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ZERMELO AND THE SKOLEM PARADOX

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Dedicated to Mrs. Gertrud Zermelo on the occasion of her 95th birthday.

On October 4, 1937, Zermelo composed a small note entitled "Der Relativismus in der Mengenlehre und der sogenannte Skolemsche Satz" ("Relativism in Set Theory and the So-Called Theorem of Skolem")[1] in which he gives a refutation of "Skolem's paradox", i.e., the fact that Zermelo-Fraenkel set theory—guaranteeing the existence of uncountably many sets—has a countable model. Compared with what he wished to disprove, the argument fails. However, at a second glance, it strongly documents his view of mathematics as based on a world of objects that could only be grasped adequately by infinitary means. So the refutation might serve as a final clue to his epistemological credo.

Whereas the Skolem paradox was to raise a lot of concern in the twenties and the early thirties, it seemed to have been settled by the time Zermelo wrote his paper, namely in favour of Skolem's approach, thus also accepting the noncategoricity and incompleteness of the first-order axiom systems. So the paper might be considered a late-comer in a community of logicians and set theorists who mainly followed finitary conceptions, in particular emphasizing the role of first-order logic (cf. [8]). However, Zermelo never shared this viewpoint: In his first letter to Gödel of September 21, 1931, (cf. [1]) he had written that the Skolem paradox rested on the erroneous assumption that every mathematically definable notion should be expressible by a finite combination of signs, whereas a reasonable metamathematics would only be possible after this "finitistic prejudice" would have been overcome, "a task I have made my particular duty".

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[1]The hitherto unpublished, three page handwritten paper is contained in the Zermelo Nachlass in the main library of the Universität Freiburg. We thank the Universitätsbibliothek for allowing us to publish it here and to cite from other papers in the Nachlass. We appreciate valuable comments on an earlier draft of this paper by U. Felgner, H. Mildenberger, and G. H. Müller. Warm thanks go to Mrs. Gertrud Zermelo for kindly supporting our study of her late husband's scientific work.

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Our paper is organized as follows. In a first section we trace the discussion that originated in the discovery of the Skolem paradox. We then, in Section 2, describe Zermelo's view concerning the nature of mathematics. Some of the citations are taken from still unpublished documents of the Nachlass. Having thus provided the necessary background, in Section 3 we give an annotated translation of the 1937 manuscript. An appendix contains the original German version.

§1. The Skolem paradox. The paper "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre" of Skolem [10] contains a number of important observations which have considerably influenced the development of set theory, and also that of the foundations of mathematics.

In a way the paper is the starting point of post-Zermelo axiomatic set theory. The particular issue is the relativity of set theoretic notions, a phenomenon that has become known as "Skolem's paradox". The paradox almost immediately became the subject of discussions in foundational circles, where it caused considerable confusion.

Skolem handled the phenomenon in a precise and definitive manner. He gave a completely satisfactory explanation.

The formulation of Skolem's paradox in the original paper runs as follows:

> If the axiom system of Zermelo in its precise form is consistent, then it must be possible to introduce an infinite sequence of symbols 1, 2, 3, …, which constitute a domain B, in which all of Zermelo's axioms are valid, if these symbols are just put together in a suitable way as pairs of the form $a \, \varepsilon \, b$.[2]

Skolem went on to point out that this fact only apparently led to paradoxical observations of the sort that the axioms allow us to prove the existence of higher infinities (höhere Mächtigkeiten), but that nonetheless one could (given the consistency) interpret axiomatic set theory in the natural numbers. In modern language: if set theory is consistent, then it has a denumerable model. He explained the paradox away by observing that a set M in the model B which could be proved non-denumerable in set theory was also non-denumerable in the sense of B, i.e., that in B there was no bijection from M to the natural numbers in B. The enumeration of M as a set of elements of B (which followed from the denumerability of B) was not, and could not be, an object in B. Skolem's conclusion was that the notions of set theory were only *relative* to the domains (models) under consideration:

[2] Ist das präzisierte Zermelo'sche Axiomensystem widerspruchfrei, so muß es möglich sein, eine unendliche Reihe von Symbolen 1, 2, 3, … so einzuführen, daß diese einen Bereich B bilden, für welchen die Zermelo'schen Axiome alle gültig sind, wenn nur diese Symbole in passender Weise zu Paaren der Form $a \, \varepsilon \, b$ zusammengestellt werden.

> The axiomatic founding of set theory leads to a relativity of the notions of sets, and this is inseparably connected with every systematic axiomatization.[3]

Skolem did not view the relativity as particularly shocking or surprising, in fact, as he states at the end of the paper, he had "already communicated it to F. Bernstein in Göttingen in the winter of 1915–16", and he concludes:

> I believed that it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics that mathematicians would, for the most part, not be very much concerned with it. But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics; therefore it seemed to me that the time had come for a critique.

Skolem became a thoroughgoing relativist who drew the ultimate (and not wholly justified) conclusion of the Skolem-Löwenheim theorem vis à vis formalized mathematics. As he put it in [11]:

> One recognizes here the same thing as before in the discussion of the theorem of Löwenheim, that there is no possibility to introduce something absolutely non-denumerable, but by a pure dogma.

The reception of Skolem's paradox illustrates the delay in the absorbing of new ideas in science. Fraenkel's influential "Einleitung in die Mengenlehre" [3] is a good example to trace the influence of Skolem. The second edition of 1923 mentions Skolem's paper, which had only just become available to Fraenkel, in a footnote. The paradox is referred to as "a difficulty which has not yet been overcome".

In the subsequent "Zehn Vorlesungen über die Grundlegung der Mengenlehre" [5], Skolem's paradox gets its own section, where it is discussed as a new, alarming attack on the axiomatic foundation of set theory. Fraenkel was not convinced of the correctness of the arguments of Skolem; he built in the *caveat* "if the conclusions of Löwenheim and Skolem proceed without gaps and errors". He did not see a solution to the paradox, but was inclined to see impredicativity as a possible source of the problem. The third edition of the "Einleitung" (1928) again questions the correctness of the Skolem argument. In spite of Skolem's crystal clear exposition, Fraenkel states (p. 333):

> Neither have the books yet been closed on the antinomy, nor has agreement on its significance and possible solution yet been reached. So we will restrict ourselves to a suggestive sketch.

[3]Die axiomatische Begründung der Mengenlehre führt zu einer Relativität der Mengenbegriffe, und diese ist mit jeder konsequenten Axiomatik untrennbar verknüpft.

The remarks show that the role of logic in set theory was not quite clear to Fraenkel, no matter how much he admired Hilbert's proof theory. Apparently Skolem's arguments were beyond his expertise.

Von Neumann was quicker to grasp what was going on; his 1925 paper "Eine Axiomatisierung der Mengenlehre" [15] shows a complete understanding of the material, notwithstanding the fact that he formalized his set theory in the traditions of Hilbert's program. Von Neumann straightforwardly acknowledged the relativity phenomenon; he ended his paper with the words:

> At present we can do no more than note that we have one more reason here to entertain reservations about set theory and that for the time being no way of rehabilitating this theory is known.

Where the majority of the mathematicians followed Fraenkel's scepticism, and a few von Neumann's resignation, the first set theoretician to surpass Cantor in his own field, Ernst Zermelo, declared the Skolem paradox a hoax.

§**2. Zermelo's foundational views.** Between 1913 and 1929, due to "lengthy illness and isolation in a foreign country" [20], Zermelo did not publish any paper on set theory. However, the year 1929 marks the beginning of a new period of activity with invited talks on the foundations of mathematics and a series of set theoretical publications. Both his talks and the subsequent papers document an engaged fight against what he thought could become a real danger for mathematics, namely the constructive and finitistic conceptions embodied, for instance, in Brouwer's intuitionism and Skolem's first-order approach to set theory. His conviction that mathematics intrinsically is of an infinitary character had already stabilized nearly a decade earlier, as clearly stated on July 17th, 1921, in five "theses about the infinite in mathematics".[4] They contain the leading ideas of his later work. We give an English translation; the German original can be found in the Appendix.

> I) Each genuine mathematical proposition has an "infinitary" character, i.e., it refers to an *infinite* domain and has to be viewed as a combination of infinitely many "elementary" sentences.
>
> II) The infinite is neither physically nor psychologically given to us in the real world, it has to be comprehended and "posited" as an idea in the Platonic sense.
>
> III) Since infinitary propositions can never be derived from finitary ones, also the "axioms" of all mathematical theories have to be infinitary, and the "consistency" of such a theory can only be

[4]The typewritten page has a handwritten number "42" added above the "21" in the date (a page number?), but probably not in Zermelo's handwriting. We doubt that the theses would have been formulated as late as 1942, because they clearly anticipate his Warsaw talks [18] and the Bad Elster talk [21].

proved by exhibiting a corresponding consistent system of infinitely many elementary sentences.

IV) By its nature, traditional "Aristotelian" logic is finitary and, hence, not suited for a foundation of the mathematical sciences. Therefore, there is a necessity for an extended "infinitary" or "Platonic" logic which rests on some kind of infinitary "intuition"—as, e.g., with the question of the axiom of choice—, but which, paradoxically, is rejected the "intuitionists" on the ground of habit.

V) Any mathematical proposition has to be conceived as a combination of (infinitely many) elementary sentences, the "ground relations", via conjunction, disjunction, and negation, and each deduction of a proposition from other propositions, in particular each "proof", is nothing but a "regrouping" of the underlying elementary sentences.

In a series of talks [18] given in Warsaw in the spring of 1929, he explicated these views more systematically. Lecture 4[5] sets the central statement: "Arithmetic—as, basically, every other mathematical discipline—essentially consists of sentences that contain infinite totalities of single propositions." The talks document that Zermelo's conception of sentences and logic are essentially of an infinitary semantical nature, mirroring a firm platonic attitude:

> Only if [such] a "model" exists or at least can be conceived, does our system count as "consistent", i.e., realizable; if it can be proved that the existence of such a model is logically impossible, the system itself is "inconsistent and contradictory".[6] (Lecture 1: *What is mathematics?*)
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> "Realizability" by models is exactly the basic assumption of all mathematical theories; without it also the question for the "non-contradictoriness" of an axiom system loses its proper meaning. For the axioms themselves do not harm each other, as long as they are not applied to one and the same (given or hypothetical) model.[7] (Lecture 2: *Disjunctive systems and the principle of the excluded middle.*)

[5]The lectures are numbered according to their order in the manuscript.

[6]Nur wenn ein [solches] "Modell" existiert oder wenigstens denkbar ist, gilt unser System als "konsistent" d.h. realisierbar; ist die Existenz eines Modells als logisch unmöglich nachweisbar, so ist damit das System selber "inkonsistent" und "widerspruchsvoll".

[7]Die "Realisierbarkeit" durch Modelle ist eben die Grundvoraussetzung aller mathematischen Theorien, und ohne sie verliert auch die Frage nach der "Widerspruchsfreiheit" eines Axiomen-Systems seine eigentliche Bedeutung. Denn die Axiome selbst tun einander nichts, bevor sie nicht auf ein und dasselbe (gegebene oder hypostasierte) Modell angewendet werden.

> True mathematics is by its nature infinitistic and based on the assumption of infinite domains; it can even be called the "logic of the infinite".[8] (Lecture 3: *Finite and infinite domains.*)
>
> ... a realization [of the assumption of infinite domains, a ground assumption of the whole of mathematics] by an explicitly given and completed model is impossible, as the infinite can nowhere tangibly be exhibited in such a way. Such an assumption can only be justified by its success, by the fact that it (alone!) has made possible the creation and development of the whole of present arithmetic which essentially is a science of the infinite.[9] (Lecture 4: *How can the assumption of the infinite be justified?*)
>
> ... the existence of the infinite as a logical postulate which ought to be the base of any "proof theory", is already secured a priori and does not need any proof. It will not do at all to base the formalism again on a formalism: at some point something has really to be thought, to be posited, to be assumed. And the simplest assumption which can be made and which is sufficient for the foundation of arithmetic (just as for the whole of classical mathematics) is precisely the idea of "infinite domain"; it compellingly suggests itself to the logical-mathematical thought and actually is the base of the whole of our science as it has developed historically.[10] (Lecture 5: *Can the consistency of arithmetic be "proved"?*)

In the time following, these views placed Zermelo outside the main foundational discussions embodied by constructivism and intuitionism on one hand and formalism on the other hand (and thus, to a certain extent, also away from Hilbert (cf. [13]). In [20] he describes his situation:

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[8] Die wahre Mathematik ist [...] ihrem Wesen nach infinitistisch und auf die Annahme unendlicher Bereiche gegründet; sie kann geradezu als die "Logik des Unendlichen" bezeichnet werden.

[9] ... unmöglich ist aber auch die Realisierung durch ein explizit gegebenes und fertig vorgelegtes Modell, weil das Unendliche eben nirgends als solches sinnfällig aufgewiesen werden kann. Rechtfertigen läßt sich eine solche Annahme lediglich durch ihren Erfolg, durch die Tatsache, daß sie (und sie allein!) die Schöpfung und Entwickelung der ganzen bisherigen Arithmetik, die eben wesentlich eine Wissenschaft des Unendlichen ist, ermöglicht hat.

[10] [...] die Existenz des Unendlichen als logisches Postulat, das jeder "Beweistheorie" zugrunde liegen müßte, [ist] bereits a priori gesichert und bedarf gar keines Beweises. Überhaupt geht es nun einmal nicht an, den Formalismus wieder auf den Formalismus zu stützen; irgend einmal muß doch wirklich gedacht, muß etwas gesetzt, etwas angenommen werden. Und die einfachste Annahme, die gemacht werden kann und die zur Begründung der Arithmetik (wie auch der gesamten klassischen Mathematik) ausreicht, ist eben jene Idee der "unendlichen Bereiche", die sich dem logisch-mathematischen Denken geradezu zwangsmäßig aufdrängt, und auf die auch tatsächlich unsere gesamte Wissenschaft, so wie sie sich historisch entwickelt hat, aufgebaut ist.

> These facts [the arguments of intuitionism] caused me to come back to investigations of foundational problems. I did not decide myself for a party in this proclaimed quarrel between "intuitionism" and "formalism"—actually I take this alternative to be a logically inadmissible application of the "Tertium non datur". But I thought I could help to clarify the questions under consideration, not as a "philosopher" by proclaiming "apodictic" principles which, by adding to already existing opinions, would merely augment confusion, but as a mathematician by showing objective mathematical connections—only they can provide a safe foundation for any philosophical theory.

Motivated by the discussion about his vague notion of definiteness as given in [16], Zermelo published a more elaborated version in [17] that adds up to second-order definability. The paper may be seen as a reaction against Fraenkel who, in the second edition of his "Einführung" (1923), had given his quasi first-order version from [4]. Besides blaming Fraenkel for the "constructive" (i.e., inductive) definition with its implicit use of the notion of finite number (which, in his opinion, should not be allowed to establish set theory but should rather be an outcome of it), Zermelo states that any attempt to make the notion precise by means of logic would be as unsuccessful as in 1908:

> A generally accepted "mathematical logic" to which I could have appealed, did not exist at that time, as it doesn't today when each foundational researcher has his own system of logic (seine eigene Logistik).

It is not quite clear whether Zermelo really knew Skolem's precise treatment in [10] or whether he was really aware of its scope when he gave his second-order definition, because he does not mention Skolem, whereas later, Skolem is *the* crystallization point of his criticism. Also Skolem, in his reply [12] hints at this question ("... he seems not to know my Helsingfors talk."). In any case, through Skolem's reply Zermelo was fully informed and clearly confronted with an opinion sharply contradicting his views as laid down in the documents cited above. Of course, Zermelo did not object to axiomatizations in general—the point of his critique was just the underlying language, where, in particular, any first-order approach was doomed to fail because the richness of mathematics could not be caught in a finitary way.

His answer to the finitary challenge was twofold: In [19] he drops any restriction in the axiom of separation, and in [21] he develops a system of infinitary logic.[11]

[11] Of course, it would be inadequate to consider these papers mainly in the light of the present discussion. They both mark essential developments in logic: [19] for the first time considers members of the cumulative hierarchy as models of set theory (cf. [7] 19ff.); [21] contains the first definition of infinitary languages; they were to be developed again only in

In the first paper he states that the separating property "may be totally *arbitrary* ... and all conclusions that have been drawn by limiting oneself to a special class [of properties] do not apply to the point of view taken here" and moreover, "to each part of a set there corresponds a set which contains all elements of this part".[12] In his 1908 paper Zermelo had introduced definiteness in order to be on the safe side with respect to the paradoxes of Richard and Russell.[13] The cancellation of any restriction is possible now as there is no longer any danger of getting involved in the paradoxes: Each "Unmenge" of a model of set theory becomes a set on the next level in the cumulative hierarchy of models:

> The "ultrafinite antinomies of set theory" ... are based only on a confusion of *set theory itself*, which rests on non-categorical axioms, with the individual *models* that represent it: an "ultrafinite non- or superset" of one model becomes a valid "set" with cardinal number and ordinal type in the next higher model and itself forms the foundation stone for the construction of the new domain.[14]

It is not clear to us, whether Zermelo has really given up the limitation of separation by definiteness. In the paper the connection of unlimited separation with the notion of definiteness and with the criticism in [12] is *expressis verbis* deferred to a later discussion. However, there seems to be evidence that the concept of the cumulative hierarchy with its free formation of subsets looked quite convincing to him:

> The diametrically opposed tendencies of the thinking spirit, the idea of creative *progress* and the idea of summarizing *conclusion* find their symbolic representation and their symbolic reconciliation in the transfinite series of numbers that, founded on the notion of well-ordering, by its unlimited progression does not have a true end, but has relative stops, namely those "limit numbers" that separate the models of higher type from the ones of lower type.[15]

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the late fifties in the group around Tarski. Concerning models of set theory some care is in order as Zermelo's models are of an "external" character; cf. footnote 26.

[12]For a discussion of other ways of reading these remarks, e.g., as making precise definiteness by second-order definability, see [13].

[13]The latter one had also been discovered independently by Zermelo himself; cf. [9].

[14]Die "ultrafiniten Antinomien der Mengenlehre" [...] beruhen lediglich auf einer Verwechselung der durch ihre Axiome nicht-kategorisch bestimmten *Mengenlehre selbst* mit den einzelnen sie darstellenden *Modellen*: was in einem Modelle als "ultrafinite Un- oder Übermenge" erscheint, ist im nächst höheren bereits eine vollgültige "Menge" mit Kardinalzahl und Ordnungstypus und bildet selbst den Grundstein zum Aufbau des neuen Bereiches.

[15]Die beiden polar entgegengesetzten Tendenzen des denkenden Geistes, die Idee des schöpferischen *Fortschrittes* und die des zusammenfassenden *Abschlusses* [...] finden ihre symbolische Darstellung und ihre symbolische Versöhnung in der auf den Begriff der

Even more, he used the loss of uniqueness that went together with the cumulative series of models, as an argument against Skolem, for in [20] he comments on his results:

> I thus ended up with some kind of "set-theoretical relativism" that is, however, basically different from Skolem's "relativism"; the latter even relativizes the notions of "subset" and "cardinality": Skolem wants to restrict the formation of subsets to special classes of defining functions, whereas I, according to the true spirit of set theory, admit *free* separation and postulate the existence of subsets in whatever way they are formed. According to Skolem the *whole* of set theory should be representable in a *countable* model, and for him already, say, the problem of the power of the continuum loses its genuine meaning.

In an appendix of his report he once more lists the main points, in the fourth point expressly stating that he "avoid[s] Skolem's "relativism" by admitting *free* formations of subsets *without* limitations of definiteness".

In his second answer to the finitary challenge, [21], with a continuation in [22], he works out the idea of infinitistic formulae and infinitistic proofs as already present in the theses of 1921 and permeating the whole of his Warsaw lectures from 1929. The introduction contains his infinitary credo:

> When starting from the assumption that all mathematical notions and theorems should be representable by a *fixed finite system of signs*, already the arithmetical continuum inevitably leads to the well-known "Paradox of Richard"; seemingly settled and buried, this paradox now has found a happy resurrection as "Skolemism", the doctrine according to which every mathematical theory, including set theory, can be realized by a *countable* model. It is well-known that inconsistent premises can prove anything one wants: however, even the strangest consequences that Skolem and others have drawn from their basic assumption, for instance the relativity of the notion of subset or equicardinality, still seem to be insufficient to raise doubts about a doctrine that, for various people, already has won the power of a *dogma* that is beyond all criticism. However, a sound "metamathematics", a true "logic of the infinite", will only be possible by thoroughly *turning away* from the assumption that I have described above and which I would name the "finitary prejudice". Generally speaking, it is not the "formations of signs" that—according to the opinion of various people—form the true subject of mathematics, but the conceptual and ideal relations between the elements of infinite varieties

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Wohlordnung gegründeten transfiniten Zahlenreihe, die in ihrem schrankenlosen Fortschreiten keinen wahren Abschluß, wohl aber relative Haltepunkte besitzt, eben jene "Grenzzahlen", welche die höheren von den niederen Modelltypen scheiden.

> that are set in a conceptual way, and our notational systems are only *defective* means of our finite intelligence to at least gradually approach and master the infinite that we cannot directly and "intuitively" survey or perceive.

And he continues that he will develop a kind of mathematical logic that, *free* of the finitistic prejudice, should allow one to build up the whole of mathematics without *arbitrary prohibitions* and *limitations*.

Besides the manuscript of the published version, the Nachlass contains a manuscript with the title "Über mathematische Systeme und die Logik des Unendlichen. Vortrag gehalten auf der Mathematiker-Tagung zu Bad Elster September 1931", which has a similar contents, but the introduction of the published version that we have just quoted is missing. At the meeting also Gödel had given a talk, presenting his incompleteness results. Apparently it was these results and, in particular, the underlying way of seeing mathematics as a finitary formal system that caused Zermelo to add the general remarks. His first letter to Gödel, less than a week after the meeting, says that, when writing the final version of his talk, he *had* to refer to Gödel's talk. Another reason for writing the letter may be seen in the fact that Zermelo believed to have detected a gap in Gödel's argument[16] which he had located as to rest on "the (erroneous) assumption that each mathematically definable notion could be expressed by a "finite combination of signs" (according to a *fixed* system)". With the possibility of winning Gödel as a comrade, he continued that "in particular your proof, if interpreted in the right way [that is, as detecting insufficiencies of any finitary approach to mathematics], could ... render an essential service for truth". The second letter to Gödel from October 29, 1931 (cf. [6]) ends with a statement about proofs and provability that once more very distinctly sets off the formal approach against his own infinitary platonic conception of mathematics:

> What does one understand by a proof? In general, a proof is understood as a system of sentences that, when accepting the premises, yields the validity of the assertion as being *reasonable*. And there remains only the question of what may be "reasonable". In any case—as you are showing yourself—*not only* the propositions of some finitary scheme that, also in your case, may always be *extended*. So in this respect we are of the same opinion; however, I a priori accept a *more general* scheme that *need* not be extended. And in *this* system, really *all* propositions are decidable.

§3. Zermelo's attempted refutation of the Skolem paradox. In [20] Zermelo lists five questions he planned to work upon. Items 3 and 4 say: "On the set theoretical relativism with Skolem and me and its significance for the

[16]cf. [2] or [6].

continuum problem" and "On mathematics as a 'logic of the infinite' and the impossibility of a 'finitistic mathematics' ". What could be more convincing with respect to this program than to really refute Skolem's paradox? The idea was not new: In a letter of May 25, 1930, addressed to "a colleague",[17] Zermelo offers to give a talk in Hamburg on his results in [19]; he writes:

> ... by "relativizing" the notion of set in this way, I believe to be able to refute Skolem's "relativism" that would like to represent the whole of set theory in a *countable* model. It simply is *impossible* to give *all* sets in a constructive way ... and any theory, founded on this assumption, would by no means be a theory of sets."[18]

On October 4, 1937, under the title "Relativism in set theory and the so-called Theorem of Skolem", he wrote down what he thought would be such a refutation:

> In set theory one usually defines the "continuum" as the set P of all subsets M_1 of a "countable" set M. However, is this definition really *unambiguous*, is the notion of "all" subsets not too indefinite? Couldn't there be *different* degrees of this totality and, hence, different "models" of the continuum; even more, could this continuum, leaving intact its formal properties (according to "Skolem's Theorem") be representable by a *countable* model? In this case it should be possible to imbed such a "meager" continuum K' in a "fatter" continuum K in such a way that all formal properties carry over to K.
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> We consider a *countable* set M and a subtotality K' of the totality K of *all* subsets N (in the sense of the "fatter" continuum) with the following property:
>
> 1. If R is a subset of K' then both its union set S_R and its intersection D_R are elements of K'. In other words: The sum and the intersection of arbitrarily many (even ∞ many) sets from K' are elements of K' again; the elements of K' form a "ring of sets".
> 2. Each element m of M is an element of at least one of the sets in K'.
>
> Now, let M_m be the *intersection* of all sets N from K' which contain m. Hence, M_m is an element of K', and two such intersections M_a and M_b are either identical or disjoint. Thus the total set M

[17] We are grateful to V. Remmert for the suggestion that the recipient was probably Emil Artin.

[18] Indem ich so den Mengenbegriff " relativiere", glaube ich andererseits den Skolemschen "Relativismus", der die ganze Mengenlehre in einem *abzählbaren* Modell darstellen möchte, widerlegen zu können. Es können eben nicht alle Mengen konstruktiv gegeben sein [...] und eine auf diese Annahme gegründete Theorie wäre überhaupt keine Mengenlehre mehr.

> is partitioned into a sum of mutually disjoint parts M_m, each of which has to be finite or denumerable. Moreover, also *each* set N from K' is partitioned into such parts N_m and, by 1), each union, each sum of such N_m has to be an element of K' again.
>
> Hence, if we denote the set of all (mutually distinct) M_a by T, then each element of K' uniquely corresponds to a subset of T, and K' is equivalent[19] to the set $\mathbf{U}T$ of all subsets of T. Now the set of parts into which a *denumerable* set M can be partitioned is either *finite or denumerable* itself, that is, equivalent to M, and hence the pseudo-continuum K' is either finite itself or equivalent to $K' \sim \mathbf{U}T \sim \mathbf{U}M = K$, that is, equivalent to the original (uncountable) continuum K. Therefore it is impossible to represent the continuum in a *denumerable* model; it then would have to be *finite*. "Skolem's Theorem" thus leads to the interesting consequence that *infinite* sets can be realised in finite models—a consequence that would not be more paradoxical than many other consequences already obtained from this nice theorem. In this way also the ideal aim of intuitionism, the abolition of the infinite from mathematics, would have been brought nearer to realization together with the aim of "formalism" that, as we know, aims at the proof of consistency. For, as we know, from absurd premises one can prove *anything*. Hence also the consistency of an arbitrary system of sentences.

Apparently, the handwritten note never reached its final stage. In the present form it contains some minor inaccuracies that Zermelo would doubtless have straightened out in the process of revising it. To see where the refutation breaks down, the modern reader might visualize the argument by thinking of a countable elementary submodel V' of a standard model V of ZF, he would take M to be ω, and pick for K and K' the respective continua in V and V'. Now, translated to this situation, Zermelo's proof makes use of *arbitrary* intersections and unions of subsets of K', i.e., arbitrary in the sense of V, where for a correct argument 'arbitrary' would have to mean 'in the sense of V''. The strong closure property, however, is not available. Should we allow it, then we know that elementary equivalence cannot apply.

So altogether Zermelo's refutation amounts to a proof of the set-theoretical statement saying that, given a denumerable set M, any subset of the powerset K of M that is closed under arbitrary unions and intersections[20] and whose union is M, either is finite or of the same cardinality as K.

How was it possible that one of the sharpest minds in set theory could have missed Skolem's point? The blame cannot be laid at Skolem's doorstep, as he had made it abundantly clear what 'relativity' meant. One explanation

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[19]'equivalent' means 'gleichmächtig', 'equipollent'

[20]and under complements with respect to M

is the fact that it took some time before the insight that set theory allowed a rich variety of models became commonplace. The next step, how to exploit this insight in practice, took even longer. It was only a small group that appreciated the possibilities of the model theoretic aspects of set theory; Skolem, von Neumann, and Gödel led the way in this respect. Apparently Zermelo completely missed this development. His increasing isolation may have been the cause of this fact. Indeed, he had become more and more isolated from the new generation of foundational researchers. Moreover, at his own university he had become the victim of the deplorable political situation. Two and a half years before the 'Relativism'-note, having been denounced by a colleague in the mathematics department, he made the honourable choice when presented with the alternative of either opening his lectures with the Hitler salute or losing his professorship.[21] In this double professional and scientific isolation he probably did not discuss his refutation of Skolem, nor did he circulate the note.

A second, and perhaps more satisfactory, explanation may be found in Zermelo's personal foundational convictions. It seems unjustified to assume that Zermelo could or did not grasp the technical details of Skolem's argument;[22] after all Zermelo introduced arguments in set theory that were far ahead of the ideas of his contemporaries. We think that, rather than claiming the theorem, or its proof, to be incorrect, Zermelo considered the whole first-order approach to set theory ill conceived.

It is reported that Brouwer's reaction to Gödel's incompleteness theorem was "This does not surprise me at all, I expected this all along". The technical details hardly interested him, as he had reached the conclusion on philosophical grounds. Similarly, Zermelo may have considered Skolem's argument of little more than scholastic interest, since the philosophical point of departure was mistaken. So his 'blind spot' may have been suggested by his foundational views. The first-order axiomatization of set theory was alien to his understanding of set theory, and strongly evoked his epistomological resistance.[23] His own infinitary axiomatisations of set theory are clearly motivated by his conviction that there was one real set-theoretic universe.[24]

[21] An 'ordentliche Honorarprofessur', usually without a salary.—Mrs. Zermelo remembers that her husband, when talking to her after the summons, said that he had also been informed that the loss of the professorship would include a ban on publishing, "a situation that he really suffered from".

[22] Granted, Fraenkel's protracted incomprehension shows that a leading expert of set theory could miss Skolem's point. It should be remembered, however, that Fraenkel had already given up technical work for a more philosophical activity.

[23] By a curious coincidence, Skolem himself was a victim of the same phenomenon: In 1922 he basically proved the completeness theorem for predicate logic (cf. Gödel's remark in [14], p. 510). It seems that his finitistic conviction prevented him from recognizing the full semantic significance of his result.

[24] We thank U. Felgner for having pointed out to us a remark from [16] that may be considered an early witness for Zermelo's absolute conception of sets. In the beginning of

So closure of K' under *all* unions, which is the crucial point in the argument, may have been so natural to him that the consideration of esoteric unions in some inadequate model of set theory was out of the question. In his own words, cf. [19]: Faced with "restriction and mutilation", he had firmly decided himself for "unfolding and enrichment".

Apparently the attempted refutation marks the end of Zermelo's endeavour to overcome the Skolem paradox. As we have tried to point out, the proper reading of his argument should perhaps not result in the plausible and superficial verdict that he slipped up, but rather in the conclusion that it establishes that in his version of set theory no Skolem phenomenon could occur.[25]

Appendix.

ZERMELO 17. Juli 1921

Thesen über das Unendliche in der Mathematik

I) Jeder echte mathematische Satz hat "infinitären" Charakter, d.h. er bezieht sich auf einen *unendlichen* Bereich und ist als eine Zusammenfassung von unendlich vielen "Elementarsätzen" aufzufassen.

II) Das Unendliche ist uns in der Wirklichkeit weder physisch noch psychisch gegeben, es muß als "Idee" im Platonischen Sinne erfaßt und "gesetzt" werden.

III) Da aus finitären Sätzen niemals infinitäre abgeleitet werden können, so müssen auch die "Axiome" jeder mathematischen Theorie infinitär sein, und die "Widerspruchslosigkeit" einer solchen Theorie kann nicht anders "bewiesen" werden als durch Aufweisung eines entsprechenden widerspruchsfreien Systems von unendlich vielen Elementarsätzen.

IV) Die herkömmliche "Aristotelische" Logik ist ihrer Natur nach finitär und daher ungeeignet zur Begründung der mathematischen Wissenschaft. Es ergibt sich daraus die Notwendigkeit einer erweiterten "infinitären" oder

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the paper the element relation seems to be introduced as an abstract relation consisting of certain *ground relations* (*Grundbeziehungen*) of the form $a \in b$ where a and b belong to the underlying universe (*Bereich*) $\mathcal{B}$ of objects. However, after having proved in 10., that every set contains a subset that is not an element of it, he states as a consequence that there is no set that contains all objects of the universe $\mathcal{B}$, "*i.e., the universe $\mathcal{B}$ itself is not a set*" ("*d.h. der Bereich $\mathcal{B}$ ist selbst keine Menge*"). However, this makes sense only in case Zermelo views the element relation $\in$ as the "real" element relation.

[25]Note that this is similar to what had happened before in the case of Gödel's incompleteness results. As indicated at the end of the second section (and explicated in the literature cited there), also here Zermelo's reaction is characterized by a certain technical superficiality coming along with a firm epistemological conviction. And there might be a similar interchange: On one hand, the epistemological engagement may have prevented his technical attention, and on the other hand, the missing technical reception may have resulted in an even stronger emphasis of his own views.

"Platonischen" Logik, die auf einer Art infinitärer "Anschauung" beruht—wie z.B. in der Frage des "Auswahlaxioms"—, aber paradoxerweise gerade von den "Intuitionisten" aus Gewohnheitsgründen abgelehnt wird.

V) Jeder mathematische Satz ist aufzufassen als eine Zusammenfassung von (unendlich vielen) Elementarsätzen, den "Grundrelationen", durch Konjunktion, Disjunktion und Negation, und jede Ableitung eines Satzes aus anderen Sätzen, insbesondere jeder "Beweis" ist nichts anderes als eine "Umgruppierung" der zugrunde liegenden Elementarsätze.

Zermelo, 4. X. 37

Der Relativismus in der Mengenlehre und der sogenannte Skolem'sche Satz

Das "Kontinuum" wird in der Mengenlehre gewöhnlich definiert durch die Menge P *aller* Untermengen M_1 einer "abzählbaren" Menge M. Aber ist diese Definition auch *eindeutig*, ist der Begriff "aller" Untermengen nicht zu unbestimmt? Könnte es nicht *verschiedene* Grade dieser Allheit und damit verschiedene "Modelle" des Kontinuums geben, könnte nicht vielleicht dieses Kontinuum unbeschadet seiner formalen Eigenschaften sogar (entsprechend dem "Skolemschen Satze") durch ein *abzählbares* Modell dargestellt werden? Dann müßte es möglich sein, ein solches "mageres" Kontinuum K' in ein "fetteres" K so einzubauen, daß sich alle formalen Eigenschaften auch auf dieses übertragen lassen.

Wir betrachten eine *abzählbare* Menge M und aus der Gesamtheit K *aller* ihrer Untermengen N (im Sinne des "fetteren" Kontinuums) eine Teilgesamtheit K' von folgender Eigenschaft:

1. Ist R eine Untermenge von K', so sind sowohl die zu R gehörige Vereinigungsmenge S_R wie ihr Durchschnitt D_R wieder Elemente von K'. M.a.W. Summe und Durchschnitt beliebig vieler (auch ∞ vieler) Mengen aus K' gehören wieder zu K', die Elemente von K' bilden einen "Mengenring".
2. Jedes Element m von M ist in mindestens einer Menge aus K' als Element enthalten.

Dann sei N_m[26] der *Durchschnitt* aller Mengen N aus K', welche m enthalten, also selbst ein Element von K', und zwei solche Durchschnitte N_0 und N_1 sind entweder identisch oder elementenfremd. Dadurch wird die Gesamtmenge M zerlegt in eine Summe von elementenfremden Bestandteilen N_m, welche einzeln entweder endlich oder abzählbar sein müssen. Ferner wird *jede* Menge N aus K' gleichfalls in solche Teile N_m zerspalten und jede Vereinigung, jede Summe solcher N_m muss nach 1) wieder ein Element von K' sein.

Bezeichnet man also mit T die Menge aller (von einander verschiedenen) M_a, so entspricht jedes Element von K' ein-eindeutig einer Untermenge

[26]Here and in the next two sentences N_m, N_0, ... should be replaced by M_m, M_0, ...

von T und K' ist äquivalent der Menge $\mathbf{U}T$ aller Untermengen von T. Nun ist die Menge der Teile, in die eine *abzählbare* Menge M zerspalten werden kann, entweder *endlich oder* selbst *abzählbar*, d.h. der Menge M äquivalent, und das Pseudo-Kontinuum K' daher entweder selbst endlich oder äquivalent $K' \sim \mathbf{U}T \sim \mathbf{U}M = K$, d.h. äquivalent dem ursprünglichen (nicht abzählbaren) Kontinuum K. Es ist daher unmöglich, das Kontinuum in einem *abzählbaren* Modell darzustellen, es müsste dann *endlich* sein. Der "Skolem'sche Satz" führt also zu der interessanten Folgerung, daß sich *unendliche* Mengen in endlichen Modellen realisieren lassen—eine Folgerung, die nicht paradoxer wäre als manche andere aus diesem schönen Satze bereits gezogene Konsequenzen. Hiermit wäre denn auch das Ideal des "Intuitionismus", die Abschaffung des Unendlichen aus der Mathematik, der Verwirklichung nahe gebracht zugleich mit dem des "Formalismus", der bekanntlich den Beweis der Widerspruchsfreiheit erstrebt. Denn aus absurden Prämissen kann man bekanntlich *alles* beweisen. Also auch die Widerspruchsfreiheit eines beliebigen Satzsystems.

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