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REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Matthew Foreman, Gerhard Jäger, Penelope Maddy, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

LOU VAN DEN DRIES, ANGUS MACINTYRE, and DAVID MARKER. *The elementary theory of restricted analytic fields with exponentiation*. ***Annals of mathematics***, ser. 2 vol. 140 (1994), pp. 183–205.

LOU VAN DEN DRIES, ANGUS MACINTYRE, and DAVID MARKER. *Logarithmic-exponential power series*. ***Journal of the London Mathematical Society***, ser. 2 vol. 56 (1997), pp. 417–434.

Since the mid-1980's, a dramatic increase in our knowledge of the model theory of various expansions of the ordered field of real numbers—and the relations of these structures to real-analytic geometry—has taken place. The results of the papers under review, as well as the techniques used therein, are some of the most significant and influential of these developments.

In order to summarize the main results, we need some definitions. Let $\boldsymbol{R}_{\mathrm{an}}$ denote the expansion of the ordered field of real numbers by all restricted analytic functions, that is, by all functions $\boldsymbol{R}^n \to \boldsymbol{R}$ (n ranging over $\boldsymbol{N}$) that vanish identically off $[-1,1]^n$ and whose restrictions to $[-1,1]^n$ are real analytic. Note that the function $\boldsymbol{R} \to \boldsymbol{R}$ given by $x \mapsto \exp(x)$ for $x \in [-1,1]$, and $x \mapsto 0$ for $x \notin [-1,1]$, is a restricted analytic function, where exp denotes the real exponential function $x \mapsto e^x : \boldsymbol{R} \to \boldsymbol{R}$. Let $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ denote the expansion of $\boldsymbol{R}_{\mathrm{an}}$ by exp, and $(\boldsymbol{R}_{\mathrm{an}}, \exp, \log)$ denote the expansion of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ by the unary function $x \mapsto \log x$ for $x > 0$, and $x \mapsto 0$ for $x \leqq 0$.

The main result of the first paper is that $(\boldsymbol{R}_{\mathrm{an}}, \exp, \log)$—more precisely, its complete theory—is explicitly universally axiomatizable, admits QE (quantifier elimination), and is

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o-minimal (that is, every parametrically definable subset of $\boldsymbol{R}$ is a finite union of points and open intervals).

In the second paper, the results from the first paper are used to exhibit a particular non-standard model of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. Certain formal properties of this non-standard model correspond closely to asymptotic properties of unary functions definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. This correspondence is exploited to affirm a conjecture made by G. H. Hardy (open since the early 1900's) and to show that certain special functions are not definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$.

Before going into more detail, I give some historical context (see also the review JSL LXIV 910). Using known facts from subanalytic geometry, van den Dries observed that $\boldsymbol{R}_{\mathrm{an}}$ is model complete, o-minimal, and that exp is not definable in $\boldsymbol{R}_{\mathrm{an}}$; see *A generalization of the Tarski–Seidenberg theorem, and some nondefinability results*, ***Bulletin of the American Mathematical Society***, n.s. vol. 15 (1986), pp. 189–193. The model-completeness result was sharpened by J. Denef and van den Dries (*p-Adic and real subanalytic sets*, JSL LXII 1481) to a quantifier-elimination result: Let $(\boldsymbol{R}_{\mathrm{an}}, {}^{-1})$ denote the expansion of $\boldsymbol{R}_{\mathrm{an}}$ by the function $0 \mapsto 0$ and $x \mapsto 1/x : \boldsymbol{R} \to \boldsymbol{R}$ for $x \neq 0$; then $(\boldsymbol{R}_{\mathrm{an}}, {}^{-1})$ admits QE. (This result is not stated explicitly; rather, it is an elementary—but somewhat tedious—variant of (4.6) of the paper.) Then came A. J. Wilkie's proof that the expansion of the ordered field of real numbers by the function exp is model complete and o-minimal; see *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, JSL LXIV 910. By extending Wilkie's methods, and employing a sophisticated result from several complex variables, it was shown soon thereafter that $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ is model complete and o-minimal; see van den Dries and Miller, *On the real exponential field with restricted analytic functions*, ***Israel journal of mathematics***, vol. 85 (1994), pp. 19–56. But even before this paper was fully completed, a remarkable idea of J.-P. Ressayre would suggest a radically different approach, leading to both stronger results and simpler proofs.

Roughly speaking, Ressayre's theorem is that, aside from a few obvious facts about exponentiation, all first-order information about the real exponential field is already present in the expansion of the real field by the restriction of exp to the interval $[-1, 1]$. (See *Integer parts of real closed exponential fields*, JSL LX 1014.) Since the restriction of exp to $[-1, 1]$ is definable in $\boldsymbol{R}_{\mathrm{an}}$, a natural extension of Ressayre's idea was that, aside from the same few obvious facts about exponentiation, all first-order information about $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ should already be present in $\boldsymbol{R}_{\mathrm{an}}$. This turned out to be a key notion in obtaining the main results of the papers under review (although, in the large, the proofs given are formally independent of those given by Ressayre).

I now outline the proof of the first main result of the first paper. The theory $T_{\mathrm{an}}(\exp)$ is defined, in an appropriate language, to be the complete theory of $\boldsymbol{R}_{\mathrm{an}}$ together with the "Ressayre axioms": (a) axioms expressing that exp is an order-preserving homomorphism of the additive group into the multiplicative group; (b) for each positive integer n, the axiom $x > n^2 \to \exp(x) > x^n$; and (c) the axiom $-1 \leqq x \leqq 1 \to \exp(x) = e(x)$, where e is the function symbol in the language of $\boldsymbol{R}_{\mathrm{an}}$ representing the restricted analytic function arising from exp. The theory $T_{\mathrm{an}}(\exp, \log)$ is then defined by adding the axioms $x > 0 \to \exp(\log(x)) = x$ and $x \leqq 0 \to \log(x) = 0$. A closer examination of the proof of QE for $(\boldsymbol{R}_{\mathrm{an}}, {}^{-1})$ reveals that an explicit universal axiomatization is obtained simply by expanding $(\boldsymbol{R}_{\mathrm{an}}, {}^{-1})$ by all root functions $x \mapsto x^{1/n}$, n ranging over all positive integers, where $x^{1/n}$ is set equal to 0 for $x \leqq 0$. It follows easily that $T_{\mathrm{an}}(\exp, \log)$ has an explicit universal axiomatization. A model-theoretic embedding test then establishes that $T_{\mathrm{an}}(\exp, \log)$ admits QE. (Although the verification of this embedding test is established by using fairly elementary techniques from valuation theory, ideas of Wilkie and Ressayre play important roles.) Then $T_{\mathrm{an}}(\exp, \log)$ is complete as well, since $(\boldsymbol{R}_{\mathrm{an}}, \exp, \log)$ embeds into every model of $T_{\mathrm{an}}(\exp, \log)$. Hence, $(\boldsymbol{R}_{\mathrm{an}}, \exp, \log)$ is axiomatized by $T_{\mathrm{an}}(\exp, \log)$, and $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ is axiomatized by $T_{\mathrm{an}}(\exp)$.

A conceptually simple proof of the o-minimality of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ is then obtained. Because a subset of the real line is a finite union of points and intervals if and only if the boundary of the set is bounded and discrete, one may show that an expansion $\mathfrak{R}$ of the real field is o-minimal by showing that every unary function definable in $\mathfrak{R}$ is ultimately of constant sign. Extensions and generalizations of known Hardy field techniques yield that o-minimality is preserved under adding exp to o-minimal expansions of the real field satisfying certain extra conditions (satisfied in particular by $\boldsymbol{R}_{\mathrm{an}}$).

A curious corollary of the proof shows that exponentiation is, in a sense, not needed for QE: The structure $(\boldsymbol{R}_{\mathrm{an}}, \log, (x^q)_{q\in\boldsymbol{Q}})$ admits QE, where, for $r \in \boldsymbol{R}$, x^r denotes the power function $x \mapsto x^r$ for $x > 0$ and $x \mapsto 0$ for $x \leqq 0$. On the other hand, the logarithm function is needed: The structure $(\boldsymbol{R}_{\mathrm{an}}, \exp, (x^r)_{r\in\boldsymbol{R}})$ does not admit QE; see Marker and Macintyre, *A failure of quantifier elimination*, ***Revista matemática de la Universidad Complutense de Madrid***, vol. 10 (1997), pp. 209–216.

The second paper is devoted to the construction and applications of a certain non-standard model of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$, namely, the field of logarithmic-exponential (LE) series, denoted by $\boldsymbol{R}((t))^{\mathrm{LE}}$. (Remark: There are changes in notation from the first to the second paper; for the sake of uniformity, I retain the notation from the first paper.) I will not attempt to define $\boldsymbol{R}((t))^{\mathrm{LE}}$ here. Roughly speaking, it consists of certain special kinds of formal sums $\sum_{\alpha\in I} f_\alpha$, with each f_α a term in the language of $(\boldsymbol{R}_{\mathrm{an}}, \exp, \log)$, and I some well-ordered index set. The explicit axiomatization obtained for $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ is used to show that $\boldsymbol{R}((t))^{\mathrm{LE}}$ is a model of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. The usefulness of $\boldsymbol{R}((t))^{\mathrm{LE}}$ as a model of $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ arises from the fact that given any function $f : \boldsymbol{R} \to \boldsymbol{R}$ definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$, one can associate to f an element $\sum_{\alpha\in I} f_\alpha \in \boldsymbol{R}((t))^{\mathrm{LE}}$ in such a way that the asymptotic behavior of f at $+\infty$ is well reflected by the formal object $\sum_{\alpha\in I} f_\alpha$.

I now list the main applications in the second paper.

A conjecture of Hardy is verified: The compositional inverse of the function $t \mapsto (\log t)(\log\log t) : (1, \infty) \to \boldsymbol{R}$ is not asymptotic at $+\infty$ to any real-valued function obtained as a finite composition of semi-algebraic functions, exp, and log.

The structure $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ is exponentially bounded, that is, given a unary function f definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$, there exists some compositional iterate $\exp_N$ of exp such that $|f|$ is bounded at $+\infty$ by $\exp_N$. (A closer look yields a sharper result; see Marker and Miller, *Levelled o-minimal structures*, ***Revista matemática de la Universidad Complutense de Madrid***, vol. 10 (1997), pp. 241–249.)

Two different methods are used to show that certain special functions are not definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. First, it is established that if a function $f : (0, \varepsilon) \to \boldsymbol{R}$, definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$, has an asymptotic expansion $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0^+$, then the power series $\sum_{n=0}^{\infty} a_n X^n$ is convergent. Consequently, no function $f : (0, \varepsilon) \to \boldsymbol{R}$ having a divergent asymptotic power series expansion at 0^+ is definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ (and similarly for a function $g : (a, \infty) \to \boldsymbol{R}$ having divergent asymptotic power series expansion at $+\infty$). This is used to show that various functions defined on $(0, \infty)$ such as $\Gamma \restriction (0, \infty)$, the error function $\int_0^x \exp(-t^2)\, dt$ and the logarithmic integral $\int_x^\infty 1/(t \exp(t))\, dt$ are not definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. The second method is based on an analysis of the sets of exponents appearing nontrivially in the LE-series associated with unary functions definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$. As a special case, if a function $f : (a, \infty) \to \boldsymbol{R}$ definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ has an asymptotic expansion of the form $\sum_{n=1}^{\infty} \exp(\alpha_n x)$ (for $\alpha_n \in \boldsymbol{R}$) as $x \to +\infty$, then the set of all α_n's is contained in some finitely generated $\boldsymbol{Q}$-vector subspace of $\boldsymbol{R}$. In particular, no unary function definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$ has an asymptotic expansion of the form $\sum_{n=1}^{\infty} \exp(-x \log n) = \sum_{n=1}^{\infty} n^{-x}$ as $x \to +\infty$. Hence, no restriction of the Riemann zeta function to a ray (c, ∞), $c \geqq 1$, is definable in $(\boldsymbol{R}_{\mathrm{an}}, \exp)$.

It is worth noting that all of the special functions mentioned in the preceding paragraph are analytic on their natural domains of definition. Hence, all restrictions to compact subsets of their natural domains are definable in $\boldsymbol{R}_{\mathrm{an}}$. Note also that for any real numbers a, b with $a > 0$ and $b > 1$, both $\Gamma \restriction (0, a)$ and $\zeta \restriction (1, b)$ are definable in $\boldsymbol{R}_{\mathrm{an}}$.

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LOU VAN DEN DRIES. ***Tame topology and o-minimal structures.*** London Mathematical Society lecture note series, no. 248. Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1998, x + 180 pp.

The concept of o-minimal structure provides an elegant generalization of semi-algebraic geometry and of the theory of semi-analytic and subanalytic sets developed by A. Gabrielov, H. Hironaka, and S. Łojasiewicz in the 60's and early 70's. An o-minimal structure $\mathcal{R} = (R, <, \dots)$ is a non-empty set R endowed with a dense linear order $<$ (without endpoints) and possibly some other operations and relations, satisfying the condition that every subset of R definable in $\mathcal{R}$ with parameters is a finite union of points and intervals (a, b) with $a, b \in R \cup \{+\infty, -\infty\}$. The name o-minimal stands for "order-minimal," and expresses the fact that the definable subsets of R can be defined in terms of the order relation alone without using the additional structure. An example is the ordered field $\boldsymbol{R}$ of the real numbers, in which case, by the Tarski–Seidenberg elimination of quantifiers, the definable subsets of $\boldsymbol{R}^n$ coincide with the semi-algebraic sets, and therefore for $n = 1$ are finite unions of points and intervals. In contrast, the ordered field of the rational numbers is not o-minimal since there is a formula, clearly with quantifiers, that defines the subset $\boldsymbol{Z} \subset \boldsymbol{Q}$ of the integers in this structure (J. Robinson).

It is remarkable that the simple condition on definable subsets of R in the definition of o-minimality entails strong regularity properties on definable subsets of R^n. Most notably every definable subset of R^n admits a cell decomposition analogous to the cylindric cell decomposition for semi-algebraic subsets of $\boldsymbol{R}^n$, with the difference that cells are bounded by graphs of definable continuous functions rather than semi-algebraic functions. An important consequence of the cell decomposition theorem is a uniform upper bound on the number of "definably connected components" of the sections $S_x = \{y \mid (x, y) \in S\} \subset R^q$ of a definable set $S \subset R^{p+q}$ for x ranging in the parameter space R^p. From the existence of uniform bounds it follows that every o-minimal structure is "strongly o-minimal" in the sense that every structure elementarily equivalent to it is o-minimal. When the underlying set R of $\mathcal{R}$ is the real line, the cell decomposition theorem was proved by van den Dries under the assumption of strong o-minimality. The cell decomposition theorem for arbitrary o-minimal structures is proved in a 1986 paper by J. Knight, A. Pillay and C. Steinhorn.

The interest in o-minimal structures is due on the one hand to the fact that definable sets in such a structure share many regularity properties of semi-algebraic sets, and on the other hand to the wealth of examples. One such is the ordered field of real numbers with the restrictions to $[-1, 1]^n$ $(n = 1, 2, \dots)$ of the real analytic functions on a neighborhood of $[-1, 1]^n$. In this case the definable subsets of $\boldsymbol{R}^n$ are the intersections with $\boldsymbol{R}^n$ of the subanalytic sets in the n-dimensional real projective space, and one has elimination of quantifiers after adding a division function (J. Denef and L. van den Dries, JSL LXII 1481; see also the preceding review). In 1991 A. Wilkie, using work of A. Khovanskiĭ, established the fundamental result that the ordered field of the real numbers with the real exponential function is model complete and o-minimal (JSL LXIV 910). Since then the subject has grown rapidly and new o-minimal structures have been found.

The book under review consists of ten chapters containing both original material and a selection of results by various authors, focusing on the consequences of o-minimality, rather than on the proof that various structures are o-minimal (which the author intends to postpone to a future volume). The central result is the cell decomposition theorem, upon which many subsequent results depend. This is proved first in the semi-linear case (Chapter 1), then in the semi-algebraic case following Łojasiewicz's proof (Chapter 2), and finally in general for every o-minimal structure $\mathcal{R}$ (Chapter 3). The proof in the general case involves different ideas from those in the semi-algebraic case. One first proves that a definable function $f : R \to R$ is piecewise monotone and continuous with respect to the order topology of R (monotonicity theorem). Then one needs a uniformity theorem stating that if all the vertical sections of a given definable subset of R^2 are finite, then there is a common finite bound on the cardinality of the sections. Finally one obtains the cell decomposition by induction on the dimension n of the ambient space R^n, making use of sections to reduce the dimension.

When the underlying set R of the structure $\mathcal{R}$ is the real line, each cell is connected and the cell decomposition theorem immediately implies that every definable set has a finite number of connected components. In the general case a cell does not need to be connected but it is "definably connected," namely it cannot be partitioned in two non-empty definable open sets.

Using cell decompositions one can define two geometric invariants of a definable set: the dimension and the Euler characteristic (Chapter 4). These are invariant under definable bijections, which need not be continuous.

In Chapter 5 the author studies a combinatorial property of the sections of a definable set that has found applications in model theory, probability, and more recently in the study of neural networks. A. Pillay and C. Steinhorn proved that the sections $S_x \subset R^q$ of a definable set $S \subset R^{p+q}$ do not have the "independence property"; hence there is a polynomial p such that the Boolean algebra generated by n sections of S has at most $p(n)$ atoms. C. Laskowski then gave a dual formulation of this result (obtained by exchanging the x-sections with the y-sections) stating that the sections of a definable set have the "Vapnik–Chervonenkis property," which is defined as follows. Given a family $\mathcal{C}$ of sets and an integer $n \in \boldsymbol{N}$, let $f_{\mathcal{C}}(n)$ be the least integer such that for all finite sets F of cardinality n, the number of subsets of F of the form $C \cap F$, with $C \in \mathcal{C}$, has cardinality at most $f_{\mathcal{C}}(n)$. S. Shelah, and independently V. Vapnik and A. Chervonenkis, proved that either $f_{\mathcal{C}}(n) = 2^n$ for all n or $f_{\mathcal{C}}$ is bounded by a polynomial. In the latter case one says that $\mathcal{C}$ has the VC-property.

In Chapter 6 the author considers the case when the given o-minimal structure $\mathcal{R}$ is an expansion of an ordered additive group. In this case one can select the midpoint of a bounded interval and derive from this the principle of definable choice, namely each definable relation contains a definable function with the same domain and each definable equivalence relation on a definable set has a definable set of representatives. Combining this with the monotonicity theorem one can prove the "curve selection lemma," stating that every point in the boundary of a definable set A is the limit of a definable continuous curve in A.

Chapters 7, 8, and 9 contain results that hold under the assumption that the given o-minimal structure is an expansion of an ordered field (necessarily real closed). In this case one can speak of differentiability and prove a definable version of the inverse function theorem (Chapter 7). Adapting to the o-minimal context the treatment of the semi-algebraic case by H. Delfs and M. Knebusch, Chapter 8 shows that each definable set has a triangulation (namely it is definably homeomorphic to a bounded semi-linear set) and in Chapter 9 a trivialization theorem for definable functions is given. Using Wilkie's result on the o-minimality of exponentiation, the author gives applications to the study of fewnomials, including the proof of a conjecture of R. Benedetti and J.-J. Risler establishing the finiteness of the number of embedded homeomorphism types of semi-algebraic sets of bounded "additive complexity." An embedding theorem for definable spaces is given in Chapter 10 by adapting the semi-algebraic treatment of R. Robson. Quotient spaces in the definable category are also considered there.

This is an excellent textbook on o-minimality, assuming only elementary algebra and analysis. The choice of material keeps the logic prerequisites to a minimum, not even requiring the compactness theorem for first-order logic (except in few exercises), at the expense, however, of omitting some topics, such as the model-theoretic treatment of dimension and its analogy with transcendence degree. The book is suitable for a graduate course or as a general reference and should be of interest to model-theorists and geometers. Many exercises lead to further developments of the theory.

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H. GARTH DALES and W. HUGH WOODIN. ***Super-real fields. Totally ordered fields with additional structure.*** London Mathematical Society monographs, n.s. no. 14. Clarendon Press, Oxford University Press, Oxford, New York, etc., 1996, xv + 357 pp.

The subject matter of this valuable book—in some sense a hybrid of a (long) research paper and an expository text—is the in-depth study of certain classes of "large" real-closed fields (rcf). The specific motivations for studying these will appear in due course.

It is widely known that rcf's are ubiquitous in all matters concerning totally ordered fields and, especially, the real numbers. For example, each totally ordered field has a unique real closure; this construction, however, adds "few" elements to the original field and is therefore not a source of "large" rcf's. Examples of genuinely "large" rcf's are obtained by (1) Hahn's construction of formal power series with well-ordered support, real coefficients, and exponents in a ("large") divisible Abelian group (S. Mac Lane, ***Bulletin of the American Mathematical Society***, vol. 45 (1939), pp. 888–890). (2) When $C(X)$ is the ring of real-valued continuous functions on a topological space X (which may be assumed completely regular in full generality, and even compact for many of the arguments in this book), the residue field $C(X)/M$ modulo a maximal ideal M is either $\mathbb{R}$ or a genuinely large rcf (in which case it is called a *hyper-real* field); cf. Theorems 13.4 and 13.8 in L. Gillman and M. Jerison, ***Rings of continuous functions***, Van Nostrand, 1960.

These examples can be considered "large" according to several criteria: the most obvious are cardinality and the existence of a large set of infinitesimal elements; more subtly, by the size of parameters (not always easy to compute) such as the minimum cardinality of an order-dense subset (called the *weight*), or cofinality, or the cofinality of the set of infinitesimals, or the possible *character* of the gaps in their orders (a *gap* is an "unfilled" Dedekind cut, and its character is the pair of cardinals ⟨cofinality of the left class, coinitiality of the right class⟩). These, and a few others, are the main parameters used in the present book; sometimes they are considered in the *value group* Γ_K of the Archimedean valuation on K, instead of the field K itself. Model-theoretic criteria for "largeness," e.g. saturation, do not appear explicitly.

Here follow some other examples, in the same spirit, of large rcf's that play a central role in this book: (3) the field of fractions, K_P, of any integral domain of the form $A_P = C(X)/P$, P a prime ideal of $C(X)$. When $K_P \ncong \mathbb{R}$, these are the *super-real* fields of the book's title (the rings A_P were first studied in Gillman and Jerison, Chapter 14). Special subclasses are obtained by imposing specific conditions on the prime P, thus: (a) the *prime z-ideals*, i.e., the primes containing all functions having the same set of zeros as any of its members; (b) the *valuation* primes, i.e., those for which A_P is a valuation ring (for every $a, b \in A_P$, either a divides b or b divides a in A_P); (c) the *strongly convex* primes, that is, those P for which the unique maximal ideal M_P of A_P is convex *in* K_P; (d) the *cut* primes, i.e., those strictly containing a strongly convex prime; and (e) the *exponential* primes, that is, those P such that $\exp(-1/f) \notin P$ for $f \in C(X)^+ \setminus P$ (this condition makes it possible to define an exponential function on the whole of K_P; cf. Theorem 6.20). None of these conditions implies that the order of the field K_P is of type η_1 (though some hit close), while hyper-real fields are always η_1. The stress in the present work is mainly on η_1-fields.

(4) There is the field $\widehat{\boldsymbol{R}}$ of formal power series over $\mathbb{R}$ with exponents in a certain group $\boldsymbol{G}$, the field $\boldsymbol{R}$ of such series with countable support, and the Cauchy completion $\widetilde{\boldsymbol{R}}$ of $\boldsymbol{R}$. Here $\boldsymbol{G}$ is the (divisible, Abelian) ordered group of formal power series over $\mathbb{R}$, with countable support, and exponents in Sierpiński's set $\boldsymbol{Q}$ of the lexicographically ordered $\{0,1\}$-sequences of length ω_1 having a last 1. With GCH, $\widetilde{\boldsymbol{R}}$ is a neat generalization of the real line $\mathbb{R}$ "one cardinal higher": it is the unique Cauchy-complete, η_1-rcf of cardinality $\aleph_2$ and weight $\aleph_1$.

The groundwork is laid in the first four chapters, the core of which is a detailed exposition of the subjects mentioned above. The central theme in Chapters 1 and 2 is Hahn's construction of formal power series and their properties. The account given here proceeds through a pleasant progression from totally ordered sets through totally ordered groups to totally ordered fields, by taking, at each step, the exponents in a structure of the preceding type (the field of coefficients remains fixed and is either $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).

The completion of totally ordered groups and fields is treated in Chapter 3; the authors follow ideas originating with Baer, as presented in Prieß-Crampe's book, ***Angeordnete Strukturen***, Springer-Verlag, 1983. (The *Cauchy completion* is obtained by "filling" only those gaps $\langle A, B\rangle$ such that the set $\{b-a \mid a \in A,\ b \in B\}$ is coinitial in the set of positive elements.) Here begins the study of the examples listed in (4) above. Some of their invariants (weight, cofinality, etc.) are computed.

Chapter 4 is an exposition of the theory of rings of continuous, real-valued functions. In treating this subject, the presentation unavoidably borrows much from Gillman and Jerison's classical textbook. The updating consists essentially in introducing and studying the basic properties and mutual relationship of the classes of prime ideals listed in (3)(a–e) above (only the z-ideals are studied in Gillman and Jerison). Information concerning the corresponding classes of rcf's at this initial state of knowledge is summarized in a useful table (p. 104), updated later in the text.

Questions concerning the existence of $\mathbb{R}$-algebra norms in the rings A_P and related structures are discussed in Chapter 5. These questions are at the origin of the authors' interest in super-real fields. They arise from a famous problem of Kaplansky, who in 1949 asked whether the uniform norm of the complex Banach algebra $C(\Omega,\mathbb{C})$ of continuous, complex-valued functions on a compact space Ω is unique, up to equivalence, amongst *all* norms (Kaplansky had proved uniqueness—in ZFC—among all *complete* norms). For each Ω, the existence of a non-maximal prime ideal P in $C(\Omega)$ such that A_P is normable, and the existence of a discontinuous homomorphism from $C(\Omega,\mathbb{C})$ into some Banach algebra, each turned out to be equivalent to non-uniqueness. Using the first of these reformulations, Dales and Esterle proved in 1978–79 non-uniqueness in ZFC + CH, while using the second, Solovay and Woodin showed the consistency of uniqueness with ZFC.

More generally, the authors deal with the question of the normability of the $\mathbb{R}$-algebra $K^{\#}$ of finite elements in totally ordered fields K containing $\mathbb{R}$. Under GCH this question can be answered (either positively or negatively) for all fields K, except when $\operatorname{card}(K)=\aleph_2$ and $\operatorname{card}(\Gamma_K)=\aleph_1$. For fields satisfying the last condition, normability of $K^{\#}$ is equivalent (in ZFC) to that of $\widehat{\boldsymbol{R}}^{\#}$ (Theorem 5.13). Likewise, normability of $\widetilde{\boldsymbol{R}}^{\#}$ is equivalent to that of $K^{\#}$, for all fields K with weight $\aleph_1$ (Theorem 5.15). The questions whether $\widehat{\boldsymbol{R}}$ and $\widetilde{\boldsymbol{R}}$ are normable remain open. Related problems of universality of (complex) non-unital integral domains are also examined in this chapter.

Thus, close examination of the fields $\boldsymbol{R}$, $\widehat{\boldsymbol{R}}$, and $\widetilde{\boldsymbol{R}}$ is a task of crucial importance. This is undertaken in Chapters 6 (for $\widehat{\boldsymbol{R}}$) and 9 (for $\widetilde{\boldsymbol{R}}$); the contents of Chapter 7 are also relevant to this end.

The main result in Chapter 6 is that the field $\widehat{\boldsymbol{R}}$ *is not super-real*. This is proved in ZFC, and shows that the η_1-super-real rcf's form a proper subclass of the η_1-rcf's. A by-product of the proof is that the additive group of $\widehat{\boldsymbol{R}}$ is not isomorphic to that of K_P for any exponential prime P. The argument proving these results goes as follows: (i) Any super-real field has a convex subgroup on which a function having the simplest properties of an exponential is defined. If P is an exponential prime, this function is defined on the whole of K_P (Theorem 6.20; Alling had long before proved this for the smaller class of hyper-real fields). (ii) $\widehat{\boldsymbol{R}}$ does not have an exponentiation defined on any convex subgroup.

On the other hand, $\boldsymbol{R}$ and $\widetilde{\boldsymbol{R}}$ do have an everywhere defined exponentiation (Theorem 6.24). The question whether these fields are super-real is, however, independent of ZFC: both are—indeed, they are hyper-real—with CH, while they are not with Martin's axiom + $\neg$CH. While for $\boldsymbol{R}$ this is a rather simple corollary of previously existing information, the proof that $\widetilde{\boldsymbol{R}}$ is hyper-real, carried out in Chapter 9, is a 45-page, highly technical *tour de force* involving the combinatorial construction of a z-ultrafilter on a space $X = Y \times \mathbb{N}$, Y compact, whose corresponding maximal ideal M in $C(X)$ makes $C(X)/M$ a Cauchy-complete field of weight $\aleph_1$. Then CH entails $C(X)/M \cong \widetilde{\boldsymbol{R}}$; see (4). It is also proved that $\widetilde{\boldsymbol{R}}$ is not an ultrapower, not even of the form K_P for any η_1-cut prime P.

The constructions just described are based on radically different principles. The existence of a partial exponentiation in super-real fields follows from the existence of an *operational calculus* which, to each real-valued Lipschitz function F on $[0,1] \subseteq \mathbb{R}$ associates a uniformly continuous map $F_{(P)}$ defined on the closed unit interval of K_P, taking values in $K_P^{\#}$. If P is a z-ideal, $F_{(P)}$ exists for each continuous F on $[0,1]$. Extension to functions F in several variables poses no problem.

On the other hand, establishing the stated properties of $\widetilde{\boldsymbol{R}}$ calls on forcing machinery and exploits the existence of arithmetical structures in super-real fields, structures that the authors have uncovered. We turn next to them.

It is widely known that by taking a non-principal ultrapower of $\mathbb{N}$ one gets a non-standard model of first-order arithmetic and, moreover, of analysis (i.e., the second-order axiom of induction for definable subsets, and the comprehension axiom), inside the corresponding ultrapower of $\mathbb{R}$. Starting from this simple observation, the authors embark in Chapter 8 on the detection of analogous structures inside super-real fields; they get models of weaker but, in some cases, still significant, fragments of arithmetic.

Their first step is to get an analogue of this construction in the fields K_P, P a (non-maximal) prime z-ideal. Though technical, this is a rather transparent generalization of the ultrapower case. In a second step, they succeed in transferring the construction to arbitrary primes P by use of the previous step on the largest prime z-ideal contained in P; this is a considerably subtler and more delicate affair, achieved by use of a certain "transfer principle" (Corollary 8.30).

In analogy with ultraproducts, the first of these constructions yields, for every subset S of $\mathbb{R}$, a "stretched" version $S_{(P)}$ of S inside K_P that carries along whatever (first-order) structure S may have (e.g. ring, field, model of arithmetic). This important point—a variant of Łoś's theorem for ultraproducts—is illustrated in specific cases (e.g. in pp. 165–167), and explicit computations are done in detail to convince the reader that the foregoing assertion is true (e.g. in pp. 162–163). But in this reviewer's opinion, this essentially "hand-waving" way of proceeding, by itself, tends to obscure the issue; not the least, for the audience of analysts to whom the book is meant to be accessible (cf. *Preface*), rather than for that of logicians who, after all, are on more familiar ground to prove the result implicit here. The book's length would not have unduly increased by the inclusion of an appendix stating (and even proving) the precise version of Łoś's theorem required at this point—and repeatedly employed elsewhere—to be used as a reference of last resort.

An outstanding application of this arithmetical machinery is the *gap theorem* (Theorem 8.61): all gaps of a (semi-) η_1 super-real field with character $(\aleph_0, \lambda)$, λ infinite, have the same right-class coinitiality λ, namely, the cofinality of the set K_P^0 of infinitesimals of K_P.

Since there are η_1-rcf's containing gaps of character $(\aleph_0, \lambda)$ for various λ's (an example with $\lambda = \aleph_1$ and $\aleph_2$ is constructed in Proposition 2.25), there exist such fields that are not even order-isomorphic to any super-real field.

The arithmetical structure constructed in Chapter 8 is essential to the work done in the later chapters. The road is full of pitfalls here, clearly signaled in the text. For example, while the structure $\mathbb{N}_{(P)}$ with arithmetical operations is a model of first-order arithmetic if P is a prime z-ideal (a point that, incidentally, could have been made clearer by use of the missing analogue of Łoś's theorem; cf. p. 250 bottom), it is not even multiplicatively closed for arbitrary P. Even for z-ideals, the natural family of subsets (called $\mathcal{P}(\mathbb{N})_{(P)}$) does not make $\mathbb{N}_{(P)}$ a model of analysis: it is not closed under unions or intersections. This difficulty is circumvented by introducing a smaller family ${}^*\mathcal{P}(\mathbb{N})_{(P)}$ having the required closure properties, and considering those prime ideals P for which ${}^*\mathcal{P}(\mathbb{N})_{(P)} = \mathcal{P}(\mathbb{N})_{(P)}$; this condition is baptized *weak Cauchy completeness* (wCc). If the second-order structure $\mathfrak{N}_{(P)} = \langle \mathbb{N}_{(P)}, \mathcal{P}(\mathbb{N})_{(P)}, +, \cdot \rangle$ is a model of analysis, then it is wCc and P is an exponential prime. In the other direction, if P is a cut prime, then the arithmetical structure contained in K_P is wCc. The field $\widetilde{\boldsymbol{R}}$ is not isomorphic to any super-real field K_P such that $\mathfrak{N}_{(P)}$ is a model of analysis. An example is given of a hyper-real field whose associated arithmetical structure is not wCc; it is obtained by adding a suitable generic Cohen real.

In Chapter 11 the authors endeavor to prove that there are super- (indeed, hyper-) real fields of cardinality $\aleph_2$ and weight $\aleph_1$ whose arithmetical structure $\mathfrak{N}_{(P)}$ is uniquely determined, up to isomorphism, by the pure order structure of K_P. This is achieved via the rather technical notion of $\mathfrak{N}_{(P)}$'s being *solid*, by means of an intricate coding of $\mathfrak{N}_{(P)}$ in terms of the order of K_P. The proof (in ZFC + GCH) of existence of a prime z-ideal P such that $\mathfrak{N}_{(P)}$ is a solid model of analysis demands a fully-fledged use of forcing techniques. Here the authors' policy of avoiding explicit use of advanced set-theoretical machinery finally breaks down (as they explicitly admit). Even so, these final results teach us important lessons: (iii) (GCH) There exist subfields of $\widetilde{\boldsymbol{R}}$—even some whose underlying arithmetical structure is a model of analysis—not order-isomorphic to any super-real field (Theorem 11.28). (iv) It is consistent with ZFC + GCH that there exist exponential primes (and valuation primes, and cut primes) P such that K_P has weight $\aleph_1$, cardinality $\aleph_2$, but is not isomorphic to any super-real field K_Q with Q a non-maximal, prime z-ideal.

By the end of their labor the authors have accomplished a quite impressive task (partially summarized in their final classification table, p. 338). It is a sobering thought that their achievement falls far short of their initial objectives, foremost, that of obtaining an intrinsic characterization of the super-real fields. Some of the problems that they have left open are listed at the end of the book (Chapter 12).

This book will be an obligatory point of passage for those who wish to become acquainted with many important structures arising from the study of rings of continuous functions, and indispensable reading for those wanting to pursue research in this area.

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Arnold Beckmann and Wolfram Pohlers. *Applications of cut-free infinitary derivations to generalized recursion theory*. ***Annals of pure and applied logic***, vol. 94 (1998), pp. 7–19.

In this paper the authors use a system of transfinite derivations of second-order infinitary propositional number theory to analyze Π^1_1-statements and definable well-orderings. For each Π^1_1-statement one may construct a tree of formulas such that if the tree is well founded then there is a transfinite derivation of the statement, while if the tree is not well founded, an infinite branch leads to an instance refuting the statement. This is an adjustment of a standard proof of completeness of ω-logic.

Given a definable ordering $\prec$, the statement $\mathrm{TI}(\prec)$ of transfinite induction over $\prec$ is a Π^1_1-statement. One important theorem in the paper states that if $\prec$ is a well-ordering, then the order type of $\prec$ is bounded by the ordinal height of the ω-proof of $\mathrm{TI}(\prec)$. This is an improved bound with interesting applications in proof theory.

The main application in the paper is a direct proof of the Σ^1_1-boundedness theorem, i.e., that any Σ^1_1-set of codes for well-orderings is bounded and that every Σ^1_1-well-ordering has a recursive ordinal as its order type, and the authors focus on the fact that they obtain alternative proofs of theorems in generalized recursion theory by analyzing proof trees.

The paper is well written. It is mainly self-contained and the proofs are easy to follow. The problem is that alternative proofs of the results from generalized recursion theory based on tree-analysis can be obtained by even more direct methods and that this was established already during the 1980's.

Based on ideas originating from Kechris and Woodin and further developed by Girard and Normann, Reinert A. Rinvold gave in his Master's thesis at the University of Oslo in 1988, a direct construction of a recursive well-ordering bounding a Σ^1_1-set of well-orderings and a direct construction of a recursive well-ordering bounding the ordinal height of a well founded Σ^1_1-relation on Baire space. He employed an amalgamation technique developed for bounding sets of dilators and ptykes in general.

Rinvold's construction was never published and not even distributed informally. But Kechris made a strong point of observing that applications of Π^1_2-logic outside its own field can be replaced by standard methods from descriptive set theory, and Rinvold's construction is a transfer of this observation to Π^1_1-logic $= \omega$-logic. For reasons not to be discussed here, the results of Kechris and Woodin and of Girard and Normann were published only much later, in 1991 and 1992, respectively (*A strong boundedness theorem for dilators*, ***Annals of pure and applied logic***, vol. 52 (1991), pp. 93–97, and *Embeddability of ptykes*, ***The journal of symbolic logic***, vol. 57 (1992), pp. 659–676).

The tree method for proving generalizations of the completeness theorem is very powerful and quite simple, and it has been used, also in generalized recursion theory, to obtain first proofs of interesting theorems. Normally the use of logic has later been removed from the proof. In the present paper the authors demonstrate this power of the completeness argument, but unfortunately, the use of logic had been removed in advance.

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ANDREAS WEIERMANN. *Sometimes slow growing is fast growing*. ***Annals of pure and applied logic***, vol. 90 (1997) pp. 91–99.

The slow-growing function hierarchy G_α was introduced by Girard (Π^1_2-*logic, part 1: dilators*, ***Annals of mathematical logic***, vol. 21 no. 2-3 (for 1981, pub. 1982), pp. 75–219), demonstrating that the ordinal ε_0 is not necessarily connected with the class of recursive functions, as the G_α catch up with the F_α, the functions of the fast-growing hierarchy, only at the much larger Bachmann–Howard ordinal. Indeed, the definition of the F_α and the G_α does not so much depend on the ordinals as on the structure of the fundamental sequences used. This is exemplified in this paper by Andreas Weiermann.

He chooses a particular system of fundamental sequences due to Cichon (*Termination orderings and complexity characterisations*, ***Proof theory, A selection of papers from the Leeds proof theory programme 1990***, Cambridge University Press, 1993 (©1992), pp. 171–193) and compares the G_α, defined by recursion along the Cichon-style fundamental sequences, with the F_α, defined by recursion along the standard assignment of fundamental sequences. He finally proves that for any $\alpha < \varepsilon_0$ and $x \geqq 3$ we have $G_{\omega^{\omega^{2+\alpha}+\omega}}(x) \geqq F_\alpha(x)$ and hence, the G_α, as defined here, grow as fast as the F_α.

The proofs are given in great detail, making the paper largely self contained. Very useful is an informal explanation of the mechanisms that govern the pointwise descent along the Cichon-style fundamental sequences, guiding the reader through the proof of the main theorem.

Basically, the Cichon norm of an ordinal $N(\sum_{i=1}^{n} \omega^{\alpha_i}) = \max\{n, 1 + N(\alpha_i)\}$ puts a uniform bound on the height and width of the ordinal notation. Now, Cichon's fundamental sequences approximate an ordinal α from below with a maximal ordinal $\alpha[x]$ of norm $N(\alpha) + x$, and consequently, each step in the pointwise descent along the xth branch of the fundamental sequence will increase the norm of the ordinal by x. Thus, $\delta + \omega^{\alpha+1}$ reduces to $\delta' + \omega^{\alpha}$ with the norm increasing by x; and $\delta + \omega^{\alpha+n}$ eventually reduces to $\delta' + \omega^{\alpha}$ with the norm increasing by nx. The interesting transition is the reduction of $\delta + \omega^{\alpha+\omega}$ to $\delta' + \omega^{\alpha+n}$ where n is asymptotically as large as $N(\delta)$. Therefore, all gains in the size of the ordinal notation, obtained during any previous descent, are preserved during this reduction. In general, the descent along Cichon's system of fundamental sequences translates the nested structure of ordinal notations into equivalently nested iterations, soon transcending the reach of elementary functions.

It has long been known that all recursive functions can be obtained in an elementary way by using only a single recursion along an elementary well-ordering of type ω, so it should not come as a surprise that all recursive functions can be obtained, even from a slow-growing function hierarchy, by using recursion along a suitably defined system of fundamental sequences. While the paper illustrates convincingly the dependency of the behaviors of function hierarchies on the underlying fundamental sequences, the more interesting question of how to restrict the set of fundamental sequences to obtain in some sense "standard" sequences is still left open; some hints at the end of the paper, however, point the reader to interesting future results.

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FRANCO MONTAGNA, GIULIA SIMI, and ANDREA SORBI. *Logic and probabilistic systems*. ***Archive for mathematical logic***, vol. 35 (1996), pp. 225–261.

This paper is motivated by ideas of Roberto Magari on *experimental logics* (*Su certe teorie non enumerabili*, ***Annali di matematica pura ed applicata***, ser. 4 vol. 48 (1974), pp. 119–152). The basic intention of his logics was to capture in a formal and rigorous fashion the notion of a scientific theory evolving via a process of trial and error. Briefly, Magari's picture of this process was as a sequence $A^0, A^1, A^2, \ldots$ of finite sets of sentences, the elements in A^s being thought of as the *provisional theses*, modulo some fixed formal system T, held at time s. As time passes, new provisional theses are added, in some specified order. These may subsequently be found to be inconsistent with T, in which case they, and all later introductions, are—at least temporarily—discarded and the introduction of provisional theses resumed at the earliest new thesis in the ordering not currently known to be inconsistent. In this way a *final theory* $A = \{x \mid (\exists t)(\forall s \geqq t)[x \in A^s]\}$ emerges, its members consisting, *à la* Popper, of sentences that have never been refuted by, rather than have been derived from, T.

In view of the informal theory development that this construction can be thought of as modeling, there is a very natural interest in the properties that A must, or might, possess. In particular, in the predominating case in the experimental logics' literature when T equals Peano's axioms (which we shall assume henceforth without further mention), how much of true arithmetic can A encompass?

In the present paper the authors consider developing Magari's ideas by assigning the proposed provisional theses *probabilities*, an innovation that seems very much in line with the way scientific theories actually evolve in practice.

To this end they introduce the notion of a *trial and error* (t.e.) *probabilistic function* which, *in essence*, is a 2-ary recursive function p such that for each n, $\lambda\varphi p(\varphi, n)$ is a rational-valued probability function with domain a finite set of sentences, $B_{\alpha(n)}$, such that every sentence is in all the $B_{\alpha(n)}$ eventually, and furthermore all theorems of T with proof Gödel numbers at most n are in $B_{\alpha(n)}$ and get value 1 according to $\lambda\varphi p(\varphi, n)$. Unfortunately a probability function is customarily conceived as being defined on *all* sentences $\varphi \in \mathit{Sent}$, precisely, to satisfy (i) if $\vdash \varphi$ then $p(\varphi) = 1$; (ii) if $\vdash \varphi \leftrightarrow \psi$ then $p(\varphi) = p(\psi)$; (iii) if $\vdash \varphi \rightarrow \neg\psi$ then $p(\varphi \vee \psi) = p(\varphi) + p(\psi)$. That, one supposes, explains the rather inelegant definition the authors are actually obliged to adopt.

Given such a t.e. probabilistic function, the natural focus of interest is the set of sentences that acquire limiting probability 1 and, more generally, for $q \in \mathbb{Q}$, the set of those sentences that eventually always receive probability at least q (even though the limit probability may not exist). With this in mind the authors introduce the relations R_p and S_p by

$$R_p(\varphi, q) \iff (\exists t)(\forall s \geqq t)[p(\varphi, s) \leqq q],$$
$$S_p(\varphi, q) \iff (\exists t)(\forall s \geqq t)[p(\varphi, s) \geqq q],$$

and their corresponding *p-outer* and *p-inner* probabilities

$$\overline{\pi}_p(\varphi) = \inf\{q \mid R_p(\varphi, q)\}, \qquad \underline{\pi}_p(\varphi) = \sup\{q \mid S_p(\varphi, q)\}.$$

Defined in this way, R_p and S_p are Σ_2 and satisfy a number of rather self-evident properties. This leads the authors to generalize still further and define relations $\langle R, S\rangle$ to constitute a *probabilistic system consistent with* T if they satisfy the following of these self-evident properties:

$$R(\varphi, q) \,\&\, S(\varphi, r) \Rightarrow r \leqq q, \qquad S(\varphi, 0) \,\&\, R(\varphi, 1),$$
$$R(\varphi, q) \,\&\, q \leqq r \Rightarrow R(\varphi, r), \qquad S(\varphi, q) \,\&\, q \geqq r \Rightarrow S(\varphi, r),$$
$$\begin{array}{lll} \text{if } \vdash \varphi \leftrightarrow \psi & \text{then} & R(\varphi, q) \Leftrightarrow R(\psi, q) \text{ and } S(\varphi, q) \Leftrightarrow S(\psi, q); \\ \text{if } \vdash \varphi \rightarrow \neg\psi & \text{then} & R(\varphi, q) \,\&\, R(\psi, r) \Rightarrow R(\varphi \vee \psi, q + r), \\ & & R(\varphi \vee \psi, q + r) \,\&\, S(\varphi, q) \Rightarrow R(\psi, r), \\ & & S(\varphi, q) \,\&\, S(\psi, r) \Rightarrow S(\varphi \vee \psi, q + r), \\ & & S(\varphi \vee \psi, q + r) \,\&\, R(\varphi, q) \Rightarrow S(\psi, r); \\ \text{if } T \vdash \varphi & \text{then} & S(\varphi, 1). \end{array}$$

(The authors suggest here that $R(\varphi, q)$, $S(\varphi, r)$ might be thought of as corresponding to a scientist's answering "yes" to the respective questions "is the probability of φ less than q?" "is the probability of φ greater than r?" where he or she is also allowed to give a "don't know" answer as well as a "no" answer.)

As the authors show, this constitutes a genuine generalization in that there are Σ_2 probabilistic systems (i.e., where both R and S are Σ_2) consistent with T that do not originate from a t.e. probabilistic function. If, however, R and S are *total* in the sense that for all $\varphi \in \mathit{Sent}$

$$\inf\{q \mid R(\varphi, q)\} = \sup\{q \mid S(\varphi, q)\}$$

then there always is a t.e. probabilistic function for which also

$$\lim_n p(\varphi, n) = \inf\{q \mid R(\varphi, q)\} = \sup\{q \mid S(\varphi, q)\}$$

for all $\varphi \in \mathit{Sent}$.

Within this formalization, the primary interest is now in the deductively closed (as the authors show) sets $\mathcal{F}_{R,S} = \{\varphi \mid S(\varphi, 1)\}$ and $\mathcal{F}'_{R,S} = \{\varphi \mid \sup\{q \mid S(\varphi, q)\} = 1\}$. With the frequent aid of a rather general procedure for constructing t.e. probabilistic functions, the authors prove a number of recursion-theoretic results, both positive and negative, relating the classes $\mathcal{F}_{R,S}$, $\mathcal{F}'_{R,S}$, etc. to the arithmetic hierarchy. In particular they show that for $\langle R, S\rangle$ a Σ_2 probabilistic system (or even simply originating from a t.e. probabilistic function), $\mathcal{F}_{R,S}$ is Σ_2 and can be Σ_2-complete whilst $\mathcal{F}'_{R,S}$ is Π_3 and can be Π_3-complete. On the other hand

$\mathcal{F}'_{R,S}$ cannot contain all the true Π_2 sentences, although it is shown that there are probabilistic systems originating from t.e. probabilistic functions where $\mathcal{F}'_{R,S}$ contains all true Σ_2 sentences and that also, for any particular fixed *finite* set of true Π_2 sentences, there are probabilistic systems originating from t.e. probabilistic functions with $\mathcal{F}'_{R,S}$ including all members of this finite set. These conclusions are particularly interesting when compared with results of Hájek from *Experimental logics and Π^0_3 theories*, ***The journal of symbolic logic***, vol. 42 no. 4 (for 1977, pub. 1978), pp. 515–522, where, within an alternate, non-probabilistic, formalization, it is shown that the Π_3 sets are exactly the weakly stable expressions of some experimental logic.

Whilst these results are, perhaps, largely of interest to the specialist, a later section of the paper contains a result of wider significance viz-a-viz theory evolution and the ultimate convergence to truth. Namely, for *any* Σ_n $(n \geqq 1)$ probabilistic system and any k there exists a true Π_{n+1} sentence φ such that $\neg S(\varphi, 1/2^{-k})$. With some additional assumptions (which force $n \geqq 2$), Π_{n+1} here can be improved to Π_{n-1}.

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Bart Jacobs. ***Categorical logic and type theory.*** Studies in logic and the foundations of mathematics, vol. 141. Elsevier, Amsterdam etc. 1999, xvii + 760 pp.

This is a book on logic, type theory, and proof theory. It should not be overlooked by any potential reader who focuses only on the word "categorical." Of course, it has a natural constituency among categorical logicians and theoretical computer scientists interested in various type theories. But a book such as this addresses issues of fundamental importance, not merely for categorical logicians, but more importantly also for more "traditional" type theorists and proof theorists, who are in danger of missing its message.

Since the late 1960's there has been considerable progress in understanding the algebraic semantics of logic and type theory, particularly because of the development of categorical analyses of most of the structures of interest to logicians. Although there have been other algebraic approaches to logic, none has been as far reaching in its aims and in its results as the categorical approach. From fairly modest beginnings, categorical logic has matured very nicely in the past three decades; the book under review is good evidence of that process.

Many readers of this journal will be acquainted with the connections between typed lambda calculus and Cartesian closed categories, and between intuitionist set theory and toposes, as outlined for example in the 1986 book ***Introduction to higher order categorical logic*** by J. Lambek and P. J. Scott (JSL LIV 1113). They may also be familiar with the now classic work on categorical logic by M. Makkai and G. E. Reyes (***First order categorical logic***, Springer-Verlag, 1977). Some readers may also know that such correspondences exist between other logics and type theories and appropriate categorical doctrines, for instance, Martin-Löf's dependent types, or polymorphic lambda calculus, to mention but two. But what has been most striking in the past decade and a half is the degree to which such work—and much that has been done since—has been given a more unified treatment by using the technical notion of fibrations. Jacobs develops a strikingly wide amount of material with this unifying notion, and presents the tools necessary for a deep understanding of the algebraic structure of the logics and type theories we are likely to consider.

"A logic is always a logic over a type theory." This is Jacobs's guiding principle, describing a type theory as a theory of types over which a logic lives. He might also say that a type theory is (often) the proof theory of a logic, under what is generally known as the Curry–Howard "proposition-as-types" isomorphism. So the treatment of logic in this book is intimately tied to the treatment of type theory. In both cases, the main technical device used is the notion of a fibration. The semantics of type theory may be presented in several ways, but attempts to give it an algebraic formulation in suitably general terms are bound to converge on a fibrational approach. Since this notion is so fundamental to this approach, let us spend a moment to consider what is involved.

One aspect of logic is that it deals with *variables*, of various types. Certain operations do not essentially alter the set of free variables involved (e.g. forming the meet or join of formulas), but others (such as quantification) do have an essential effect. There are also many type theories involving types with variables (such as dependent type theories or polymorphic type theories); these have a similar structure. To take variables into consideration properly, some notion of "indexing" is necessary. Consider for instance the simplest example: multisorted predicate logic. A formula $\varphi(x)$ with a free variable x of type X may be thought to be indexed by X—think of $\varphi(x)$ as an X-indexed collection of "objects" φ_x. One can imagine richer examples of this phenomenon: e.g. a type might depend on term variables, which in turn could have types that depend on term variables of previously constructed types. For example, we might have a subtype $\mathsf{N}(k)$ of the natural numbers from 1 to k, for a natural number $k \in \mathsf{N}$. In a similar vein, we might consider the type of lists (of some type) of a fixed length k, the type of integers mod k, the type of vectors in R^k, or even (given a k and a vector $\boldsymbol{v} \in \mathsf{R}^k$) the type of linear transformations $\mathsf{R}^k \to \mathsf{R}^k$ that send $\boldsymbol{v}$ to $\boldsymbol{0}$, the zero vector. In short, examples of such dependency are to be found throughout ordinary mathematics and computer science. So the "index types" can have quite a rich structure themselves.

Fibrations provide precisely the kind of structure necessary for such indexing. There are other notions, but fibrations provide serious technical advantages. Apart from matters of theoretical elegance, one of the most significant advantages is perhaps the ability to treat logics (and type theories) in a modular fashion. This is illustrated throughout the book, but nowhere as strikingly as in the last couple of chapters which deal with notions of dependent type theories: extensions of predicate logic over simple type theory (e.g. logic over dependent types or over polymorphic types) may be dealt with by treating the logic as a "module" over the appropriate type theory.

There are two fairly obvious ways to treat "indexing": thinking of sets and functions for the moment, we can regard an indexed family of sets $\{X_i \mid i \in I\}$ either as a map $X : I \to \mathcal{X}$, so that $X(i) = X_i$ and $\mathcal{X}$ is (or contains) the collection of the X_i, or as a map $X : \mathcal{X} \to I$, so that $X^{-1}(i) = X_i$ and $\mathcal{X}$ is the disjoint union of the X_i (where $X(x)$ is the index of the set to which x belongs). The first approach leads to the notion of "indexed category," and has the advantage that initially it seems more intuitive and technically simpler; its main conceptual drawback is that it places the emphasis on the individual "fibres" X_i. The second approach, which emphasizes the global entity that represents the family of sets, leads to the notion of fibration. It has the advantage that it avoids the axiom of choice in places where it really is unnecessary, and so permits the description of canonical constructions; essentially, this means that the principal notions of importance are *properties* of a structure, rather than being additional structure. Also fibrations make the modular approach mentioned above more straightforward. For example, if a logic is given by a fibration "above" a type theory, and the type theory itself involves variables (e.g. types may have term or type variables), then the type theory is itself a fibration, and the overall structure of the logic over the type theory is simply a fibration over a fibration. The ability to compose fibrations makes this approach to semantics technically more flexible and more powerful; it is the essence of this algebraic approach to type theory and proof theory, and compared to it, other approaches often seem contrived. This is particularly true where dependencies are made over dependencies, as with polymorphic and dependent types, but even with logics over simple type theory (for instance), the fibrational approach allows a unity of view that is not achieved with a more traditional treatment.

The theory of fibrations goes back to 1960–61 work of Grothendieck (*Catégories fibrées et descente*, published in 1971 by Springer-Verlag, Lecture notes in mathematics, vol. 224, pp. 145–194), but in the early 1970's, Jean Bénabou and his students realized its significance for foundational studies, particularly for the foundations of category theory (***The journal of symbolic logic***, vol. 50 (1985) pp. 10–37). Right from the start, it was realized that "in many domains of mathematics we have important notions of 'families', e.g. vector bundles, homotopies, families of analytic manifolds ... but the family is the global entity.... Why should the emphasis be made on the fibers ...?" One might well add type theories and logics to this list: if anything, the need to stack one structure on top of another is even more pronounced in logic, and so the need for this global algebraic structure to simplify and modularize the semantics is more striking. The research on the fibrational approach to type theory and logic in the past decade and a half, culminating in the present book, offers a sound justification of this viewpoint.

Jacobs gives a thorough account of the semantics of several type theories and of logics over these. His pattern of exposition is quite consistent: he starts with a concrete description of the theory or logic at hand, in a manner that would be familiar to most logicians, and then introduces abstraction as needed, always with clear motivation. For example, the reasons for using fibrations as an organizing device will be clear to the reader well before the technicalities arise.

Jacobs's experience as a computer scientist and as a logician makes him sensitive to the amount of care that is necessary in presenting logical systems. He pays considerable attention to the minutiae of syntax, without belaboring the point, seeking the *right* level of syntactic precision. The text can be read at an informal level which may help the beginner get the gist of the matter, but when careful attention to detail becomes necessary, the reader does not have to work this out from careless prose, but rather can depend on the author's care in getting the details right the first time.

A strong point of this book is its potential pedagogical value as a graduate text. It covers a very wide range of topics, touching upon virtually all the standard topics in (categorical) logic, proof theory, type theory, and model theory. Of course, to cover so wide a range, many topics are merely introduced, but references to the extensive bibliography allow a student to follow up on topics of interest. This book ought to help make more logicians aware of the value of the fibrational approach to semantics; were it to succeed in that alone, it would have served a useful purpose. Some experience with elementary category theory and logic are probably necessary prerequisites for reading this book, and so this would be best for a second semester graduate course.

Rather than systematically go through the table of contents (which the reader may do on the publisher's web site), let me describe some key topics covered by Jacobs's book. Initially, he gives an introduction to the theory of fibrations, motivated particularly by the notions of indexing and substitution. A number of concrete examples not only illustrate the notions being developed, but will later be crucial in describing the semantics of such structures as simple type theory and dependent type theory. The following chapters cover the basic semantics of simple type theory, equational logic, and first-order and higher-order predicate logic. Much of classical "categorical logic" is handled (at least briefly) in these chapters, all covered from the uniform perspective given by fibrations. For example, regular and exact categories, toposes, etc. are characterized as categories whose subobject fibrations satisfy evident logical axioms, or equivalently, which have appropriate structure and properties. He introduces the notion of functorial semantics (the insight that a model is a structure-preserving functor), and extends it to the fibrational context so he can handle the various logics and type theories being considered. In these chapters, we can see one of the virtues of the fibered approach: we can have several different logics suitable for some structure; this simply amounts to having different fibrations over the same base. In his 1985 article (op. cit.), Bénabou made the point that equality is a *structure* that an entity (set, category, ...) may or may not carry; implicit in this is the notion that such an entity may in fact carry several different notions of equality. As a simple example, consider the category $\mathcal{R}el$ of sets and relations. We may regard a relation as a "multifunction" (a function with many outputs for a single input). Such multifunctions have several reasonable notions of equality: e.g. $R = S$ if and only if $R(x) = S(x)$ for any x, or alternatively $R = S$ if and only if $R(x)$ and $S(x)$ are not disjoint. These different notions of equality "live in" different fibrations built on top of $\mathcal{R}el$.

In the chapter on higher-order predicate logic Jacobs introduces the notion of a "generic object," that will prove to be central to the semantics of polymorphic type theory. Consider the more familiar setting of higher-order logic: in order to quantify over propositions and predicates, we may use a type Prop of propositions, with an axiom $\mathsf{Prop} \vdash \mathsf{Type}$. There is a bijection between predicates $x : \sigma \vdash \varphi : \mathsf{Prop}$ with a free variable of type σ, and terms $\varphi : \sigma \to \mathsf{Prop}$. We may say that Prop is a "generic object." Given abstractly, this is the necessary extra ingredient to a first-order fibration over a Cartesian closed base in order to make it higher order. Jacobs goes on in later chapters to use this to give suitable functorial semantics for logic over dependent types and over polymorphic types.

In Chapter 9, Jacobs returns to the theory of fibrations, in order to develop the more sophisticated tools that will be necessary to handle dependent types. He starts by showing that there is a close connection between "global" structure (products, coproducts, Cartesian closed structure) and "fibred" structure (such structure over each index). At times the precise conditions for a "good" correspondence are somewhat subtle, and this section needs close attention. He develops a general theory of quantification and equality in terms of comonadic structure based on weakening and contraction (though contraction is actually induced by the rest of the structure). Again, the details are sufficiently subtle that the novice reader should study the simple examples to see how this generalizes the straightforward quantification from earlier chapters. Finally he introduces "double indexing"—e.g. Prop over Type over Kind—which corresponds to having fibrations over fibrations. To do this properly we need to develop "category theory" over a fibration. This section studies this setup, using the notions originally due to J. Bénabou from the 1970's.

This material is then used to analyze dependent type theories and logic over dependent types; one famous example of this is D. Scott's closure operators on $\mathcal{P}\omega$; this is a $\lambda\omega$-fibration (a fibration over higher-order polymorphic type theory). A richer example is given by partial equivalence relations (symmetric, transitive, not necessarily reflexive) in the effective topos; in an earlier chapter much of the structure of the effective topos is studied, and the book ends with this famous example (in categorical logic) of a structure that models (albeit in a subtle manner) full higher-order dependent type theory. One technical device worth mentioning is the notion of comprehension categories. These have a dual role: they may be interpreted as "domains of quantification," or as "models of dependent type theory," and this dual view allows a modular (or incremental) approach to the semantics (and so to the syntax, via term models) of these fully dependent type theories.

The reader of this review will have gathered by now that I like this book. It covers a vast amount of material, although it leaves even more out, so any reader familiar with the field will regret the omission of his or her favorite topics. Its bibliography and the references in the text to the bibliography do compensate to some extent, but complaints will be inevitable. As an introductory text, however, it is exemplary; as a reference it will prove invaluable. There is one omission that strikes me: the author is a theoretical computer scientist, who is experienced with the many applications this material has in that domain. Although there are many references in the text to such applications, very little actual development of these topics is made, and generally when such material is exposed, this is only in an introductory manner, more substantive material being referred to only bibliographically. A more subjective quibble: on a number of occasions Jacobs draws a distinction between "traditional" logic and the (categorical) logic treated by this book. For example, he says that traditional logic is single sorted, unlike the multisorted logic necessary for applications in computer science (and for categorical applications in general). In effect, he is alluding to issues that were raised and settled mathematically twenty to thirty years ago; that they persist is not a mathematical but rather a sociological fact, and there is no need for apology in using the more general (categorical) setting where its superiority is manifest.

There are many minor typographical errors, although very few that caused me any serious misunderstanding. Overall, the production is of a high quality (a feature that seems more due to the author than to his publisher, I think). The exercises amplify many of the topics very well, and enable the book to cover far more material than would otherwise be possible. This book will be the standard reference in its field for some time to come.

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