTIME (this result is due to Neil Immerman—see JSL LIV 287—and Moshe Vardi, and very similar results have been established independently by Yuri Sazonov, Yuri Gurevich, and Alexander Livchak).

Even if a linear order is not explicitly present in a class of finite models, but can be uniformly defined in each of the models by a *simple* formula, the Immerman–Vardi result will still hold. The following considerations are important here: First, no linear order can be defined by a logical formula in a model with nontrivial automorphisms; hence, we restrict ourselves to rigid models only. Secondly, elements of any rigid model can be ordered, say, by their first-order types; however, we are interested in simple ordering formulas, to continue to stay within the complexity class PTIME. Thirdly, it is one of these deadly complexity-theoretical problems whether rigid graphs can be canonically oriented (in essence, linearly ordered) in polynomial time. Hence, further constraints should be imposed on the classes of rigid finite models.

The reviewer had tackled the problem in *Axiomatizable classes of finite models and definability of linear order* (***Proceedings of the seventh annual IEEE symposium on logic in computer***

science, IEEE Computer Society Press, 1992, pp. 64–70). Motivated by a conjecture of Boris Zilber, he considered finitely axiomatizable classes of rigid models and definability of linear order by a first-order formula. (The class of all rigid finite models is not finitely axiomatizable.) The feeling was, since the first-order axiomatization is restrictive enough to catch automorphisms, it should be possible to massage somehow the axiomatization into an order-defining formula. The intuition proved to be wrong, and the paper actually constructed a finitely axiomatizable class $\mathcal{K}$ of finite structures in which no first-order formula would define linear order. The models of the class $\mathcal{K}$ were, however, observed to admit ordering easily either by LFP, or by first-order implicit definability (recall that over finite models, Beth's definability theorem fails). This result yielded two conjectures, the conjecture by Dawar cited above, and a similar but weaker conjecture by the reviewer that, under the same conditions, linear order can always be defined implicitly.

The paper under review refutes Dawar's conjecture by presenting a class $\mathcal{M}$ of rigid finite models called *odd multipedes* (actually called in the paper *odd* 4-*multipedes*) that is finitely axiomatizable, and yet no LFP formula linearly orders elements of every odd multipede.

An odd multipede is a three-layered structure consisting of *segments*, *feet*, and *sets*, as follows. *Segments* is a linearly ordered set. *Sets* contains exactly one element for each subset of segments, and the relation $s \,\varepsilon\, S$ holds iff the segment s belongs to the set S. Finally, each segment has exactly two *feet*, attached to it by a binary relation.

It is clear that because of the linear order on the segments, naturally expanded to the sets, any imaginable automorphism of an odd multipede consists of permuting the feet of some segments. To eliminate such movements of feet, there is an additional hypergraph structure imposed on the segments, and on the feet, separately, as follows: (a) for every foot hyperedge in the foot hypergraph, the corresponding segment hyperedge belongs to the segment hypergraph; (b) for every segment hyperedge, exactly four of the induced eight foot hyperedges belong to the foot hypergraph, and the other four do not; (c) for every foot hyperedge in the foot hypergraph, every other foot hyperedge obtained by permuting the feet of exactly two segments also belongs to the hypergraph; (d) for every subset S of segments there exists a hyperedge H in the segment graph whose intersection with S is of odd cardinality. In other words, either $|S \cap H| = 1$, or $|S \cap H| = 3$.

For the odd multipedes described in this way, it is easy to see both that the class of odd multipedes is finitely axiomatizable, and that every odd multipede is rigid. Indeed, if θ were a nontrivial automorphism, it would permute the feet of either one or three segments in a certain segment hyperedge, and thus transform a foot hyperedge into a non-hyperedge.

Further, the paper shows that for any k, no formula of $L^k_{\infty,\omega}$ with counting quantifiers distinguishes between the two feet of any segment in a large enough k-*meager* odd multipede. (A multipede is called k-meager if any subset of segments of cardinality $c \leqq 2k$ contains at most $2c$ segment hyperedges.) This is done by employing the standard technique of playing a version of an infinite k-pebble game that is known to characterize the expressive power of $L^k_{\infty,\omega}$ with counting quantifiers. The technically most difficult part of the paper is an asymptotic argument showing that, for every k, k-meager multipedes of arbitrarily large size do exist.

Anuj Dawar, Lauri Hella, and Phokion Kolaitis observed (*Implicit definability and infinitary logic in finite model theory*, ***Automata, languages and programming***, Springer, 1995, pp. 624–635) that in the class $\mathcal{M}$ of odd multipedes linear order is definable *implicitly* by a first-order formula. This provides further support for the conjecture that linear order is implicitly definable in any finitely axiomatizable class of rigid finite models. Implicit definability by itself, however, is believed to be beyond PTIME.

In a sense, the result of this paper signals failure of the entire approach to dealing with PTIME definability over rigid structures as outlined in this review. Employing languages more powerful than least fixed point logic (or indeed $L^k_{\infty,\omega}$ with counting quantifiers) may succeed as far as defining linear order goes, but this would likely drive us outside the PTIME class thus defeating the purpose. (It was because of these considerations that the paper by Stolboushkin and Taitslin, *Normalizable linear orders and generic computations in finite models*, ***Archive for mathematical logic***, vol. 38 (1999), pp. 257–271, explored an alternative approach of *normalizing* linear order rather than *defining* it.)

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LAURI HELLA, PHOKION G. KOLAITIS, and KERKKO LUOSTO. *How to define a linear order on finite models*. ***Annals of pure and applied logic***, vol. 87 (1997), pp. 241–267.

Background. It has long been known that for matters of *descriptive complexity theory*, the presence of linear orderings is of eminent importance. Descriptive complexity, as the study of logical versus computational complexity, is concerned with the relationship between the definability of properties of finite structures (in certain natural logics) and the computational complexity of deciding those properties algorithmically (in terms of standard complexity classes). Extensive research in this area has highlighted a deep divide between the two settings, that of structures carrying a linear order and that of structures without. In the ordered setting, tight correspondences have long been established between the major complexity classes and very natural extensions of first-order logic, e.g. by various fixed point mechanisms. For the unordered setting, it remains open whether, for instance, PTIME admits any characterization in terms of a logic. It is therefore quite natural to investigate the issue of under what circumstances and by what logical means a linear ordering can definably be obtained. The present paper addresses this issue systematically, proves some interesting new results, and establishes equally interesting connections with known results. The new overall picture which thus emerges is rounded off with a number of links to complexity theory proper, indicating likely limits for further progress along the present lines.

Main issues and results. Linear orderings can be definable only on rigid structures. Among classes of rigid structures, there is an important distinction concerning the level of "locality" at which rigidity manifests itself. This distinction of classes of *bounded* versus *unbounded rigidity* can be based on the behaviour of k-variable first-order logic on these classes. For a more algebraic view of the concept, consider systems of partial automorphisms, closed under restrictions and with the back-and-forth property for extensions up to some fixed finite size k. The *rigidity number* of a finite rigid structure is then the minimal k for which no such system contains any non-trivial mappings. A class of rigid structures is of bounded or unbounded rigidity according to the rigidity numbers of its members. Linearly ordered structures, for instance, have rigidity number at most 2. In the sense of asymptotic probabilities, all classes of bounded rigidity are small. In fact, the class of all rigid finite structures as well as for each k the class of all (rigid) structures whose rigidity number is at least k are asymptotically dense. The analysis of definability of linear order subject to this major distinction, on classes of bounded rigidity versus unbounded rigidity, yields characteristic and distinctive results. If the analysis is based on back-and-forth systems that respect multiplicities of extensions, one obtains a corresponding notion of *bounded rigidity with counting*, or C-bounded rigidity. Unlike the case of plain bounded rigidity, even the class of rigid structures of C-rigidity number 2 is asymptotically dense. There are, however, first-order definable classes of rigid structures that are of unbounded C-rigidity (by a sophisticated probabilistic construction by Gurevich and Shelah; see the preceding review).

Bounded rigidity. Bounded rigidity can actually be characterized by uniform definability of a linear ordering in least fixed point logic LFP (Dawar), but also by LFP-definability of the isomorphism query in restriction to that class. These characterizations carry over to the case of C-bounded rigidity, if LFP is correspondingly augmented by a basic means of cardinality comparison. A very nice application of these results on C-bounded rigidity concerns a particularly strong failure of Beth's definability property for first-order logic in the sense of finite model theory. An implicit first-order definition over finite (rigid) structures is exhibited, that is not equivalent to an explicit definition even in infinitary finite variable logic. An interesting open problem is whether, over classes of rigid structures, the condition that LFP captures PTIME also characterizes bounded rigidity.

Unbounded rigidity. Unbounded rigidity should be seen as the generic case. Much less structural information is available in this case, and it is only to be expected that few results can be obtained that would not be conditional on complexity-theoretic assumptions. The main contribution in this general setting concerns a detailed analysis of the definability (and complexity) of the canonical *minimal order* on rigid structures. The strongest unconditional result is that this canonical ordering is *implicitly* definable in partial fixed-point logic PFP. (Another way of saying this would be that implicit definability in PFP captures PSPACE over the class of all rigid structures.)

Complexity of universal orderings. In the ordered setting, implicit definability in first-order or least fixed point logic has a complexity theoretic counterpart in the class UP $\cap$ co-UP (Kolaitis), where UP is *unambiguous non-deterministic* PTIME. It is shown that the minimal order is implicitly first-order definable over the class of all rigid structures if and only if its canonical total extension is computable in UP. This fact is used to show that one cannot rule out the possibility that some universal order is computable in UP $\cap$ co-UP without separating this class (and therefore PTIME) from NP. Conversely, it is shown that the existence of a universal order in UP $\cap$ co-UP would imply corresponding upper bounds for the graph automorphism problem. Similar consequences are derived for the existence (or for the impossibility) of an implicitly LFP definable universal order.

Remark. The study of universal orderings on asymptotically dense classes of rigid structures is a theme of *Almost everywhere equivalence of logics in finite model theory*, this BULLETIN, vol. 2 (1996), pp. 422–443, by the same authors.

Minutiae. Theorem 3.2 could have been formulated with k the maximum of 2 (rather than 3) and the width of the vocabulary, which is an appealing strengthening in the important case of graphs. Similarly for Corollary 3.6.

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GRIGORI MINTS, SERGEI TUPAILO, and WILFRIED BUCHHOLZ. *Epsilon substitution method for elementary analysis*. ***Archive for mathematical logic***, vol. 35 (1996), pp. 103–130.

Prior to G. Gentzen's approach to Hilbert's program via cut elimination, there was the technique of so-called epsilon substitution developed by D. Hilbert and worked out further by W. Ackermann, J. von Neumann, and P. Bernays. This development, however, suffered from a serious defect in Ackermann's treatment of Peano arithmetic PA, which was exhibited by von Neumann, and it was only after Gentzen's successful ordinal analysis of Peano arithmetic that Ackermann was able to find the correct formulation of the epsilon substitution for PA and to prove its termination by transfinite induction up to ε_0. Despite some further important work in this direction by G. Kreisel and W. Tait in connection with the no-counterexample interpretation, Hilbert's original program to extend his epsilon-substitution method to stronger systems like analysis and analysis with the axiom of choice was not pursued for quite some time. It was only in the 1980's that the first author of the paper under review took up this program again systematically, first single-handedly and then in recent years with the collaboration of the second author, which resulted in a series of important papers which also clarified the connection between cut elimination and epsilon substitution, thereby showing how to use cut elimination to construct terminating epsilon substitutions.

In the paper under review this is carried out for elementary analysis EA and two proofs of termination are given. The first termination proof uses cut elimination for a ramified system into which the epsilon-extension EAε of EA is embedded, whereas the second—much shorter—termination proof is based on a non-constructive continuity argument. Here elementary analysis EA is the fragment of second-order arithmetic with full induction but with comprehension restricted to arithmetical formulas which may, however, contain free set variables. The proof-theoretic ordinal of EA is known to be $\varepsilon_{\varepsilon_0}$. The first (constructive) proof of convergence of the epsilon-elimination process is based on previous work by Mints (*Gentzen-type systems and Hilbert's epsilon substitution method, I*, ***Logic, methodology and philosophy of science IX***, Elsevier, 1994, pp. 91–122) in which a cut-elimination argument is used to show the termination of an epsilon-substitution process for the epsilon extension of Peano arithmetic PAε. By this general technique, the rate of convergence is measured by the ordinal needed to carry out the cut elimination by transfinite induction. For EA, the rate of convergence therefore is measured by $\varepsilon_{\varepsilon_0}$. As an intermediate step, a Gentzen-style formulation εEA of EAε with a special ω-rule is used.

As an application of this constructive epsilon elimination, it is shown how this procedure can be viewed as an algorithm to compute a witnessing numeral n (or a finite predicate) from an EAε-proof of a purely existential sentence $\exists x\, F(x)$ (or $\exists X\, F(X)$, respectively). The main results are due to the first two authors and were presented first as a technical report (Report No. CSLI-93-175, CSLI, Stanford University, 1993). The core parts of that report were rewritten by the third author with many improvements and simplifications, however, and the clarity of the presentation in the paper under review is very much due to him.

Subsequently, Mints and Tupailo extended their method to analysis with the Δ^1_1 comprehension rule (*Epsilon-substitution method for the ramified language and Δ^1_1-comprehension rule*, ***Logic and foundations of mathematics***, Kluwer Academic Publishers, 1999, pp. 107–130).

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Michael Rathjen. *Recent advances in ordinal analysis: Π^1_2-CA and related systems*. ***The bulletin of symbolic logic***, vol. 1 (1995), pp. 468–485.

Rathjen announces a solution to what many perceive to have been the outstanding problem of proof theory for some time: providing an ordinal analysis for theories of Π^1_2-comprehension. The reader is reminded that the *ordinal* of a (Π^1_1-sound) theory T is the least ordinal that is not the order type of a recursive well-ordering of the natural numbers that can be proved to be so in T. The boundedness theorem of descriptive set theory will imply that the ordinal of T is recursive. An *ordinal analysis* of T, in the crudest sense, consists of a recursive well-ordering $\preceq$ of a set of natural numbers, the *ordinal notation system*, and a proof that the length of the given well-ordering is the ordinal of T.

While one expects much more from an ordinal analysis than the description above requires (e.g. the reduction of the arithmetic part of T to transfinite induction over initial segments of $\preceq$), there is no more refined notion that is commonly accepted, and Rathjen prefers to discuss the following goals and uses (quoted from the paper) rather than the precise meaning of "ordinal analysis": (I) Hilbert's Program relativized: reduction (interpretation) of impredicative theories to (in) constructive frameworks. (II) Finding currently untapped resources of set theory: new combinatorial principles. (III) Development of a proof theory of set theory.

Important examples from categories (I) and (II) by Rathjen and others are mentioned. The contributions described in the paper under review belong to category (III). While the title concerns Π^1_2-CA, proof theorists have been analyzing such theories by first reducing them to set theories for some time. Rathjen mentions the possibility that ordinal analysis could eventually allow for a better explanation of Kunen's theorem that there is no nontrivial elementary embedding $j : V \to V$, perhaps leading to a clearer understanding of where inconsistency first appears in the hierarchy of reflection and embeddings. A "bold optimistic view," as Rathjen says, but not so bold that one should not take it seriously.

There is another, perhaps intermediate, goal of ordinal analysis which is even less precise but which nobody who has worked in the area would fail to recognize. When constructing an ordinal analysis, one feels that one is uncovering the "bare bones" of the theory in question and gains a perspective that is transparent in a very satisfying way. This perspective is akin to what set theorists studying large cardinals experience when working with inner models. It is behind the semantic interpretations of the notation systems given in Section 5.

After a whirlwind introduction in Section 3 to the methods developed by the likes of Pohlers, Buchholz, and Jäger for theories between Kripke–Platek set theory, KP, and KP + "the universe is recursively inaccessible" (for a detailed account, see W. Buchholz, *A simplified version of local predicativity*, ***Proof theory***, Cambridge University Press, 1992, pp. 115–147), an outline of the additional techniques needed to analyze KP + Σ_1-separation (the set-theoretic equivalent of Π^1_2-CA + BI) is given in Section 4. The underlying idea is that higher realms in the notation system correspond to higher degrees of set-theoretic reflection. The highest such notion needed in this case is that of a stable ordinal (an ordinal α is stable if $L_\alpha \prec_{\Sigma_1} L$). Other notions of reflection impose themselves: the existence of an ordinal α such that $L_\alpha \prec_{\Sigma_1} L_{\alpha+1}$ requires smaller ordinals that are Π_n-reflecting for each natural number n; the existence of an ordinal α such that $L_\alpha \prec_{\Sigma_1} L_{\alpha+2}$ requires smaller ordinals that are "1-Π_n-reflecting" for each natural number n, etc. More refined degrees of reflection needed for the analysis are hinted at in Definition 4.8, Lemma 4.9, and comments made after Lemma 4.9. The delineation of all necessary intermediate forms of reflection is a remarkable *tour de force*, a substantial part of which Rathjen had already carried out in his *Proof theory of reflection* (***Annals of pure and applied logic***, vol. 68 (1994), pp. 181–224).

Section 5 outlines the notation system for Π^1_2-CA + BI assuming the existence of ω + 1 stable ordinals (such semantic methods requiring strong assumptions—the existence of stable ordinals in this case—are important motivation and simplify the development, but are ultimately dispensable). The construction generalizes to give other systems, depending on how many stable ordinals are used. The verification that these systems are recursive is undoubtedly hard work. An alternative construction is given in Section 6 where stable ordinals are replaced by certain large cardinals; the strength of the large cardinals is said to allow proofs to be simplified (there is also the rather mysterious connection at play between large cardinals and their recursive counterparts—apparently, the new large cardinal axioms introduced could be considered the large cardinal analogues of forms of stability). The section closes with the statement of the main theorem: the notation systems provide ordinal analyses for the theories Π^1_2-CA$_0$, Π^1_2-CA + BI, and Δ^1_3-CA.

The paper is nicely written though the pace is extremely fast and necessarily sketchy. Rathjen has given us a tantalizing glimpse at what promises to be a cornerstone of the proof theory of the future.

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MICHAEL WOODS. ***Conditionals.*** Edited by David Wiggins. Clarendon Press, Oxford University Press, Oxford, New York, etc., 1997, ix + 152 pp.

DOROTHY EDGINGTON. *Commentary*. Therein, pp. 95–137.

This is a very interesting investigation of natural language conditionals. Woods displays both a remarkably sensitive insight into examples of usage and an excellent grasp of the relevant literature. He makes a very strong case against material conditional interpretations, and offers interesting new support for the sort of conditional assertion accounts of simple conditionals offered by Adams and Gibbard and for the sort of prior propensity account of other conditionals that has been offered by Skyrms.

The first chapter is a careful introduction to the two broad classes of conditional assertions. Woods uses Adams's famous example,

'If Oswald did not assassinate Kennedy, someone else did.'
'If Oswald had not assassinated Kennedy, someone else would have.'

as well as an earlier example from Ramsey, to illustrate the distinction between what are often called "counterfactual" and "indicative" conditionals. He points out that these terms can be misleading, and uses the term "simple conditionals" to characterize the sort of conditionals exhibited in the indicative sentence in such examples.

The second chapter reviews the long history of attempts by Grice, Jackson, and others to defend the interpretation of simple conditionals as material conditionals (or as material conditionals plus some further conditions) in the face of the fact that assertibility conditions of natural language conditionals do not coincide with truth conditions of material conditionals.

In the third chapter Woods points out that Adams's theory—which takes the conditional probability of the consequent on the antecedent as the appropriate measure of assertibility for conditionals—explains intuitive examples where simple conditionals violate inferences valid for material conditionals.

The fourth chapter begins by pointing out that Lewis's famous triviality result can motivate approaches like that of Grice where assertibility is distinguished from probability of truth. It then provides examples of assertible conditionals where the probability of the material conditional is only slightly higher than that of the negation of the antecedent, which are problems for Grice. He also shows how Jackson's requirement of robustness with respect to the antecedent makes his proposal do better than Grice's for such examples.

The fifth chapter offers a brief overview of the nearness of possible worlds approaches made salient by the work of Stalnaker and Lewis. Woods suggests that appeal to possible worlds is not necessary for the analysis of simple conditionals.

The sixth chapter is a quite informative investigation of the extent to which compound conditionals raise problems for conditional assertion accounts. On such accounts to assert (or to accept) a conditional is a conditional assertion (or acceptance) of the consequent given the antecedent, rather than the assertion or acceptance of a conditional proposition. The problems come with cases with disjunctions or conditionals as antecedents. For conditionals with disjunctive antecedents, Woods offers a promising Gricean suggestion that would make the conditional assertion account agree with intuitive examples that tell quite heavily against material conditional accounts.

Conditionals in antecedents of conditionals present the greatest problem, but here again the problem for the material conditional is worse. Woods discusses an especially telling example due to Gibbard:

'If the cup broke if it was dropped, it was fragile.'

This may be very assertible in a context where the cup is assumed to be not dropped and not fragile, and so can be assertible even when the corresponding material conditional is assumed to be false. Accounts like Grice's and Jackson's offer rationale for not asserting or accepting "If P then Q" even though the corresponding material conditional would be assumed true. They, therefore, are no help at all against Gibbard's example.

Woods suggests that conditional assertion accounts can accommodate such cases by taking the antecedent proposition to be relevant circumstances that would make the antecedent conditional assertion acceptable, for example sufficient hardness of the floor and sufficient height from which the cup is dropped.

Woods also has interesting things to say about negations of conditionals, which, of course, have long been a problem for interpreting "If P then Q" as a material conditional.

In Chapter 7 Woods reviews arguments from Dorothy Edgington against the view that simple conditionals have truth-conditions and proposes a conditional assertion account. On this account acceptability, assertibility, and belief in "If P then Q" go by the epistemic conditional probability $\mathrm{Prob}(Q \mid P)$. He expands somewhat on his earlier suggestion that context can fix the non-conditional grounding beliefs that make it possible to take embedded simple conditionals as antecedents of complex conditionals. He also explores some nice parallels that his account offers between conditional assertions and conditional commands and questions.

In Chapter 8 Woods offers his account of conditionals that require more than his simple conditional analysis. He argues that sentences like

'If John catches the train, he will arrive on time.'

should not be interpreted as simple conditionals, even though they carry no implication of counterfactuality of their antecedents. Woods suggests that the purport of such sentences is that matters are now such as to support a high objective conditional probability of Q given P. Such objective conditional probabilities are to be thought of as epistemic conditional probabilities from an *ideal* perspective—one in which the relevant factors are all known.

According to Woods, Adams's famous past subjunctive sentence should be acceptable if at some appropriate time in the past

'If Oswald does not shoot Kennedy someone else will do so.'

was acceptable from the standpoint of someone apprised of all the relevant facts.

The book also contains a quite useful commentary by Dorothy Edgington. She raises several questions among which are some reservations about the sharpness with which Woods distinguishes will-conditionals from simple conditionals that are to be evaluated by the Ramsey test. I think that Woods is quite correct and that this can be seen clearly in applications of conditionals in decision and game theory.

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