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ACCESSIBLE RECURSIVE FUNCTIONS

STANLEY S. WAINER

Abstract. The class of all recursive functions fails to possess a natural hierarchical structure, generated predicatively from "within". On the other hand, many (proof-theoretically significant) sub-recursive classes do. This paper attempts to measure the limit of predicative generation in this context, by classifying and characterizing those (predictably terminating) recursive functions which can be successively defined according to an autonomy condition of the form: allow recursions only over well-orderings which have already been "coded" at previous levels. The question is: how can a recursion code a well-ordering? The answer lies in Girard's theory of dilators, but is reworked here in a quite different and simplified framework specific to our purpose. The "accessible" recursive functions thus generated turn out to be those provably recursive in $(\Pi^1_1 - CA)_0$.

Introduction. Before one accepts a computable function as being recursive, a proof of totality is required. This will generally be an induction over the tree of sub-computations unravelled from the defining algorithm, since termination corresponds to well-foundedness along the computation-branches. If the tree is well-founded, the strength of the induction principle over its Kleene-Brouwer well-ordering thus serves as a measure of the proof-theoretical complexity of the given function.

The aim of this paper is to isolate and characterize those recursive functions which may be termed "predictably terminating" or "predicatively accessible" according to the following hierarchy principle: *one is allowed to generate a function at a new level only if it is provably recursive over a well-ordering already coded in a previous level, i.e., only if one has already constructed a method to prove its termination.*

This begs the question: what should it mean for a well-ordering to be "coded in a previous level"? Certainly it is not enough merely to require that the characteristic function of its ordering relation should have been generated at an earlier stage, since the resulting hierarchy would then collapse in the sense that all recursive functions would appear immediately once the Δ_0 relations had been produced. This is simply because every recursive

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function φ_e has a Δ_0 termination ordering $<_e$ of length ω (for example, define $(n,s) <_e (n',s')$ if and only if either (i) $n < n'$ or (ii) $n = n'$, $s > s'$ and $\varphi_e(n)$ is undefined at step s'). Since the well-foundedness of $<_e$ is just a restatement of the assumption that φ_e is total, it provides no prior information as to why that function is recursive. In order to avoid this circularity, a more delicate notion of "code" for well-orderings is needed, but one which is still finitary in that it should be determined by number theoretic functions only. The crucial idea is the one underpinning Girard's Π^1_2-Logic [7], and this paper can be viewed as a reconstruction of some of his main results concerning subrecursive hierarchies which were not proven in detail (see his concluding remarks A.4.4). However our approach is quite different and, since the concern is with only those parts of the general framework specific to subrecursive hierarchies, it can be developed in a simpler and more basic context. The slogan is: *code well-orderings by number theoretic functors whose direct limits are (isomorphic to) the well-orderings themselves*. This functorial connection is easily explained.

A well-ordering is an "intensional ordinal". If the ordinal is countable then the additional intensional component should amount to a particular choice of enumeration of its elements. Thus, by a *presentation* of a countable ordinal α we shall mean a chosen sequence of *finite subsets* of it, denoted $\alpha[n]$, $n \in \mathbb{N}$, such that

$$\forall n\ (\alpha[n] \subset \alpha[n+1]) \quad \text{and} \quad \forall \beta < \alpha\ \exists n\ (\beta \in \alpha[n]).$$

It will later be convenient to require also that for all β

$$\beta + 1 \in \alpha[n] \Longrightarrow \beta \in \alpha[n] \quad \text{and} \quad \beta \in \alpha[n] \Longrightarrow \beta + 1 \in \alpha[n+1]$$

provided $\beta + 1 < \alpha$. Notice that a presentation of α immediately induces a sub-presentation for each $\beta < \alpha$ by $\beta[n] := \alpha[n] \cap \beta$, and consequently a system of "rank functions" given by

$$G(\beta, n) := \operatorname{card} \beta[n]$$

for $\beta \leq \alpha$, so that if β belongs to $\alpha[n]$ then it is the $G(\beta,n)$-th element in ascending order. Thus $G(\gamma,n) < G(\beta,n)$ whenever $\gamma \in \beta[n]$.

This system G, called the "slow-growing hierarchy" on the given presentation, determines a functor

$$G(\alpha)\colon \mathbb{N}_0 \to \mathbb{N}$$

where $\mathbb{N}_0$ is the category $\{0 \to 1 \to 2 \to 3 \to \dots\}$ in which there is only one arrow $i_{mn}\colon m \to n$ if $m \leq n$, and where $\mathbb{N}$ is the category of natural numbers in which the morphisms between m and n are all strictly increasing maps from $\{0,1,\dots,m-1\}$ to $\{0,1,\dots,n-1\}$. The definition of $G(\alpha)$ is straightforward. On numbers we take $G(\alpha)(n) = \operatorname{card} \alpha[n]$ and on arrows we take $G(\alpha)(i_{mn})$ to be the map $p\colon G(\alpha,m) \to G(\alpha,n)$ such

that if $k < G(\alpha, m)$ and the k-th element of $\alpha[m]$ in ascending order is β, then $p(k) = G(\beta, n)$.

It is easy to check that if instead we view $G(\alpha)$ as a functor from $\mathbb{N}_0$ into the larger category of all linear orderings, with the order-preserving maps as morphisms, then $G(\alpha)$ has a direct limit which will be a well-ordered structure isomorphic to the presentation of α we started with. We shall therefore write

$$(\alpha, [\ ]) = \varinjlim G(\alpha)$$

or more loosely, when the presentation is understood

$$\alpha = \varinjlim G(\alpha).$$

$G(\alpha)$ will be taken as the canonical functorial code of the given presentation.

Note that, given two presentations $(\alpha, [\ ])$ and $(\alpha', [\ ]')$, the existence of a natural transformation from $G(\alpha)$ to $G(\alpha')$ is equivalent to the existence of an order-preserving map ν from α into α' such that for every n, ν takes $\alpha[n]$ into $\alpha'[n]'$. Thus although the notion of "natural well-ordering" or "natural presentation" remains unclear (see Feferman [6] for a discussion of this bothersome problem) there is nevertheless a "natural" partial ordering of them.

We can now begin to describe what is meant by an *accessible recursive function*. Firstly we need to develop recursion within a robust hierarchical framework, one which closely reflects provable termination on the one hand, and complexity on the other. That is, if a function is provably recursive over a well-ordering of order-type α then the chosen hierarchy should provide a complexity bound for it, at or near level α. *The so-called "fast growing hierarchy" has this property*—for details on such function classifications and their proof-theoretic relevance, see, e.g., Fairtlough and Wainer [5], Buchholz, Cichon and Weiermann [3], and Weiermann [14]. There are several different versions of the fast-growing hierarchy. The one we shall work with consists of bounding functions B_β obtained by transfinitely extending the exponential function, i.e., $B_\beta(n) = n + 2^\beta$ when β is finite.

DEFINITION. Given a presentation of α, define for each $\beta \leq \alpha$ the function $B_\beta : \mathbb{N} \to \mathbb{N}$ as follows:

$$B_\beta(n) = n + 1 \quad \text{if } \beta[n] = \emptyset \quad \text{and} \quad B_\beta(n) = B_\gamma \circ B_\gamma(n) \quad \text{if } \beta[n] \neq \emptyset$$

where γ is the maximum element of $\beta[n]$.

THEOREM. *For a suitably large class of ordinal presentations α, the function B_α naturally extends to a functor on $\mathbb{N}$. This functor is, in the sense described earlier, a canonical code for a (larger) ordinal presentation α^+. Thus $B_\alpha = G(\alpha^+)$ and hence*

$$\alpha^+ = \varinjlim B_\alpha.$$

DEFINITION. The *accessible part* of the fast-growing hierarchy is $(B_\alpha)_{\alpha<\tau}$ where $\tau = \sup \tau_i$ and the presentations τ_i are generated as follows:

$$\tau_0 = \omega \quad \text{and} \quad \tau_{i+1} = \varinjlim B_{\tau_i} = \tau_i^+.$$

The *accessible recursive functions* are those computable within B_α-bounded time or space, for any $\alpha < \tau$ (or those Kalmar-elementary in B_α's, $\alpha < \tau$).

THEOREM. *τ is a presentation of the proof-theoretic ordinal of the theory $(\Pi^1_1 - CA)_0$. The accessible recursive functions are therefore the provably recursive functions of this theory.*

The main effort below will lie in computing the operation $\alpha \mapsto \alpha^+$ and establishing the functorial identity $B_\alpha = G(\alpha^+)$. This requires the recursive definition of systems of "fast-growing" operations on ordinal presentations. It will therefore be convenient to develop a more explicitly computational theory of ordinal presentations within a uniform inductive framework. This is where "structured tree-ordinals" come into play.

§1. Structured tree ordinals. The sets $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \dots$ of finite, countable and higher-level *tree ordinals* (hereafter denoted by lower-case greek letters) are generated by the following iterated inductive definition:

$$\alpha \in \Omega_k \Leftarrow \alpha = 0 \vee \exists \beta \in \Omega_k\ (\alpha = \beta + 1) \vee \exists i < k\ (\alpha \colon \Omega_i \to \Omega_k)$$

where $\beta + 1$ denotes $\beta \cup \{\beta\}$, and if $\alpha \colon \Omega_i \to \Omega_k$ we call it a limit and often write, more suggestively, $\alpha = \sup_{\Omega_i} \alpha_\xi$, the subscript ξ denoting evaluation of the function at ξ. The *subtree partial ordering* $\prec$ on Ω_k is the transitive closure of $\beta \prec \beta + 1$ and $\forall \xi\ (\alpha_\xi \prec \alpha)$. The identity function on Ω_i will be denoted ω_i, so that $\omega_i = \sup_{\Omega_i} \xi \in \Omega_k$ whenever $i < k$.

DEFINITION 1.1. For each $\alpha \in \Omega_k$ and $\vec{\gamma} \in \Omega_{k-1} \times \Omega_{k-2} \times \cdots \times \Omega_0$ define the finite linearly-ordered set $\alpha[\vec{\gamma}]$ of $\prec$-predecessors of α by induction as follows:

$$0[\vec{\gamma}] = \emptyset; \quad \alpha + 1[\vec{\gamma}] = \alpha[\vec{\gamma}] \cup \{\alpha\}; \quad \sup_{\Omega_i} \alpha_\xi[\vec{\gamma}] = \alpha_{\gamma_i}[\vec{\gamma}].$$

DEFINITION 1.2. The subset Ω^S_k of *structured tree ordinals* at level k is defined by induction on k. If each Ω^S_i has already been defined for $i < k$, let $\prec^S \subset \Omega_k \times \Omega_k$ be the transitive closure of $\beta \prec^S \beta + 1$ and $\alpha_\xi \prec^S \alpha$ for every $\xi \in \Omega^S_i$, in the case where $\alpha \colon \Omega_i \to \Omega_k$. Then Ω^S_k consists of those $\alpha \in \Omega_k$ such that for every $\lambda \preceq^S \alpha$ with $\lambda = \sup_{\Omega_i} \lambda_\xi$, the following condition holds:

$$\forall \vec{\gamma} \in \Omega^S_{k-1} \times \Omega^S_{k-2} \times \cdots \times \Omega^S_0\ .\ \forall \xi\ (\xi \in \omega_i[\vec{\gamma}] \Longrightarrow \lambda_\xi \in \lambda[\vec{\gamma}]).$$

REMARKS 1.3. The structuredness condition above ensures that "fundamental sequences" mesh together appropriately. In particular, since $\Omega^S_o =$

$\Omega_0 = \mathbb{N}$ and since $\omega_0[x] = \{0, 1, 2, \ldots, x-1\}$ for each $x \in \Omega_0$, the condition for countable tree-ordinals $\lambda = \sup_{\Omega_0} \lambda_z$ simply amounts to

$$\forall x\ \forall z\ (z < x \Longrightarrow \lambda_z \in \lambda_x[x]).$$

Note also that if $\alpha \in \Omega_k^S$ and $\beta \prec^S \alpha$ then $\beta \in \Omega_k^S$, that $\omega_0, \omega_1, \ldots, \omega_{k-1}$ are structured at level k, and that $\Omega_i^S \subset \Omega_k^S$ whenever $i < k$.

LEMMA 1.4. *If $\alpha \in \Omega_1^S$ then*

$$\forall x\ (\alpha[x] \subset \alpha[x+1]) \quad \textit{and} \quad \forall \beta \prec \alpha\ \exists x\ (\beta \in \alpha[x]).$$

PROOF. By induction over the generation of countable tree ordinals α. The $\alpha = 0$ and $\alpha = \beta + 1$ cases are trivial. Suppose $\alpha = \sup \alpha_z$ where z ranges over $\Omega_0 = \mathbb{N}$, and assume inductively that the result holds for each α_z individually. Since α is structured, we have for each $x \in \mathbb{N}$, $\alpha_x \in \alpha[x+1]$ and hence $\alpha_x[x+1] \subset \alpha[x+1]$. Thus $\alpha[x] = \alpha_x[x] \subset \alpha_x[x+1] \subset \alpha[x+1]$. For the second part, if $\beta \prec \alpha$, then $\beta \preceq \alpha_z$ for some z. So by the induction hypothesis $\beta \in \alpha_z[x] \cup \{\alpha_z\}$ for all sufficiently large x. Therefore choosing $x > z$ we have $\alpha_z \in \alpha[x]$ by structuredness and hence $\alpha_z[x] \cup \{\alpha_z\} \subset \alpha[x]$ and hence $\beta \in \alpha[x]$. ⊣

THEOREM 1.5. *If α is a countable structured tree ordinal then $\{\, \beta : \beta \prec \alpha \,\}$ is well-ordered by $\prec$, and if $\beta + 1 \prec \alpha$ we have*

$$\forall x\ (\beta + 1 \in \alpha[x] \Longrightarrow \beta \in \alpha[x])$$

and

$$\forall x\ (\beta \in \alpha[x] \Longrightarrow \beta + 1 \in \alpha[x+1]).$$

Therefore by associating to each $\beta \preceq \alpha$ its set-theoretic ordinal

$$|\beta| = \text{lub}\{\, |\gamma| + 1 : \gamma \prec \beta \,\},$$

it is clear that α determines a presentation of the countable ordinal $|\alpha|$ given by

$$|\alpha|[n] = \{\, |\beta| : \beta \in \alpha[n] \,\}.$$

PROOF. By Lemma 1.4, if $\beta \prec \alpha$, and $\gamma \prec \alpha$ then for some large enough x, β and γ both lie in $\alpha[x]$, and so $\beta \prec \gamma$ or $\gamma \prec \beta$ or $\beta = \gamma$. Hence $\prec$ well-orders $\{\, \beta : \beta \prec \alpha \,\}$. The rest is quite straightforward. ⊣

Thus Ω_1^S provides a convenient structure over which ordinal presentations can be computed. The reason for introducing higher level tree ordinals Ω_k^S is that they will enable us to name large elements of Ω_1^S in a uniform way, by higher level versions of the fast-growing hierarchy.

DEFINITION 1.6. The φ-hierarchy at level k is the function $\varphi^{(k)}: \Omega_{k+1} \times \Omega_k \to \Omega_k$ defined by the following recursion over $\alpha \in \Omega_{k+1}$:

$$\varphi^{(k)}(0,\beta) = \beta + 1$$
$$\varphi^{(k)}(\alpha+1,\beta) = \varphi^{(k)}(\alpha, \varphi^{(k)}(\alpha,\beta))$$
$$\varphi^{(k)}(\sup_{\Omega_i} \alpha_\xi, \beta) = \begin{cases} \sup_{\Omega_i} \varphi^{(k)}(\alpha_\xi,\beta) & \text{if } i < k \\ \varphi^{(k)}(\alpha_\beta,\beta) & \text{if } i = k. \end{cases}$$

When the context is clear, the superscript (k) will be supressed. Also $\varphi(\alpha,\beta)$ will sometimes be written $\varphi_\alpha(\beta)$.

NOTE 1.7. At level $k = 0$ we have $\varphi^{(0)}_\alpha = B_\alpha$ where B_α is the fast-growing function defined in the introduction, according to the presentation determined by $\alpha \in \Omega^S_1$. This is because $\alpha[n] = \alpha_n[n]$ if α is a limit and the maximum element of $\alpha + 1[n]$ is α.

LEMMA 1.8. *For any level k, all α, $\alpha' \in \Omega_{k+1}$, all $\beta \in \Omega_k$, and all $\vec{\gamma} \in \Omega_{k-1} \times \cdots \times \Omega_0$,*

$$\alpha' \in \alpha[\beta,\vec{\gamma}] \Longrightarrow \varphi(\alpha',\beta) \in \varphi(\alpha,\beta)[\vec{\gamma}].$$

PROOF. By induction on $\alpha \in \Omega_{k+1}$. The implication holds vacuously if $\alpha = 0$ since $0[\beta,\vec{\gamma}]$ is empty. For the successor step α to $\alpha + 1$, if $\alpha' \in \alpha+1[\beta,\vec{\gamma}]$ then $\alpha' \in \alpha[\beta,\vec{\gamma}] \cup \{\alpha\}$, so by the induction hypthesis,

$$\varphi(\alpha',\beta) \in \varphi(\alpha,\beta)[\vec{\gamma}] \cup \{\varphi(\alpha,\beta)\}.$$

But $\delta \in \varphi(\alpha,\delta)[\vec{\gamma}]$ for any δ, so putting $\delta = \varphi(\alpha,\beta)$ gives

$$\varphi(\alpha',\beta) \in \varphi(\alpha,\varphi(\alpha,\beta))[\vec{\gamma}] = \varphi(\alpha+1,\beta)[\vec{\gamma}]$$

as required. Now suppose $\alpha = \sup_{\Omega_i} \alpha_\xi$. If $i < k$ then from the definitions, $\alpha[\beta,\vec{\gamma}] = \alpha_{\gamma_i}[\beta,\vec{\gamma}]$ and $\varphi(\alpha,\beta)[\vec{\gamma}] = \varphi(\alpha_{\gamma_i},\beta)[\vec{\gamma}]$ so the result follows immediately from the induction hypothesis for $\alpha_{\gamma_i} \prec \alpha$. The final case $i = k$ follows in exactly the same way, but with γ_i replaced by β. ⊣

THEOREM 1.9. *φ preserves structuredness. For any level k, if $\alpha \in \Omega^S_{k+1}$ and $\beta \in \Omega^S_k$ then $\varphi(\alpha,\beta) \in \Omega^S_k$.*

PROOF. By induction on $\alpha \in \Omega^S_{k+1}$. The zero and successor cases are immediate, and so is the limit case $\alpha = \sup_{\Omega_k} \alpha_\xi$ since if $\beta \in \Omega^S_k$ then by definition, $\alpha_\beta \in \Omega^S_{k+1}$ and hence $\varphi(\alpha,\beta) = \varphi(\alpha_\beta,\beta) \in \Omega^S_k$. Suppose then, that $\alpha = \sup_{\Omega_i} \alpha_\xi$ where $i < k$. Then for every $\xi \in \Omega^S_i$ we have $\alpha_\xi \in \Omega^S_{k+1}$ and so $\varphi(\alpha_\xi,\beta) \in \Omega^S_k$ by the induction hypothesis. It remains only to check the structuredness condition (1.2) for $\lambda = \varphi(\alpha,\beta) = \sup_{\Omega_i} \varphi(\alpha_\xi,\beta)$. Assume $\vec{\gamma} \in \Omega^S_{k-1} \times \cdots \times \Omega^S_0$ and $\xi \in \omega_i[\vec{\gamma}]$. Then by the structuredness of α we have $\alpha_\xi \in \alpha[\beta,\vec{\gamma}]$ because $\xi \in \omega_i[\vec{\gamma}]$ implies $\xi \in \omega_i[\beta,\vec{\gamma}]$ when $i < k$. Therefore by 1.8 above, $\varphi(\alpha_\xi,\beta) \in \varphi(\alpha,\beta)[\vec{\gamma}]$. So $\varphi(\alpha,\beta) \in \Omega^S_k$. ⊣

COROLLARY 1.10. *Define, for each positive integer* k,

$$\tau_k = \varphi^{(1)}(\varphi^{(2)}(\ldots\varphi^{(k)}(\omega_k, \omega_{k-1})\ldots, \omega_1), \omega_0)$$

and set $\tau_0 = \omega_0$. *Thus* $\tau_1 = \varphi^{(1)}(\omega_1, \omega_0)$, $\tau_2 = \varphi^{(1)}(\varphi^{(2)}(\omega_2, \omega_1), \omega_0)$, *etc. Then each* $\tau_k \in \Omega_1^S$. *Furthermore, since* $\omega_{k-1} \in \varphi^{(k)}(\omega_k, \omega_{k-1})[\omega_{k-2}, \ldots, \omega_0, k]$ *we have* $\tau_{k-1} \in \tau_k[k]$ *by repeated application of* 1.8. *Therefore* $\tau = \sup \tau_k$ *is also structured.*

Our notion of structuredness is closely related to the work of Schmidt [10] on "step-down" relations and "built-up" systems of fundamental sequences. See also Kadota [8] for an earlier alternative treatment of τ.

§2. The collapsing properties of G. For the time being we set structuredness to one side and review some of the "arithmetical" properties of the slow-growing G-function developed in Wainer [12]. These will be fundamental to what follows later.

Recall that $G(\alpha, n)$ measures the size of $\alpha[n]$. Since we are now working with tree ordinals α and the presentations determined by them according to Definition 1.1 and Theorem 1.5, it is clear that G can be defined by recursion over $\alpha \in \Omega_1$ as follows:

$$G(0, n) = 0, \quad G(\alpha + 1, n) = G(\alpha, n) + 1, \quad G(\sup \alpha_z, n) = G(\alpha_n, n).$$

Notice that the parameter n does not change in this recursion, so what we are actually defining is a function $G_n\colon \Omega_1 \to \Omega_0$ for each fixed n where $G_n(\alpha) := G(\alpha, n)$. We need to lift this to higher levels.

DEFINITION 2.1. Fix $n \in \mathbb{N}$ and, by induction on k, define $G_n\colon \Omega_{k+1} \to \Omega_k$ and $L_n\colon \Omega_k \to \Omega_{k+1}$ as follows:

$$\begin{aligned} G_n(0) &= 0 & L_n(0) &= 0 \\ G_n(\alpha + 1) &= G_n(\alpha) + 1 & L_n(\beta + 1) &= L_n(\beta) + 1 \\ G_n(\sup_{\Omega_0} \alpha_z) &= G_n(\alpha_n) \\ G_n(\sup_{\Omega_{i+1}} \alpha_\xi) &= \sup_{\Omega_i} G_n(\alpha_{L_n\zeta}) & L_n(\sup_{\Omega_i} \beta_\zeta) &= \sup_{\Omega_{i+1}} L_n(\beta_{G_n\xi}). \end{aligned}$$

LEMMA 2.2. *For all* $\beta \in \Omega_k$, $G_n \circ L_n(\beta) = \beta$. *Hence for every positive integer* k *we have* $G_n(\omega_k) = \omega_{k-1}$.

PROOF. By induction on $\beta \in \Omega_k$. The zero and successor cases are immediate, and if $\beta = \sup_{\Omega_i} \beta_\zeta$ then, assuming we have already proved $G_n \circ L_n(\zeta) = \zeta$ for every $\zeta \in \Omega_i$ $(i < k)$, we can simply unravel Definition 2.1 to obtain

$$G_n \circ L_n(\beta) = G_n(\sup_{\Omega_{i+1}} L_n(\beta_{G_n\xi})) = \sup_{\Omega_i} G_n \circ L_n(\beta_{G_n L_n \zeta}) = \sup_{\Omega_i} \beta_\zeta = \beta.$$

Hence $G_n \circ L_n =$ identity and since $\omega_k = \sup_{\Omega_k} \xi$ we have, again by Definition 2.1,

$$G_n(\omega_k) = \sup_{\Omega_{k-1}} G_n(L_n\zeta) = \sup_{\Omega_{k-1}} \zeta = \omega_{k-1}.$$

Note that this Lemma holds for every fixed n. ⊣

DEFINITION 2.3. For each fixed n define the subset $\Omega_k^G(n)$ of Ω_k as follows by induction on k. Set $\Omega_0^G(n) = \Omega_0$ and assume $\Omega_i^G(n)$ defined for each $i < k$. Take $\prec_n^G \subset \Omega_k \times \Omega_k$ to be the transitive closure of $\beta \prec_n^G \beta + 1$ and $\alpha_\xi \prec_n^G \alpha$ for every $\xi \in \Omega_i^G(n)$, if $\alpha\colon \Omega_i \to \Omega_k$. Then $\Omega_k^G(n)$ consists of those $\alpha \in \Omega_k$ such that for every $\lambda = \sup_{\Omega_i} \lambda_\xi \preceq_n^G \alpha$ the following condition holds:

$$\forall \xi \in \Omega_i^G(n)\ (G_n(\lambda_\xi) = G_n(\lambda_{L_n G_n \xi})).$$

Call this the "G-condition". Note that for every n, $\Omega_0^G(n) = \Omega_0$ and $\Omega_1^G(n) = \Omega_1$ since $\xi = L_n G_n \xi$ if $\xi \in \Omega_0$.

LEMMA 2.4. *For each fixed $n \in \mathbb{N}$,*

(i) *if $\lambda = \sup_{\Omega_{i+1}} \lambda_\xi \in \Omega_k^G(n)$ then for every $\xi \in \Omega_{i+1}^G(n)$,*

$$G_n(\lambda_\xi) = G_n(\lambda)_{G_n(\xi)}.$$

(ii) $\omega_0, \omega_1, \ldots, \omega_{k-1} \in \Omega_k^G(n)$.

(iii) $L_n\colon \Omega_k \to \Omega_{k+1}^G(n)$.

PROOF.

(i) If $\lambda = \sup_{\Omega_{i+1}} \lambda_\xi \in \Omega_k^G(n)$ then by Definition 2.1,

$$G_n(\lambda) = \sup_{\Omega_i} G_n(\lambda_{L_n\zeta})$$

so if $\xi \in \Omega_{i+1}^G(n)$ we can put $\zeta = G_n(\xi)$ to obtain

$$G_n(\lambda)_{G_n(\xi)} = G_n(\lambda_{L_n G_n \xi}) = G_n(\lambda_\xi)$$

by the G-condition (2.3).

(ii) First note $\omega_0 \in \Omega_1 = \Omega_1^G(n) \subset \Omega_k^G$ if $k > 0$. If $k > i+1$ and $\lambda \prec_n^G \omega_{i+1}$ then $\lambda \in \Omega_{i+1}^G$ so it satisfies the G-condition. If $\lambda = \omega_{i+1}$ then it is just the identity function on Ω_{i+1} and so the G-condition amounts to $G_n(\xi) = G_n(L_n G_n \xi)$, which holds by Lemma 2.2. Hence $\omega_{i+1} \in \Omega_k^G(n)$.

(iii) We show $L_n(\beta) \in \Omega_{k+1}^G(n)$ for every $\beta \in \Omega_k$, by induction on β. The zero and successor cases are immediate. If $\beta = \sup_{\Omega_i} \beta_\zeta$ then $L_n(\beta) = \sup_{\Omega_{i+1}} L_n(\beta_{G_n\xi})$ and $L_n(\beta_{G_n\xi}) \in \Omega_{k+1}^G(n)$ by the induction hypothesis. For $L_n(\beta) \in \Omega_{k+1}^G(n)$, it remains to check the G-condition:

$$G_n(L_n(\beta_{G_n\xi})) = G_n(L_n(\beta_{G_n L_n G_n \xi})).$$

But this holds because $G_n \circ L_n$ is the identity. ⊣

THEOREM 2.5. *For each fixed $n \in \mathbb{N}$ and every k, if $\alpha \in \Omega^G_{k+2}(n)$ and $\beta \in \Omega^G_{k+1}(n)$ then*

(i) $\varphi^{(k+1)}(\alpha, \beta) \in \Omega^G_{k+1}(n)$

(ii) $G_n(\varphi^{(k+1)}(\alpha, \beta)) = \varphi^{(k)}(G_n(\alpha), G_n(\beta))$.

PROOF. By induction on $\alpha \in \Omega^G_{k+2}(n)$. The zero and successor cases are straightforward, and so is the case where $\alpha = \sup_{\Omega_{k+1}} \alpha_\xi$ because then we have:

(i) $\varphi^{(k+1)}(\alpha, \beta) = \varphi^{(k+1)}(\alpha_\beta, \beta) \in \Omega^G_{k+1}(n)$ by the induction hypothesis, and

(ii) $G_n(\alpha_\beta) = G_n(\alpha)_{G_n(\beta)}$ by Lemma 2.4 (i),

so by the induction hypothesis and the definition of the φ-functions,

$$G_n(\varphi^{(k+1)}(\alpha_\beta, \beta)) = \varphi^{(k)}(G_n(\alpha)_{G_n(\beta)}, G_n(\beta)) = \varphi^{(k)}(G_n(\alpha), G_n(\beta)).$$

Now suppose $\alpha = \sup_{\Omega_i} \alpha_\xi$ with $i \leq k$. Then

(i) $\varphi^{(k+1)}(\alpha, \beta) = \sup_{\Omega_i} \varphi^{(k+1)}(\alpha_\xi, \beta)$ and for each $\xi \in \Omega^G_i(n)$,

$$\varphi^{(k+1)}(\alpha_\xi, \beta) \in \Omega^G_{k+1}(n)$$

by the induction hypothesis. Furthermore, $\alpha_{L_n G_n \xi} \in \Omega^G_{k+2}(n)$ by Lemma 2.4 (iii), and $G_n(\alpha_\xi) = G_n(\alpha_{L_n G_n \xi})$. Therefore by part (ii) of the induction hypothesis $G_n(\varphi^{(k+1)}(\alpha_\xi, \beta))$ and $G_n(\varphi^{(k+1)}(\alpha_{L_n G_n \xi}, \beta))$ are identical. So $\varphi^{(k+1)}(\alpha, \beta) \in \Omega^G_{k+1}(n)$.

(ii) If $i = 0$ then $G_n(\alpha) = G_n(\alpha_n)$ so by the induction hypothesis,

$$\begin{aligned} G_n(\varphi^{(k+1)}(\alpha, \beta)) &= G_n(\varphi^{(k+1)}(\alpha_n, \beta)) = \varphi^{(k)}(G_n(\alpha_n), G_n(\beta)) \\ &= \varphi^{(k)}(G_n(\alpha), G_n(\beta)). \end{aligned}$$

If $i > 0$ then for every $\zeta \in \Omega_{i-1}$ we have $L_n \zeta \in \Omega^G_i(n)$ by Lemma 2.4, so $\alpha_{L_n \zeta} \in \Omega^G_{k+2}(n)$ and $G_n(\alpha_{L_n \zeta}) = G_n(\alpha)_\zeta$. Therefore, using the induction hypothesis once more,

$$\begin{aligned} G_n(\varphi^{(k+1)}(\alpha, \beta)) &= \sup_{\Omega_{i-1}} G_n(\varphi^{(k+1)}(\alpha_{L_n \zeta}, \beta)) \\ &= \sup_{\Omega_{i-1}} \varphi^{(k)}(G_n(\alpha_{L_n \zeta}), G_n(\beta)) \\ &= \sup_{\Omega_{i-1}} \varphi^{(k)}(G_n(\alpha)_\zeta, G_n(\beta)) \\ &= \varphi^{(k)}(G_n(\alpha), G_n(\beta)). \end{aligned}$$

This completes the proof. ⊣

COROLLARY 2.6. *Recalling the definition*:

$$\tau_k = \varphi^{(1)}(\varphi^{(2)}(\ldots \varphi^{(k)}(\omega_k, \omega_{k-1}) \ldots, \omega_1), \omega_0)$$

and the fact that $\varphi^{(0)} = B$, *we have for each* $k > 0$

$$G(\tau_k, n) = B_{\tau_{k-1}}(n) \quad \textit{for all } n \in \mathbb{N}.$$

Our next task is to extend this to a functorial identity. The following simple lemma plays a crucial role.

LEMMA 2.7. *For all* n *and* k, *if* $\alpha \in \Omega_k^G(n)$ *and* $\gamma_i \in \Omega_i^G(n)$ *for* $1 \leq i \leq k-1$, *then* G_n *bijectively collapses*

$$\alpha[\gamma_{k-1}, \ldots, \gamma_1, n]$$

onto

$$G_n(\alpha)[G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)].$$

PROOF. By an easy induction on α, noting that $G_n(\alpha+1) = G_n(\alpha) + 1$ and $G_n(\alpha_\gamma) = G_n(\alpha)_{G_n(\gamma)}$ for $\gamma \in \Omega_i^G(n)$. ⊣

§3. The functors G, B and φ. Henceforth we shall restrict attention to those tree ordinals which simultaneously are structured and possess the G-collapsing properties of section 2.

DEFINITION 3.1.

$$\mathcal{O}_k := \Omega_k^S \cap \bigcap_{n \in \mathbb{N}} \Omega_k^G(n).$$

Thus $\mathcal{O}_0 = \mathbb{N}$, $\mathcal{O}_1 = \Omega_1^S$ and if $\alpha = \sup_{\Omega_i} \alpha_\xi \in \mathcal{O}_k$ then $\alpha_\xi \in \mathcal{O}_k$ whenever $\xi \in \mathcal{O}_i$.

DEFINITION 3.2. Set

$$\boldsymbol{\mathcal{O}}^{<k} = \mathcal{O}_{k-1} \times \mathcal{O}_{k-2} \times \cdots \times \mathcal{O}_0$$

for each $k > 0$. Make $\boldsymbol{\mathcal{O}}^{<k}$ into a category by choosing as morphisms

$$\sigma\colon (\gamma_{k-1}, \ldots, \gamma_0) \to (\gamma'_{k-1}, \ldots, \gamma'_0),$$

all $\prec$-preserving maps from $\omega_{k-1}[\gamma_{k-1}, \ldots, \gamma_0]$ into $\omega_{k-1}[\gamma'_{k-1}, \ldots, \gamma'_0]$. Note that $\omega_{k-1}[\gamma_{k-1}, \ldots, \gamma_0]$ is the same as $\gamma_{k-1}[\gamma_{k-2}, \ldots, \gamma_0]$.

DEFINITION 3.3. The *functor* $G\colon \boldsymbol{\mathcal{O}}^{<k+1} \to \boldsymbol{\mathcal{O}}^{<k}$ is given by:
(i) $G(\gamma_k, \ldots, \gamma_1, n) = (G_n(\gamma_k), \ldots, (G_n(\gamma_1))$
(ii) If $\sigma\colon (\gamma_k, \ldots, \gamma_1, n) \to (\gamma'_k, \ldots, \gamma'_1, m)$ then

$$G(\sigma) = G_m \circ \sigma \circ G_n^{-1}\colon (G_n(\gamma_k), \ldots, G_n(\gamma_1)) \to (G_m(\gamma'_k), \ldots, G_m(\gamma'_1))$$

where G_n^{-1} denotes the inverse of the bijection G_n from $\omega_k[\gamma_k, \ldots, \gamma_1, n]$ on to $\omega_{k-1}[G_n(\gamma_k), \ldots, G_n(\gamma_1)]$ as in Lemma 2.7.

NOTE 3.4. Each $\alpha \in \mathcal{O}_1$ can be made into a functor $\boldsymbol{\alpha}$ from

$$\mathbb{N}_0 = \{0 \to 1 \to 2 \to 3 \to \dots\}$$

into $\boldsymbol{\mathcal{O}}^{<2}$ by defining $\boldsymbol{\alpha}(n) = (\alpha, n)$ and $\boldsymbol{\alpha}(i_{nm})$ to be the identity embedding of $\alpha[n]$ into $\alpha[m]$. Then $G \circ \boldsymbol{\alpha}$ is exactly the functor $G(\alpha)\colon \mathbb{N}_0 \to \mathbb{N}$ defined in the introduction.

Before defining φ_α as a functor we need a "normal form lemma".

LEMMA 3.5. *For all* $\alpha \in \Omega_{k+1}$, $\beta \in \Omega_k$ *and* $\vec{\gamma} \in \Omega_{k-1} \times \cdots \times \Omega_0$: *if* $\delta \in \varphi^{(k)}(\alpha, \beta)[\vec{\gamma}]$ *then either* $\delta \in \beta[\vec{\gamma}] \cup \{\beta\}$ *or else* δ *is expressible uniquely in the "normal form"*

$$\delta = \varphi(\alpha_r, \varphi(\alpha_{r-1} \dots \varphi(\alpha_1, \varphi(\alpha_0, \beta)) \dots)).$$

where $\alpha_0 \in \alpha[\beta, \vec{\gamma}]$ *and* $\alpha_{i+1} \in \alpha_i[\varphi(\alpha_i, \dots \varphi(\alpha_0, \beta) \dots), \vec{\gamma}]$.

Furthermore, if α, β *and* $\vec{\gamma}$ *are structured then for each* $i < r$,

$$\alpha_{i+1} \in \alpha[\varphi(\alpha_i, \dots \varphi(\alpha_0, \beta) \dots), \vec{\gamma}].$$

PROOF. By induction on α. If $\alpha = 0$ then $\varphi(\alpha, \beta)[\vec{\gamma}] = \beta[\vec{\gamma}] \cup \{\beta\}$. If α is a limit then $\varphi(\alpha, \beta)[\vec{\gamma}] = \varphi(\alpha_\xi, \beta)[\vec{\gamma}]$ where $\xi = \beta$ or γ_{k-1} or $\dots$ or γ_0, so the result follows immediately. For the successor step α to $\alpha + 1$ note that $\varphi(\alpha + 1, \beta)[\vec{\gamma}] = \varphi(\alpha, \beta_1)[\vec{\gamma}]$ where $\beta_1 = \varphi(\alpha, \beta)$. Therefore if $\delta \in \varphi(\alpha + 1, \beta)[\vec{\gamma}]$ then by the induction hypothesis for α, either $\delta \in \beta_1[\vec{\gamma}]$, in which case the unique normal form is as stated, or $\beta_1 \preceq \delta$ in which case the normal form is as stated but with β replaced by $\beta_1 = \varphi(\alpha, \beta)$.

If, furthermore, α, β and $\vec{\gamma}$ are structured, then by induction on $i = 0, 1, 2, \dots, r - 1$ we show

$$\alpha_{i+1} \in \alpha[\varphi(\alpha_i, \dots \varphi(\alpha_0, \beta) \dots), \vec{\gamma}].$$

Firstly, we have $\alpha[\beta, \vec{\gamma}] \subset \alpha[\varphi(\alpha_0, \beta), \vec{\gamma}]$ by a simple induction on α. The only non-trivial case is where $\alpha\colon \Omega_k \to \Omega_{k+1}$, so $\alpha[\beta, \vec{\gamma}] = \alpha_\beta[\beta, \vec{\gamma}] \subset \alpha_\beta[\varphi(\alpha_0, \beta), \vec{\gamma}]$. But $\beta \in \varphi(\alpha_0, \beta)[\vec{\gamma}] = \omega_k[\varphi(\alpha_0, \beta), \vec{\gamma}]$ so by structuredness $\alpha_\beta \in \alpha[\varphi(\alpha_0, \beta), \vec{\gamma}]$ and hence $\alpha_\beta[\varphi(\alpha_0, \beta), \vec{\gamma}] \subset \alpha[\varphi(\alpha_0, \beta), \vec{\gamma}]$. Thus for the base-case $i = 0$, $\alpha_0 \in \alpha[\beta, \vec{\gamma}] \subset \alpha[\varphi(\alpha_0, \beta), \vec{\gamma}]$ and hence $\alpha_1 \in \alpha_0[\varphi(\alpha_0, \beta), \vec{\gamma}] \subset \alpha[\varphi(\alpha_0, \beta), \vec{\gamma}]$. And for the induction step i to $i + 1$, replacing β by $\varphi(\alpha_i, \dots, \varphi(\alpha_0, \beta) \dots)$ and α_0 by α_{i+1} gives

$$\alpha_{i+2} \in \alpha_{i+1}[\varphi(\alpha_{i+1}, \dots \varphi(\alpha_0, \beta) \dots), \vec{\gamma}] \subset \alpha[\varphi(\alpha_{i+1}, \dots, \varphi(\alpha_0, \beta) \dots), \vec{\gamma}]$$

as required. This completes the proof. ⊣

NOTE 3.6. Conversely, if δ has the normal form of 3.4, then

$$\delta \in \varphi^{(k)}(\alpha, \beta)[\vec{\gamma}]$$

by repeated application of Lemma 1.8.

DEFINITION 3.7. The *functor* $\varphi_\alpha^{(k)}\colon \boldsymbol{\mathcal{O}}^{<k+1} \to \boldsymbol{\mathcal{O}}^{<k+1}$ is defined as follows:

First, assume $\alpha \in \mathcal{O}_{k+1}$ has already been made into a functor $\boldsymbol{\alpha}\colon \boldsymbol{\mathcal{O}}^{<k+1} \to \boldsymbol{\mathcal{O}}^{<k+2}$ such that $\boldsymbol{\alpha}(\gamma_k, \ldots, \gamma_0) = (\alpha, \gamma_k, \ldots, \gamma_0)$ and $\boldsymbol{\alpha}(i_{\gamma\gamma'}) = i_{\boldsymbol{\alpha}(\gamma)\boldsymbol{\alpha}(\gamma')}$ where $i_{\gamma\gamma'}$ denotes the identity embedding of $\omega_k[\gamma_k, \ldots \gamma_0] = \gamma_k[\gamma_{k-1}, \ldots, \gamma_0]$ as a subset of $\gamma'_k[\gamma'_{k-1}, \ldots, \gamma'_0]$ when it exists. Note that this amounts to a monotonicity condition: if σ is a subfunction of σ' then $\boldsymbol{\alpha}(\sigma)$ will be a subfunction of $\boldsymbol{\alpha}(\sigma')$. Girard called such functors "flowers". We can now define $\varphi_\alpha^{(k)}$ as a functor on $\boldsymbol{\mathcal{O}}^{<k+1}$; the superscript (k) will be omitted.

(i) $\varphi_\alpha(\gamma_k, \gamma_{k-1}, \ldots, \gamma_0) = (\varphi_\alpha(\gamma_k), \gamma_{k-1}, \ldots, \gamma_0)$

(ii) If $\sigma\colon (\gamma_k, \ldots, \gamma_0) \to (\gamma'_k, \ldots, \gamma'_0)$ then

$$\varphi_\alpha(\sigma)\colon \varphi_\alpha(\gamma_k)[\gamma_{k-1}, \ldots, \gamma_0] \to \varphi_\alpha(\gamma'_k)[\gamma'_{k-1}, \ldots, \gamma'_0]$$

is the map $\delta \mapsto \delta'$ built up inductively on $\delta \in \varphi_\alpha(\gamma_k)[\vec{\gamma}]$ according to its normal form as in Lemma 3.5:

- if $\delta \in \gamma_k[\gamma_{k-1}, \ldots, \gamma_0]$ then $\delta' = \sigma(\delta)$
- if $\delta = \gamma_k$ then $\delta' = \gamma'_k$
- if $\delta = \varphi_{\alpha_r} \circ \varphi_{\alpha_{r-1}} \circ \cdots \circ \varphi_{\alpha_1} \circ \varphi_{\alpha_0}(\gamma_k)$ then set

$$\delta' = \varphi_{\alpha'_r} \circ \varphi_{\alpha'_{r-1}} \circ \cdots \circ \varphi_{\alpha'_1} \circ \varphi_{\alpha'_0}(\gamma'_k)$$

where for each $i = 0, 1, 2, \ldots, r$, $\alpha'_i = \boldsymbol{\alpha}(\sigma_i)(\alpha_i)$ with σ_i the previously determined subfunction taking

$$\xi \in \varphi_{\alpha_{i-1}} \circ \cdots \circ \gamma_{\alpha_0}(\gamma_k)[\vec{\gamma}]$$

to

$$\xi' \in \varphi_{\alpha'_{i-1}} \circ \cdots \circ \varphi_{\alpha'_0}(\gamma'_k)[\vec{\gamma}'].$$

Notice that σ_i is a subfunction of σ_{i+1} and so $\boldsymbol{\alpha}(\sigma_i)$ is a subfunction of $\boldsymbol{\alpha}(\sigma_{i+1})$. This means that α_{i+1} occurs below α_i in the domain of $\boldsymbol{\alpha}(\sigma_{i+1})$, and hence α'_{i+1} occurs below α'_i in $\alpha[\varphi_{\alpha'_i} \circ \cdots \circ \varphi_{\alpha'_0}(\gamma'_k), \vec{\gamma}']$. Thus

$$\alpha'_{i+1} \in \alpha'_i[\varphi_{\alpha'_i} \circ \cdots \circ \varphi_{\alpha'_0}(\gamma'_k), \vec{\gamma}']$$

for each i. So $\delta' \in \varphi_\alpha(\gamma'_k)[\vec{\gamma}']$ as required, by Note 3.6. This completes the definition.

NOTE 3.8. A careful reading of Definition 3.7 should convince the reader that the maps $\varphi_\alpha^{(k)}(\sigma)$ do in fact constitute a functor, that is:

$$\varphi_\alpha(\mathrm{id}_\gamma) = \mathrm{id}_{\varphi_\alpha(\gamma)} \quad \text{and} \quad \varphi_\alpha(\sigma \circ \sigma') = \varphi_\alpha(\sigma) \circ \varphi_\alpha(\sigma').$$

This depends, of course, on the assumed functoriality of $\boldsymbol{\alpha}$. Furthermore, φ_α also satisfies the "flower" property, in the sense that if $\gamma_k = \gamma'_k$ and σ is the identity function from $\gamma_k[\gamma_{k-1}, \ldots, \gamma_0]$ into $\gamma_k[\gamma'_{k-1}, \ldots, \gamma'_0]$ then $\varphi_\alpha(\sigma)$ is the identity function from $\varphi_\alpha(\gamma_k)[\vec{\gamma}']$ into $\varphi_\alpha(\gamma_k)[\vec{\gamma}']$. Again, this depends on the assumption that $\boldsymbol{\alpha}$ is a "flower".

THEOREM 3.9. *Fix $k > 0$ and suppose $\alpha \in \mathcal{O}_{k+1}$ satisfies the assumptions of Definition* 3.7. *Suppose also that there is a $\beta \in \mathcal{O}_k$ such that $G_n(\alpha) = \beta$ for every n, and β determines a functor $\boldsymbol{\beta}: \boldsymbol{\mathcal{O}}^{<k} \to \boldsymbol{\mathcal{O}}^{k+1}$ satisfying $G \circ \boldsymbol{\alpha} = \boldsymbol{\beta} \circ G$. Then*

$$G \circ \varphi_\alpha^{(k)} = \varphi_\beta^{(k-1)} \circ G.$$

PROOF. Firstly, if $(\gamma_k, \ldots, \gamma_1, n) \in \boldsymbol{\mathcal{O}}^{<k+1}$ then by Theorem 2.5 and the definition of the functor G in 3.3,

$$\begin{aligned} G \circ \varphi_\alpha^{(k)}(\gamma_k, \ldots, \gamma_1, n) &= G(\varphi_\alpha^{(k)}(\gamma_k), \gamma_{k-1}, \ldots, \gamma_1, n) \\ &= (G_n\varphi_\alpha^{(k)}(\gamma_k), G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)) \\ &= (\varphi_\beta^{(k-1)}(G_n\gamma_k), G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)) \\ &= \varphi_\beta^{(k-1)}(G_n(\gamma_k), G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)) \\ &= \varphi_\beta^{(k-1)} \circ G(\gamma_k, \ldots, \gamma_1, n). \end{aligned}$$

Secondly, if $\sigma: (\gamma_k, \ldots, \gamma_1, n) \to (\gamma'_k, \ldots, \gamma'_1, m)$ is in $\boldsymbol{\mathcal{O}}^{<k+1}$ then using the notation of Definition 3.7 and again the definition of the functor G,

$$G \circ \varphi_\alpha^{(k)}(\sigma): \varphi_\beta^{(k-1)} \circ G(\gamma_k, \ldots, \gamma_1, n) \to \varphi_\beta^{(k-1)} \circ G(\gamma'_k, \ldots, \gamma'_1, m)$$

is the map sending $G_n(\delta)$ to $G_m(\delta')$ whenever $\delta \mapsto \delta'$ under $\varphi_\alpha^{(k)}(\sigma)$. Therefore in order to prove

$$G \circ \varphi_\alpha^{(k)}(\sigma) = \varphi_\beta^{(k-1)} \circ G(\sigma)$$

we have to check that for every $\delta \in \varphi_\alpha(\gamma_k)[\gamma_{k-1}, \ldots, \gamma_1, n]$,

$$G_m(\delta') = \varphi_\beta^{(k-1)}(G(\sigma))(G_n(\delta)).$$

Recall throughout that by Lemma 2.7, G_n always collapses

$$\gamma_k[\gamma_{k-1}, \ldots, \gamma_1, n]$$

bijectively onto

$$G_n(\gamma)[G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)].$$

Now according to the definition of $\varphi_\alpha^{(k)}(\sigma)$ in 3.7, there are three cases to consider:

(i) if $\delta \in \gamma_k[\gamma_{k-1}, \ldots, \gamma_1, n]$ then $\delta' = \sigma(\delta)$ and so

$$G_m(\delta') = G_m \circ \sigma(\delta) = G(\sigma)(G_n(\delta)) = \varphi_\beta^{(k-1)}(G(\sigma))(G_n(\delta))$$

because in this case we have

$$G_n(\delta) \in G_n(\gamma_k)[G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)].$$

(ii) if $\delta = \gamma_k$ then $\delta' = \gamma'_k$ and so $G_n(\delta) = G_n(\gamma_k)$ and in this case

$$G_m(\delta') = G_m(\gamma'_k) = \varphi_\beta^{(k-1)}(G(\sigma))(G_n(\delta)).$$

(iii) if $\delta = \varphi^{(k)}_{\alpha_r} \circ \cdots \circ \varphi^{(k)}_{\alpha_0}(\gamma_k)$ is in $\varphi^{(k)}_{\alpha}(\gamma_k)[\gamma_{k-1}, \ldots, \gamma_1, n]$ where each α_{i+1} occurs below α_i in $\alpha[\varphi^{(k)}_{\alpha_i} \circ \cdots \circ \varphi^{(k)}_{\alpha_0}(\gamma_k), \vec{\gamma}]$, then

$$G_n(\delta) = \varphi^{(k-1)}_{\beta_r} \circ \cdots \circ \varphi^{(k-1)}_{\beta_0}(G_n(\gamma_k))$$

with $\beta_i = G_n(\alpha_i)$ for each $i \leq r$. The collapsing property (2.7) of G_n ensures that β_{i+1} occurs below β_i in

$$\beta[\varphi^{(k-1)}_{\beta_i} \circ \cdots \circ \varphi^{(k-1)}_{\beta_0}(G_n\gamma_k), G_n\vec{\gamma}]$$

and that

$$G_n(\delta) \in \varphi^{(k-1)}_{\beta}(G_n\gamma_k)[G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)].$$

Furthermore, every element of $\varphi^{(k-1)}_{\beta}(G_n\gamma_k)[G_n(\gamma_{k-1}), \ldots, G_n(\gamma_1)]$ occurs as such a $G_n(\delta)$. In this case, we have $\delta' = \varphi^{(k)}_{\alpha'_r} \circ \cdots \circ \varphi^{(k)}_{\alpha'_0}(\gamma'_k)$ where $\alpha'_i = \boldsymbol{\alpha}(\sigma_i)(\alpha_i)$, and σ_i is the previously generated subfunction taking

$$\xi \in \varphi^{(k)}_{\alpha_{i-1}} \circ \cdots \circ \varphi^{(k)}_{\alpha_0}(\gamma_k)[\vec{\gamma}]$$

to

$$\xi' \in \varphi^{(k)}_{\alpha'_{i-1}} \circ \cdots \circ \varphi^{(k)}_{\alpha'_0}(\gamma'_k)[\vec{\gamma}'].$$

Therefore $G_m(\delta') = \varphi^{(k-1)}_{\beta'_r} \circ \cdots \circ \varphi^{(k-1)}_{\beta'_0}(G_m\gamma'_k)$ where $\beta'_i = G_m(\alpha'_i)$ for each $i \leq r$. But since $G \circ \boldsymbol{\alpha} = \boldsymbol{\beta} \circ G$, it follows that

$$\beta'_i = G \circ \boldsymbol{\alpha}(\sigma_i)(\beta_i) = \boldsymbol{\beta} \circ G(\sigma_i)(\beta_i)$$

and $G(\sigma_i)$ is the subfunction taking

$$G_n(\xi) \in \varphi^{(k-1}_{\beta_{i-1}} \circ \cdots \circ \varphi^{(k-1)}_{\beta_0}(G_n\gamma_k)[G_n\vec{\gamma}]$$

to

$$G_m(\xi') \in \varphi^{(k-1)}_{\beta'_{i-1}} \circ \cdots \circ \varphi^{(k-1)}_{\beta'_0}(G_m\gamma'_k)[G_m\vec{\gamma}'].$$

All of this means that $G_m(\delta') = \varphi^{(k-1)}_{\beta}(G(\sigma))(G_n(\delta))$ according to the definition of $\varphi^{(k-1)}_{\beta}(G(\sigma))$. ⊣

THEOREM 3.10. *Again recall the definition*

$$\tau_k = \varphi^{(1)}(\varphi^{(2)}(\ldots\varphi^{(k)}(\omega_k, \omega_{k-1})\ldots, \omega_1), \omega_0).$$

As before, write $B = \varphi^{(0)}$ *and, assuming* $\alpha \in \mathcal{O}_1$ *has been made into a functor* $\boldsymbol{\alpha}: \boldsymbol{\mathcal{O}}^{<1} \to \boldsymbol{\mathcal{O}}^{<2}$, *write* $G(\alpha) = G \circ \boldsymbol{\alpha}$. *Then we have the functorial identity*:

$$G(\tau_k) = B_{\tau_{k-1}}.$$

Hence for each $k > 0$,

$$\varinjlim B_{\tau_{k-1}} = \tau_k.$$

PROOF. Fix $k > 0$ and for each $i = 1, \ldots, k$ set

$$\alpha_i = \varphi^{(i)}(\varphi^{(i+1)}(\ldots \varphi^{(k)}(\omega_k, \omega_{k-1})\ldots), \omega_i), \omega_{i-1}).$$

Then from Sections 1 and 2, we have $\alpha_i \in \mathcal{O}_i$. Make ω_i into a functor $\boldsymbol{\omega}_i : \boldsymbol{\mathcal{O}}^{<i+1} \to \boldsymbol{\mathcal{O}}^{<i+2}$ by setting

(i) $\omega_i(\gamma_i, \ldots, \gamma_0) = (\omega_i, \gamma_i, \ldots, \gamma_0)$
(ii) $\omega_i(\sigma) = \sigma$ whenever $\sigma : (\gamma_i, \ldots, \gamma_0) \to (\gamma'_i, \ldots, \gamma'_0)$.

This makes good sense because $\omega_i[\gamma_i, \ldots, \gamma_0]$ is the same set of tree-ordinals as $\gamma_i[\gamma_{i-1}, \ldots, \gamma_0]$. Since $\boldsymbol{\omega}_i$ is just the identity functor it automatically has the "flower" property. Therefore starting with $\boldsymbol{\omega}_k$ and repeatedly applying Definition 3.7 and Note 3.8, we can make each α_i into a functor $\boldsymbol{\alpha}_i : \boldsymbol{\mathcal{O}}^{<i} \to \boldsymbol{\mathcal{O}}^{<i+1}$, again with the flower property, by setting

$$\boldsymbol{\alpha}_k = \boldsymbol{\varphi}^{(k)}_{\omega_k} \circ \boldsymbol{\omega}_{k-1}$$

and for each $i = k-1, \ldots, 1$ in turn,

$$\boldsymbol{\alpha}_i = \boldsymbol{\varphi}^{(i)}_{\alpha_{i+1}} \circ \boldsymbol{\omega}_{i-1}.$$

Exactly the same thing can be done with

$$\beta_i = \varphi^{(i-1)}(\varphi^{(i)}(\ldots \varphi^{(k-1)}(\omega_{k-1}, \omega_{k-1})\ldots, \omega_{i-1}), \omega_{i-2})$$

for $1 < i \leq k$, and we claim that for each such i,

$$G \circ \boldsymbol{\alpha}_i = \boldsymbol{\beta}_i \circ G.$$

The proof is by downward induction on $i = k, k-1, \ldots, 2$. If $i = k$ then since $G_n(\omega_k) = \omega_{k-1}$ for every n, and $G \circ \boldsymbol{\omega}_k = \boldsymbol{\omega}_{k-1} \circ G$, we can apply Theorem 3.9 to get

$$G \circ \boldsymbol{\alpha}_k = \boldsymbol{\varphi}^{(k-1)}_{\omega_{k-1}} \circ G \circ \boldsymbol{\omega}_{k-1} = \boldsymbol{\varphi}^{(k-1)}_{\omega_{k-1}} \circ \boldsymbol{\omega}_{k-2} \circ G = \boldsymbol{\beta}_k \circ G.$$

The induction step from $i+1$ to i is similar. First note that by Theorem 2.5, $G_n(\alpha_{i+1}) = \beta_{i+1}$ for every n, and $G \circ \boldsymbol{\alpha}_{i+1} = \boldsymbol{\beta}_{i+1} \circ G$ by the induction hypothesis. So Theorem 3.9 applies again, giving

$$G \circ \boldsymbol{\alpha}_i = \boldsymbol{\varphi}^{(i-1)}_{\beta_{i+1}} \circ G \circ \boldsymbol{\omega}_{i-1} = \boldsymbol{\varphi}^{(i-1)}_{\beta_{i+1}} \circ \boldsymbol{\omega}_{i-2} \circ G = \boldsymbol{\beta}_i \circ G.$$

This proves the claim, and the theorem follows from $G \circ \boldsymbol{\alpha}_2 = \boldsymbol{\beta}_2 \circ G$ by one more application of 3.9:

$$G(\tau_k) = G \circ \boldsymbol{\alpha}_1 = \boldsymbol{\varphi}^{(0)}_{\beta_2} \circ G \circ \boldsymbol{\omega}_0 = \boldsymbol{B}_{\tau_{k-1}}$$

since $\alpha_1 = \tau_k$, $\beta_2 = \tau_{k-1}$, $\varphi^{(0)} = B$ and $G \circ \boldsymbol{\omega}_0 = \mathrm{id}_{\mathbb{N}}$. ⊣

REMARK 3.11. Girard's dilators are functors on the category of (set-theoretic) ordinals which commute with direct limits and with pull-backs. Commutation with direct limits provides number-theoretic representation systems for countable ordinals named by the dilator, and commutation with pull-backs ensures uniqueness of representation. Although our context is

very different, the φ-functors above are dilator-like, since Theorem 3.9 essentially expresses preservation of "limits" under G, and Lemma 3.5, gives uniqueness of representation with respect to φ.

A SIMPLE EXAMPLE 3.12. B_{ω_0} is the following functor on $\mathcal{O}^{<1}$:

(i) $B_{\omega_0}(n) = n + 2^n$

(ii) $B_{\omega_0}(\sigma : n \to m)$ is the map taking

$$n + 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r} \mapsto n + 2^{\sigma(n_0)} + 2^{\sigma(n_1)} + \cdots + 2^{\sigma(n_r)}.$$

Thus

$$\varinjlim B_{\omega_0} = \varphi^{(1)}_{\omega_1}(\omega_0) = \varphi^{(1)}_{\omega_0}(\omega_0) = \omega_0 + 2^{\omega_0}.$$

So B_{ω_0} constructs a presentation of the ordinal $\omega.2$.

§4. Accessible recursive functions. The *accessible part* of the fast-growing hierarchy is generated from ω_0 by iteration of the principle: *given* α, *form* B_α *and then*

$$\alpha^+ = \varinjlim B_\alpha.$$

Note that the equation $\alpha^+ = \operatorname{Lim}_{\longrightarrow} B_\alpha$ is really only an isomorphism of presentations. However, this is enough to ensure that the B-functions are uniquely determined. So by Theorem 3.10 we may take $\omega_0^+ = \tau_1$, $\omega_0^{++} = \tau_2$, $\omega^{+++} = \tau_3$, etc. Thus:

THEOREM 4.1. *The accessible recursive functions are exactly those which are Kalmar elementary in the functions* B_α, $\alpha \prec \tau$, *where* $\tau = \sup \tau_i$. *Similarly they are exactly those which are elementary in the functions* G_α, $\alpha \prec \tau$. τ *is the first point in this scale at which the elementary closures of* $\{ B_\alpha : \alpha \prec \tau \}$ *and* $\{ G_\alpha : \alpha \prec \tau \}$ *are equal.*

THEOREM 4.2. *The accessible recursive functions are exactly those which are provably recursive in the theory* $(\Pi^1_1 - CA)_0$.

Theorem 4.2 clearly needs some explanation. At this point, however, we can lean heavily on the work of Buchholz in chapter IV of Buchholz, Feferman, Pohlers and Sieg [4], and in Buchholz [2], where he provides detailed ordinal analyses for theories of iterated induction definitions. (See also Arai [1]). We shall only be concerned with the finitely iterated levels ID_k since their union $ID_{<\omega}$ turns out to be as far as our notion of accessibility reaches, and it proves the same arithmetic formulas as $(\Pi^1_1 - CA)_0$. Buchholz's analysis proceeds by embedding the theory ID_k into an infinitary system, based on his so-called Ω_i-rules, and then computing tree-ordinals which "majorize" normal derivations. These bounds lie initially in Ω_{k+1}, but by successively "collapsing" them according to a certain hierarchy of functions D, not unlike the fast-growing φ's above, they can be brought down to the countable level Ω_1. At this stage one has bounds on the amount of transfinite induction (for

arithmetic formulas) available in ID_k, and hence on the functions which can be proved recursive within it. The optimality of these bounds is also treated in detail in the references mentioned above, but see section 5 below. There is no reason to repeat the fine technicalities presented so carefully in those papers. The essential point is that the analysis may be carried through in the same way using the φ-functions instead of D. This is fairly easy to see. The D-hierarchy consists of systems of "collapsing" functions $D^{(k)}\colon \Omega_{k+1} \to \Omega_k$ defined as follows (for the time being we omit the superscript k):

$$\begin{aligned} D(0) &= 0 \\ D(\alpha+1) &= D(\alpha)+1 \\ D(\lambda) &= \begin{cases} \sup D(\lambda_\xi) & \text{if } \lambda\colon \Omega_i \to \Omega_{k+1},\ i<k \\ D(\lambda_{D(\lambda_0)}) & \text{if } \lambda\colon \Omega_k \to \Omega_{k+1}. \end{cases} \end{aligned}$$

Now suppose we generalize this slightly by allowing an arbitrary $\beta \in \Omega_k$ as the starting point instead of the fixed 0. Then with β as a parameter we obtain a binary version of D:

$$\begin{aligned} D(0,\beta) &= \beta \\ D(\alpha+1,\beta) &= D(\alpha,\beta)+1 \\ D(\lambda,\beta) &= \begin{cases} \sup D(\lambda_\xi,\beta) & \text{if } \lambda\colon \Omega_i \to \Omega_{k+1},\ i<k \\ D(\lambda_{D(\lambda_0,\beta)},\beta) & \text{if } \lambda\colon \Omega_k \to \Omega_{k+1}. \end{cases} \end{aligned}$$

DEFINITION 4.3. Addition and exponentiation of tree-ordinals in Ω_{k+1} are given by:

$$\begin{aligned} \alpha+0 &= \alpha, \\ \alpha+(\gamma+1) &= (\alpha+\gamma)+1, \\ \alpha+\lambda &= \sup(\alpha+\lambda_\xi) \\ 2^\circ &= 1, \\ 2^{\alpha+1} &= 2^\alpha+2^\alpha, \\ 2^\lambda &= \sup 2^{\lambda_\xi}. \end{aligned}$$

DEFINITION 4.4. $\psi^{(k)}(\alpha,\beta) := D^{(k)}(2^\alpha,\beta)$.

LEMMA 4.5. *For $\alpha,\gamma \in \Omega_{k+1}$ and $\beta \in \Omega_k$ we have*:

(i) $D(\alpha+\gamma,\beta) = D(\gamma, D(\alpha,\beta))$

(ii) $$\begin{aligned} \psi(0,\beta) &= \beta+1 \\ \psi(\alpha+1,\beta) &= \psi(\alpha,\psi(\alpha,\beta)) \\ \psi(\alpha,\beta) &= \begin{cases} \sup \psi(\alpha_\xi,\beta) & \textit{if } \alpha\colon \Omega_i \to \Omega_{k+1},\ i<k \\ \psi(\alpha_{\psi(\alpha_0,\beta)},\beta) & \textit{if } \alpha\colon \Omega_k \to \Omega_{k+1}. \end{cases} \end{aligned}$$

PROOF. (i) is by a straightforward induction on γ and (ii) then follows from the definition of 2^α. ⊣

Note the striking similarity between ψ and φ. The first three clauses in their definitions are identical; only the "big limit" clause differs, with ψ having the somewhat stronger diagonalization. However φ "catches up" at the immediate successor of $\alpha\colon \Omega_k \to \Omega_{k+1}$ because then

$$\varphi(\alpha+1,\beta) = \varphi(\alpha,\varphi(\alpha,\beta)) = \varphi(\alpha_{\varphi(\alpha,\beta)},\varphi(\alpha,\beta)).$$

Alternatively if, in the generation of $\alpha \in \Omega_{k+1}$, every big limit $\lambda\colon \Omega_k \to \Omega_{k+1}$ is successively transformed into $\lambda'\colon \Omega_k \to \Omega_{k+1}$, where for each $\beta \in \Omega_k$, $\lambda'_\beta = (\lambda_{\psi(\lambda_0,\beta)})'$ then for all α, β,

$$\varphi(\alpha',\beta) = \psi(\alpha,\beta).$$

Thus the differences between D and φ are inessential as far as ordinal analysis is concerned.

Buchholz's results provide a tree-ordinal presentation of the proof-theoretic ordinal of ID_k in the form:

$$D^{(1)}D^{(2)}\dots D^{(k)}\varepsilon_{\omega_k+1}.$$

(Actually, he chooses to collapse down to the countable level all in one go, whereas the above representation does it one step at a time.) The tree-ordinal $\varepsilon_{\omega_k+1} \in \Omega_{k+1}$ is an iterated exponential form representing the next epsilon number above the k-th regular cardinal $|\omega_k|$. There are many ways in which such a form can be computed. One way, using the φ-functions, is to set

$$\varepsilon_{\omega_k+1} := \varphi^{(k+1)}(\varphi^{(k+2)}(\omega_0,\omega_{k+1}),\omega_k)$$

because the right-hand expression is equal to

$$\begin{aligned}\varphi^{(k+1)}(\omega_{k+1}+2^{\omega_0},\omega_k) &= \sup_{n\in\mathbb{N}} \varphi^{(k+1)}(\omega_{k+1}+2^n,\omega_k)\\ &= \sup_{n\in\mathbb{N}} \varphi^{(k+1)}_{\omega_{k+1}} \circ \varphi^{(k+1)}_{\omega_{k+1}} \circ \dots \circ \varphi^{(k+1)}_{\omega_{k+1}}(\omega_k)\end{aligned}$$

where for each n, there are 2^{2^n}-many iterates of $\varphi^{(k+1)}_{\omega_{k+1}}$ and each application $\varphi^{(k+1)}_{\omega_{k+1}}(\beta)$ gives $\beta + 2^\beta$.

Finally then, if we substitute the above φ-form for ε_{ω_k+1}, and replace each application of $D^{(i)}$ by $\varphi^{(i)}$ with the fixed second parameter ω_{i-1}, then the above Buchholz form of the tree-ordinal for ID_k becomes:

$$\varphi^{(1)}(\varphi^{(2)}(\dots\varphi^{(k+1)}(\varphi^{(k+2)}(\omega_0,\omega_{k+1}),\omega_k)\dots,\omega_1),\omega_0)$$

which, because of the diagonalization at big limits, is the same as

$$\tau_{k+2} = \varphi^{(1)}(\varphi^{(2)}(\dots\varphi^{(k+1)}(\varphi^{(k+2)}(\omega_{k+2},\omega_{k+1}),\omega_k)\dots,\omega_1),\omega_0).$$

THEOREM 4.6. *By defining the sequence* (τ_k) *recursively*:

$$\tau_0 = \omega_0 \quad \textit{and} \quad \tau_{k+1} = \varinjlim B_{\tau_k}$$

we have constructed tree-ordinal presentations τ_{k+2} *of the proof-theoretic ordinals of the theories* ID_k. *Thus* $\tau = \sup \tau_k$ *gives a presentation of the proof-theoretic ordinal of* $(\Pi^1_1 - CA)_0$.

§5. Provable recursiveness of B_α, $\alpha < \tau$. Since the systems of φ functions, and the number-theoretic functions B_α indexed by them, are all defined by (admittedly somewhat abstract) recursion equations, they are, at least in an intuitive sense, recursive. Furthermore, since it takes k iterated inductive definitions to produce Ω_k and hence name all the B_α's in fact for $\alpha < \tau_{k+2}$, it is intuitively reasonable to believe that these functions are provably recursive in ID_k, and hence that the B_α's for $\alpha < \tau$, are all provably recursive in $ID_{<\omega}$ and thus in $\Pi^1_1 - CA_0$. Here we indicate briefly (at the referee's suggestion) why this is really the case. In fact the axioms for $ID_{<\omega}$ are essentially designed for this purpose.

Rathjen [9] shows how the use of uncountable "cardinals" ω_k in naming large proof-theoretic ordinals can be replaced by their (countable) admissible analogues ω_k^{CK}. Therefore it must be possible to replace the uncountable sets of tree ordinals Ω_k used above, by corresponding sets of Kleene-style ordinal notations which we shall call W_k. Then the fast-growing systems of tree-ordinal functions φ can be replaced by recursive functions operating on the sets of numbers W_k. In $ID_{<\omega}$ we can therefore construct notations for the τ_k's and prove by transfinite induction that the functions B_α, for $\alpha < \tau$, are indeed recursive.

For each formula $A(a)$ with one distinguished free variable, let $F_k(A, a)$ denote the "operator form" which is the disjunction:

$$a = 0 \vee \exists b\ (a = \langle 0, b\rangle \wedge A(b)) \bigvee_{i<k} \exists e\ (a = \langle i+1, e\rangle \wedge \forall x\ (W_i(x) \to \exists y\ (\{e\}(x) = y \wedge A(y)))).$$

Then the theories ID_k are built up as follows: ID_0 is Peano Arithmetic and if $k > 0$ then ID_k is obtained from ID_{k-1} by adding the new unary predicate W_k together with the inductive closure axiom:

$$\forall a\ (F_k(W_k, a) \to W_k(a))$$

and the least-fixed-point axiom schema:

$$\forall a\ (F_k(A, a) \to A(a)) \to \forall a\ (W_k(a) \to A(a)).$$

$ID_{<\omega}$ is then the union of the ID_k's.

The Recursion Theorem now allows us to mimic the definition of each $\varphi^{(k)}\colon \Omega_{k+1} \times \Omega_k \to \Omega_k$ as a partial recursive function $f^{(k)}\colon W_{k+1} \times W_k \to W_k$ where $\langle 0, b\rangle$ is the notation for the successor of b and $\langle i+1, e\rangle$ is a

notation for the limit of the sequence $\{e\}(x)$ taken over x in W_i. Therefore taking $\langle i+1, e_0\rangle$ as a notation for ω_i where e_0 is an index for the identity function, we obtain a notation in W_1 for each of the ordinals τ_k, by repeated applications of the $f^{(i)}$ functions analogous to the tree-ordinal definition given in 1.10. However we need to do this provably inside $ID_{<\omega}$.

Let $A(a)$ be the formula expressing $f_a^{(k)}\colon W_k \to W_k$ for given $a \in W_{k+1}$, i.e.,

$$\forall b\ (W_k(b) \to \exists c\ (f^{(k)}(a,b) = c \wedge W_k(c))).$$

Then the recursive definition of $f^{(k)}$ together with the inductive closure axiom for W_k enable one to formally prove

$$\forall a\ (F_{k+1}(A,a) \to A(a))$$

and hence, by the least-fixed-point axiom for W_{k+1},

$$\forall a\ (W_{k+1}(a) \to A(a)).$$

Thus, provably in $ID_{<\omega}$ we have $f^{(k)}\colon W_{k+1} \times W_k \to W_k$ for each k, and hence in $ID_{<\omega}$ we can construct a notation for each τ_k and prove transfinite induction up to it. This is enough to prove the recursiveness of B_α for every $\alpha < \tau$.

§6. Concluding remarks. Many of the results contained here have their origins in Girard [7] and subsequent work with his students (see especially the sequence of papers culminating in Vauzeilles [11] where a characterization is given of the Howard ordinal, our τ_3). However, our context and methods are entirely different to theirs, and some of their results were bedded deeply in a more elaborate category-theoretic setting which made it rather difficult to establish relationships with known proof-theoretic ordinal systems. The intention here has been to present the results as succinctly as seems possible, and in a more standard proof/recursion-theoretic framework. An underlying theme is the search for an acceptable definition of "natural well-ordering", and basic notions of category theory do seem useful here. One seeks a large class of well-orderings (or ordinal presentations) and a "natural" partial ordering $\lhd$ on them which ensures that any recursion or proof "majorized" by α will be majorized by α' whenever $\alpha \lhd \alpha'$. This would avoid the problem of "subrecursive collapse" mentioned in the introduction, where even simply defined well-orderings on $\mathbb{N}$ of order-type ω may encode arbitrarily high computational complexity. Clearly $\lhd$ should be some kind of controlled embeddability relation, and in terms of ordinal presentations there is an obvious choice, viz. natural transformation: Write $(\alpha, [\ ]) \lhd (\alpha', [\ ]')$ to mean that there is an order-preserving map ν from α into α' such that for every n, ν takes $\alpha[n]$ into $\alpha'[g(n)]'$ where g is a strictly increasing function of limited complexity (elementary

or primitive recursive, say). Then as in Wainer [13], hierarchies defined by recursion over $(\alpha, [\ ])$ will be bounded by hierarchies defined recursively over $(\alpha', [\ ]')$. One could wish for more, however—that recursions over $(\alpha, [\ ])$ are not only bounded by, but subrecursively reducible to recursions over $(\alpha', [\ ]')$. This requires some form of representability of the presentations as well-orderings of $\mathbb{N}$, such that the notations for elements of $\alpha[n]$, $\alpha'[n]'$ are computable uniformly in n. Furthermore, it requires the natural transformation ν to be effective in the sense that for each n its restriction $\nu : \alpha[n] \to \alpha'[g(n)]'$ is uniformly computable on notations. With this in mind one could define an *accessible* presentation or well-ordering to be any representable one for which there is an effective natural transformation from it into some τ_k (with its induced presentation). Recursions over accessible well-orderings will then always produce accessible recursive functions.

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SCHOOL OF MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS LS2 9JT, UK
E-mail: s.s.wainer@leeds.ac.uk