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COMPLETENESS BEFORE POST: BERNAYS, HILBERT, AND THE DEVELOPMENT OF PROPOSITIONAL LOGIC

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Abstract. Some of the most important developments of symbolic logic took place in the 1920s. Foremost among them are the distinction between syntax and semantics and the formulation of questions of completeness and decidability of logical systems. David Hilbert and his students played a very important part in these developments. Their contributions can be traced to unpublished lecture notes and other manuscripts by Hilbert and Bernays dating to the period 1917–1923. The aim of this paper is to describe these results, focussing primarily on propositional logic, and to put them in their historical context. It is argued that truth-value semantics, syntactic ("Post-") and semantic completeness, decidability, and other results were first obtained by Hilbert and Bernays in 1918, and that Bernays's role in their discovery and the subsequent development of mathematical logic is much greater than has so far been acknowledged.

§1. Introduction. Paul Bernays is best known today for being Hilbert's primary collaborator on foundational matters in the Göttingen of the 1920s. He both shaped and helped execute the research project now known as Hilbert's program. The *Grundlagenbuch* [46, 47], the decidability of the so-called Bernays-Schönfinkel class of first-order formulas [11], and his work on axiomatic set theory [8] are considered to be his major contributions to the foundations of mathematics. Bernays is also the author of a number of influential papers on philosophy of mathematics, and the details and refinements of Hilbert's mature philosophical views certainly owe much to

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him. His mathematical work in the early 1920s however, is little known and even less appreciated.

Bernays came to Göttingen in the Fall of 1917, at Hilbert's invitation.[1] For the following 17-odd years, Bernays worked in Göttingen as his assistant. His main task was to collaborate with Hilbert in his foundational work, in particular, to assist in the preparation of Hilbert's lecture courses and in preparing polished typescripts of these lectures. Many of these lecture notes are preserved at the library of the Department of Mathematics at the University of Göttingen, and in Hilbert's *Nachlaß* at the Niedersächsische Staats- und Universitätsbibliothek. Hilbert's lectures have recently received much attention, since they provide a much more nuanced and detailed way of understanding the development not only of Hilbert's views on the foundations of mathematics, but of the development of first-order logic in the 20s. Moore [60] and Sieg [69] discuss, inter alia, the lecture notes for the course on the "Principles of Mathematics" [33].[2] I, too, want to focus on these notes, and on Bernays's *Habilitationsschrift* [5], of which only parts were published [6]. My central concern, however, shall be the results on propositional logic contained therein. These results include: explicit semantics for propositional logic using truth values, decidability of the set of valid propositional formulas, completeness of the axiom systems considered relative to that semantics, as well as what is now called Post completeness, consistency and independence results, general three- and four-valued matrices, and rule-based derivation systems.

All these results were obtained independently of logicians to whom they are usually credited (notably Pierce, Wittgenstein, Post, and Łukasiewicz).[3] Far be it from me to dispute their priority. After all, Hilbert and Bernays's work remained unpublished, and in some respects the work by those other logicians investigates the questions at hand more deeply or is more precise than Hilbert and Bernays's. I do think, however, that a detailed exposition of the results may provide clues to the development of logic in the 1920s, in particular in the Hilbert school.

While I believe that all of the results on propositional logic in question are interesting in their own right, some of my discussion also has significant bearing on the understanding of the development of first-order logic and Hilbert's foundational program as a whole. For instance, one of the conclusions of a close look at the historical record will be that the seminal early results on propositional *and first-order* logic were in large part due to Bernays.

About his *Habilitationsschrift* of 1918, Bernays said:

> [It] was certainly of a mathematical character. But the opinion at the time was that foundational investigations connected to mathematical logic were not taken seriously. They were considered amusing, playful. I had a similar tendency, and so did not take it seriously either.

> I was not very ambitious to get it published in time, and it appeared only much later, and then only in part [. . .] And so some of what I had achieved there was not duly recognized in the expositions of the development of mathematical logic.[4]

The present paper is in part an attempt to answer this complaint.

In §2, I give an exposition of the ideas contained in the lecture notes and in the *Habilitationsschrift* concerning semantics and completeness. Since there is significant overlap between Hilbert's lecture and Bernays's *Habilitationsschrift*, a discussion of the issue of authorship of the relevant passages is in order. This is the topic of §3. In §4, I present the parts of the *Habilitationsschrift* dealing with dependence and independence of axioms. §5 deals with Bernays's efforts to provide an axiomatization of propositional logic based on rules as opposed to axioms, an approach influencing later axiomatic developments and also Gentzen's sequent calculus. In §6, I try to provide several hints as to how this early work by Bernays and Hilbert influenced the further direction that logical investigations took in the Göttingen of the 1920s.

§2. Semantics, Normal Forms, Completeness.

2.1. Prehistory: Hilbert's lectures on *Logical Principles of Mathematical Thought* 1905. In the Summer semester of 1905, Hilbert holds a course on "Logical principles of mathematical thought" [31]. A detailed exposition of the lectures and their historical context is given by Peckhaus [62], to whom much of the discussion in this section is indebted (see also [63, 64]). These lectures are highly interesting, for they contain developments of axiom systems not only for arithmetic and geometry, but also thermodynamics and probability theory. In them, Hilbert first discusses set theory and the paradoxes. In Chapter V ("The logical calculus"), we then read: "The paradoxes we have just introduced show sufficiently that an examination and redevelopment of the foundations of mathematics and logic is urgently necessary."[5]

Following a discussion of the purpose of logic and of the significance of contradictions, Hilbert develops propositional logic algebraically, using ideas from his first Heidelberg lecture given the year before [32]. Hilbert lays down the following axioms:

> Axiom I. If $X \equiv Y$[6] then one can always replace X by Y and Y by X.
>
> Axiom II. From 2 propositions X, Y a new one results ("additively")
>
> $$Z \equiv X + Y$$
>
> Axiom III. From 2 propositions X, Y a new one results in a different way ("multiplicatively")
>
> $$Z \equiv X \cdot Y$$

> The following identities hold for these "operations":
>
> IV. $X + Y \equiv Y + X$ VI. $X \cdot Y \equiv Y \cdot X$
> V. $X + (Y + Z) \equiv (X + Y) + Z$ VII. $X \cdot (Y \cdot Z) \equiv (X \cdot Y) \cdot Z$
> VIII. $X \cdot (Y + Z) \equiv X \cdot Y + X \cdot Z$
>
> [...] There are 2 definite propositions 0, 1, and for each proposition X a different proposition $\overline{X}$ is defined, so that the following identities hold:
>
> IX. $X + \overline{X} \equiv 1$ X. $X \cdot \overline{X} \equiv 0$
> XI. $1 + 1 \equiv 1$ XII. $1 \cdot X \equiv X$[7]

Hilbert's intuitive explanations make clear that X, Y, and Z stand for propositions, $+$ for conjunction, $\cdot$ for disjunction, $\overline{\cdot}$ for negation, 1 for falsity, and 0 for truth.[8] The axioms are followed by a discussion of the system from an algebraic standpoint. Hilbert points out how the axioms with the exception of (XI) also apply to arithmetic, and discusses the correspondence between negation and subtraction. Then he poses the main metatheoretical questions:

> It would now have to be investigated in how far the axioms are dependent and independent of one another [...] What would be most important here, however, is the proof that the 12 axioms do not contradict each other, i.e., that using the process defined one cannot obtain a proposition which contradicts the axioms, say, $X + \overline{X} = 0$. These are only hints which have not been carried out completely as of yet, and one still has free reign in the details; generally speaking this whole section supplies material for the ultimate solution of the interesting questions, rather than give the ultimate solution.[9]

These questions are to be solved 12 years later in the lectures from 1917–18 and in Bernay's *Habilitationsschrift*. It is interesting to note that Hilbert has all the tools in hand to give the solution already in 1905. We even find a nonderivability proof using an arithmetical interpretation of the axioms on p. 233: The axioms (XI) and (XII) are not derivable from the other axioms together with $X + 0 \equiv X$ and $X \cdot 0 \equiv 0$ (interpret $+$ and $\cdot$ as ordinary sum and product of reals, and take $\overline{X}$ to be $1 - X$.)

Hilbert proceeds to establish a number of consequences of the axioms in the style of algebraic proofs, in particular, de Morgan's laws. There is no distinction between consequence and the material conditional, $X \mid Y$[10] "Y follows from X [aus X folgt Y]" is defined by $\overline{X} \cdot Y \equiv 0$. Given this definition, it seems problematic to use nested conditionals, but subsequent examples indicate that $X \mid Y$ is intended also as an abbreviation for $\overline{X} \cdot Y$, not only for the equation $\overline{X} \cdot Y = 0$.

Hilbert then proves that every propositional formula can be brought into one of two normal forms. First one uses de Morgan's laws repeatedly to see that every sentence can be written as sums and products of primitive propositions and their negations. Using the distributive law, this can be rewritten as a sum of products. Hilbert then uses a number of ways to simplify these, and claims (erroneously) that the resulting conjunctive normal form is unique up to reordering of conjuncts.[11] Using duality, it is then proved that every expression can also be brought into a disjunctive normal form.

Hilbert also discusses consequence at length. The system of propositional logic is intended as a background framework for other axiomatic theories. The axioms of those theories are interpreted as "correct" propositions, and the calculus is intended to make clear which propositions follow from the axioms according to the definition of consequence: Y follows from X if $\overline{X} \cdot Y = 0$. Hilbert proves the following about this notion of consequence:

> A proposition Y follows from another proposition X if and only if it is of the form $A \cdot X$, where A is some proposition. To deduce is to multiply correct propositions with arbitrary propositions.[12]

This theorem leads Hilbert to identify proofs with such factors A. The normal form theorem then provides the first proof of decidability of the propositional calculus. In the lecture on mathematical problems [28, p. 262], Hilbert discussed the issue of the decidability of every mathematical problem and proclaimed that "in mathematics there is no ignorabimus." The decidability of the propositional calculus is an example of what Hilbert is looking for:

> I now want to point out what is probably the most important application of the normal form of a proposition and its uniqueness. We will—and this is a restriction we have to impose for the time being—take a finite number of propositions a, b, c, ... (axioms about the things considered or proper names) as given. Then there can be only a finite number of propositions (that is, propositions built up from these basic propositions), for every one can be brought into the form of a sum of products [conjunction of disjunctions] in basically a unique way. Every basic proposition appears in any summand [conjunct] only in the first dimension and any product [disjunction] appears only once as a summand [conjunct]. Every correct proposition must follow from the sum of the axioms $a + b + \cdots$ by multiplication with a certain factor A (Proof) and for this A there are only finitely many [possible] forms by what has just been said. So it turns out that for every theorem there are only *finitely many possibilities of proof*, and thus we have solved, in the most primitive case at hand, the old problem that it must be possible to achieve any correct result by a *finite proof*. This problem was the original starting point of all my investigations in our field, and the solution to this problem in the most general case[,] the proof that there can be no "ignorabimus" in mathematics, has to remain the ultimate goal.[13]

There are many difficulties with this passage. First of all, if one takes the axioms of a theory to be a finite set of unanalyzed propositions a, b, c, ... , the propositional consequences of such a theory will not cover any significant number of their logical consequences. Taking the passage at face value, what we get is essentially a decision procedure for the propositional consequences of a set of variables. The argument can, however, easily be modified to apply to consequences of a finite set of propositional formulas.[14] This would not get us too far either, but Hilbert after all acknowledges that we are here dealing only with "a most primitive case." The next difficulty arises from Hilbert's earlier error of claiming that the normal form for a given formula is unique. For Hilbert's procedure to work, we would not only have to be able to enumerate all possible proofs A, but also be able to check if $A \cdot (a + b + \cdots) = Y$. This would presumably have to be done by comparing normal forms, since no other method—e.g., truth tables—is available. But normal forms are not unique, so there is no guarantee that the left and right side will result in the same one.[15] Lastly, the worry about the existence of a finite proof of any correct proposition is puzzling. It is not that the proof itself has to be finite what is important, but that there are only finitely many possibilities for a proof; we may decide, after finitely many steps, whether there is a proof or not.

All these difficulties aside, the main point is still notable. Here, in 1905, one of Hilbert's aims in the foundations of mathematics is made almost explicit, namely the aim to provide decision procedures for logic on the one hand, and particular systems of mathematics and science, e.g., arithmetic, on the other.

2.2. The structure of *Prinzipien der Mathematik*. In the years following 1905, Hilbert's interest in the foundations of mathematics seems to have subsided. He does not follow up his groundbreaking ideas of 1905 until around 1917, when he returns with full force to his work on axiomatics.[16] In September 1917, Hilbert delivers his lecture on "axiomatic thought" in Zürich, and invites Bernays to come to Göttingen as his assistant. In the Winter semester 1917–18 Hilbert teaches a course on the "Principles of mathematics." The lecture notes to that course are preserved in the library of the Department of Mathematics at the University of Göttingen.[17] They are divided into two parts: Part A (62 pages) on the axiomatic method contains an exposition of axiomatic geometry; Part B (pp. 63–246, 184 pages) deals with mathematical logic. The material in Part B is new and interesting. It starts out with a discussion of propositional calculus in the style of algebraic logic in Section 1 (pp. 63–80). The propositional calculus is extended to a calculus of classes in Section 2 (pp. 81–107), and a theory of syllogisms is developed. In Section 3, the limitations of the class calculus are used to motivate the introduction of the calculus of functions, i.e., first-order logic with quantifiers (pp. 108–129). This calculus of functions is formally introduced and studied in Section 4 (pp. 129–187). Section 5 (pp. 188–246) deals with the extended calculus of functions (i.e., second-order logic), as well as with induction, the definition of identity, the paradoxes, type theory, and the axiom of reducibility.[18]

From a historical point of view, the last two sections of Part B are the most interesting ones. The development of geometry in Part A is standard, and overlaps both with the *Foundations of Geometry* [30] and the material presented in numerous courses on axiomatic geometry taught by Hilbert at Göttingen. The propositional calculus presented in Section 1 of Part B is exactly the same as the one developed in Hilbert's 1905 course. There are two notable differences in the presentation. The 1917–18 notes contain an independence proof similar to the one in [31], as well as a proof of consistency of the axioms of propositional logic. In contrast to the independence proof, which uses an arithmetic interpretation, consistency is proved by restricting the range to only the propositions 0 and 1, and defining sum and product case-by-case:

> Restrict the domain of propositions by allowing only the propositions 0 and 1, and interpret the equations in accordance with this as proper identities. Furthermore, define sum and product by the 8 equations
>
> $$\begin{array}{ll} 0+0=0 & 0\times 0=0 \\ 0+1=1 & 0\times 1=0 \\ 1+0=1 & 1\times 0=0 \\ 1+1=1 & 1\times 1=1 \end{array}$$
>
> which are characterized by turning into correct arithmetical equations, if one replaces the symbolic sum by the maximum of the summands and the symbolic product by the minimum of the factors. Declare the proposition 1 to be the negation of the proposition 0 and the proposition 0 to be the negation of 1.
>
> These definitions in any case do not lead to a contradiction, for each one of them defines a new symbol. On the other hand, one can establish by finitely many tries that all the axioms I–XII are satisfied by these definitions. These axioms therefore cannot result in a contradiction either. Thus the question of consistency of our calculus can be completely resolved.[19]

What is interesting here is that, while Hilbert thought that an arithmetical interpretation is good enough to establish independence results, something more basic is needed to show consistency. The first sentence in the last paragraph just quoted indicates that Hilbert had scruples regarding the use of arithmetic correctness of equations to establish consistency. He simply wanted to avoid appeal to infinite structures at this point.

The second difference is a much more elaborate discussion of consequence. The definition is the same as in 1905 (only the symbol for implication changes to $\rightarrow$), but now a number of properties are proved that one would expect of a system of logic: For any X and Y, $X \rightarrow X$, $X + Y \rightarrow X$, if $X \rightarrow Y$ then $\overline{Y} \rightarrow \overline{X}$, and others. A discussion of "proofs as multiplication" and of decidability is missing, however.

Taking this notion of consequence as a starting point, Hilbert takes on an investigation of how much of mathematical reasoning can be accommodated in the propositional calculus. In Section 2 (Predicate calculus and class calculus), the propositional calculus is reinterpreted as first, a calculus of predicates, and second, a calculus of classes (extensions of predicates). These reinterpretations are then used to account for the Aristotelian syllogisms in the framework of the calculus. Naturally it is ultimately found (in Section 3: Transition to the calculus of functions) to be insufficient for a foundation of mathematics, for it is unable to deal with relations between individuals or with nested quantifiers. This leads Hilbert to introduce the function calculus, first by example (the difference between convergence and uniform convergence), and then finally, as an axiom system.

Section 4, entitled "Systematic presentation of the function calculus," contains a presentation of the function calculus, i.e., first-order logic, organized as follows:

4.1. Axioms of the function calculus (pp. 129–140)
4.2. The system of logical propositional formulas (pp. 140–153)
4.3. The complete system of logical formulas (pp. 154–179)
4.4. Examples of applications of the function calculus (pp. 180–187)

Section 5 of Part B of the lecture notes discusses the extended function calculus, i.e., higher-order logic. It includes discussions of definitions of number, set theory, paradoxes and type theory.

Let me now turn to a discussion of the propositional fragment of the function calculus as developed in 4.1 and 4.2. For discussion of the full first-order logic and the later parts of the lecture notes, the interested reader is referred to the papers by Moore [60] and Sieg [69].

2.3. The propositional calculus. The propositional fragment of the function calculus is investigated separately in Subsection 2 of Section 4. Syntax and axioms are modeled after the propositional fragment of *Principia Mathematica* [74]. The language consists of propositional variables [Aussage-Zeichen] X, Y, Z, ..., as well as signs for particular propositions, and the connectives $\overline{\cdot}$ (negation) and $\times$ (disjunction). The conditional, conjunction, and equivalence are introduced as abbreviations. Expressions are defined by recursion:

1. Every propositional variable is an expression.
2. If α is an expression, so is $\overline{\alpha}$.
3. If α and β are expressions, so are $\alpha \times \beta$, $\alpha \rightarrow \beta$, $\alpha + \beta$ and $\alpha = \beta$.[20]

Hilbert introduces a number of conventions, e.g., that $X \times Y$ may be abbreviated to XY, and the usual conventions for precedence of the connectives. Finally, the logical axioms are introduced. Group I of the axioms of the function calculus gives the formal axioms for the propositional fragment (unabbreviated forms are given on the right, recall that XY is "X or Y"):

1. $XX \to X \qquad \overline{XX}X$
2. $X \to XY \qquad \overline{X}(XY)$
3. $XY \to YX \qquad \overline{XY}(YX)$
4. $X(YZ) \to (XY)Z \qquad \overline{X(YZ)}((XY)Z)$
5. $(X \to Y) \to (ZX \to ZY) \qquad \overline{\overline{X}Y}(\overline{ZX}(ZY))$

The formal axioms are postulated as correct formulas [richtige Formel], and we have the following two rules of derivation ("contentual axioms"):

a. Substitution: From a correct formula another one is obtained by replacing all occurrences of a propositional variable with an expression.
b. If α and $\alpha \to \beta$ are correct formulas, then β is also correct.

Although the calculus is very close to the one given in *Principia Mathematica*, there are some important differences. Russell uses (2′) $X \to YX$ and (4′) $X(YZ) \to Y(XZ)$ instead of (2) and (4). *Principia* also does not have an explicit substitution rule.[21] The fact that Hilbert realizes that such a rule must be included in the calculus illustrates how Hilbert's axiomatic method makes the presentation of logic in 1917–18 much clearer than Schröder's algebra of logic and much closer to the modern conception of logic as calculus than Russell's *Principia*. But the division between syntax and semantics is not quite complete. The calculus is not regarded as concerned with uninterpreted formulas; it is not separated from its interpretation. (This is also true of the first-order part, see [69], B3.) Also, the notion of a "correct formula" which occurs in the presentation of the calculus is intended not as a concept defined, as it were, by the calculus (as we would nowadays define the term "provable formula" for instance), but rather should be read as a semantic stipulation: The axioms are true, and from true formulas we arrive at more true formulas using the rules of inference.[22] Read this way, the statement of modus ponens is not that much clearer than the one given in *Principia:* "Everything implied by a true proposition is true." (*1.1)

Hilbert goes on to give a number of derivations and proves additional rules. These serve as stepping stones for more complicated derivations. First, however, he proves a normal form theorem, just as he did in the 1905 lectures, to establish decidability and completeness. In the new propositional calculus, however, Hilbert needs to prove that arbitrary subformulas can be replaced by equivalent formulas, that is, that the rule of replacement is a dependent rule.[23] He does so by establishing the admissibility of rule (c): If $\varphi(\alpha)$, $\alpha \to \beta$, and $\beta \to \alpha$ are provable, then so is $\varphi(\beta)$.[24] With that, the admissibility of using commutativity, associativity, distributivity, and duality inside formulas is quickly established, and Hilbert obtains the normal form theorem just as he did for the first propositional calculus in the 1905 lectures. Normal forms again play an important role in proofs of decidability and now also completeness.

2.4. Consistency and completeness. "This system of axioms would have to be called inconsistent if it were to derive two formulas from it which stand in the relation of negation to one another."[25] That the system of axioms is not inconsistent in this sense is proved, again, using an arithmetical interpretation. The propositional variables are interpreted as ranging over the numbers 0 and 1, $\times$ is just multiplication and $\overline{X}$ is just $1 - X$. One sees that the five axioms represent functions which are constant equal to 0, and that the two rules preserve that property. Now if α is derivable, $\overline{\alpha}$ represents a function constant equal to 1, and thus is underivable.

Why did Hilbert not use this straightforward arithmetical interpretation to prove consistency for the first propositional calculus in 1905 or earlier in the lectures (Section 4.1)? If it was his concern that an infinite interpretation should not be used to establish consistency of such a basic system as that of propositional logic, then the numbers 0 and 1 alone would do just as well. One possible explanation is that up until the introduction of the new propositional calculus based on the *Principia* system, conjunction and disjunction were both primitives. Giving an arithmetical interpretation for these systems would thus have required finding an interpretation which also satisfies $1 + 1 = 1$. Simply taking congruences modulo 2 does not do the trick here. Only when $+$ is taken as a defined symbol can one take the congruences modulo 2 as an interpretation of the axioms. Compared to the consistency proof in Section 4.1 using true and false propositions, the arithmetical interpretation is further away again from truth-value semantics for propositional logic.

> Let us now turn to the question of *completeness*. We want to call the system of axioms under consideration complete if we always obtain an inconsistent system of axioms by adding a formula which is so far not derivable to the system of basic formulas.[26]

This is the first time that completeness is formulated as a precise mathematical question to be answered for a system of axioms. Before this, Hilbert [31, p. 13] had formulated completeness as the question of whether the axioms suffice to prove all "facts" of the theory in question. Aside from that, completeness had always been *postulated* as one of the axioms. In the *Foundations of Geometry*, for instance, we find axiom V(2), stating that it is not possible to extend the system of points, lines, and planes by adding new entities so that the other axioms are still satisfied. In [31], such an axiom is also postulated for the real numbers. Following its formulation, we read:

> This last axiom is of a general kind and has to be added to every axiom system whatsoever in some form. It is of special importance in this case, as we shall see. Following this axiom, the system of numbers has to be so that whenever new elements are added contradictions arise, regardless of the stipulations made about them. If there are things which can be adjoined to the system without contradiction, then in truth they already belong to the system.[27]

We see here that the formulation of completeness of the axioms arises directly out of the completeness axioms of Hilbert's earlier axiomatic systems, only that this time completeness is a theorem *about* the system. I shall return to this issue in the final section.

The completeness proof in the 1917–18 lectures itself is an ingenious application of the normal form theorem: Every formula is interderivable with a conjunctive normal form. As has been proven earlier, a conjunction is provable if and only if each of its conjuncts is provable. A disjunction of propositional variables and negations of propositional variables is provable only if it represents a function which is constant equal to 0, as the consistency proof shows. A disjunction of this kind is equal to 0 if and only if it contains a variable and its negation, and conversely, every such disjunction is provable. So a formula is provable if and only if every conjunct in its normal form contains a variable and its negation. Now suppose that α is an underivable formula. Its conjunctive normal form β is also underivable, so it must contain a conjunct γ where every variable occurs only negated or unnegated but not both. If α were added as a new axiom, then β and γ would also be derivable. By substituting X for every unnegated variable and $\overline{X}$ for every negated variable in γ, we would obtain X as a derivable formula (after some simplification), and the system would be inconsistent.

In a footnote, the result is used to establish the converse of the characterization of provable formulas used for the consistency proof: every formula representing a function which is constant equal to 0 is provable. For, supposing there were such a function which was not provable, following the consistency proof above, adding this formula to the axioms would not make the system inconsistent, and this would contradict syntactic completeness [33, p. 153].

2.5. The contribution of Bernays's *Habilitationsschrift*. We have seen that the lecture notes to *Principles of Mathematics* 1917–18 contain consistency and completeness proofs (relative to a syntactic completeness concept) for the propositional calculus of *Principia Mathematica*. They also implicitly contain the familiar truth-value semantics and a proof of semantic soundness and completeness. In his *Habilitationsschrift* [5], Bernays fills in the last gaps between these remarks and a completely modern presentation of propositional logic.

Bernays introduces the propositional calculus in a purely formal manner. The concept of a formula is defined and the axioms and rules of derivation are laid out almost exactly as done in the lecture notes. §2 of [5] is entitled "Logical interpretation of the calculus. Consistency and completeness." Here Bernays first gives the interpretation of the propositional calculus, which is the motivation for the calculi in Hilbert's earlier lectures [31, 33]. The reversal of the presentation—first calculus, then its interpretation—makes it clear that Bernays is fully aware of a distinction between syntax and semantics, a distinction not made precise in Hilbert's earlier writings.[28] There, the calculi were always introduced with the logical interpretation built in, as it were. Bernays writes:

> The axiom system we have set up would not be of particular interest, were it not capable of an important contentual interpretation.
>
> Such an interpretation results in the following way:
>
> The variables are taken as symbols for *propositions* (sentences).
>
> That propositions are either true or false, and not both simultaneously, shall be viewed as their characteristic property.
>
> The symbolic product shall be interpreted as the connection of two propositions by "or," where this connection should not be understood in the sense of a proper disjunction, which excludes the case of both propositions holding jointly, but rather so that "X or Y" holds (i.e., is true) if and only if at least one of the two propositions X, Y holds.[29]

Similar truth-functional interpretations of the other connectives are given as well. Bernays then defines what a provable and what a valid formula is, thus making the syntax-semantics distinction explicit:

> The importance of our axiom system for logic rests on the following fact: If by a "provable" formula we mean a formula which can be shown to be correct according to the axioms [footnote in text: It seems to me to be necessary to introduce the concept of a provable formula in addition to that of a correct formula (which is not completely delimited) in order to avoid a circle], and by a "valid" formula one that yields a true proposition according to the interpretation given for any arbitrary choice of propositions to substitute for the variables (for arbitrary "values" of the variables), then the following theorem holds:
>
> *Every provable formula is a valid formula and conversely.*
>
> The first half of this claim may be justified as follows: First one verifies that all basic formulas are valid. For this one only needs to consider finitely many cases, for the expressions of the calculus are all of such a kind that in their logical interpretation their truth or falsehood is determined uniquely when it is determined of each of the propositions to be substituted for the variables whether it is true

or false. The content of these propositions is immaterial, so one only needs to consider truth and falsity as values of the variables.[30]

Everything one would expect of a modern discussion of propositional logic is here: A formal system, a semantics in terms of truth values, soundness and completeness relative to that semantics. As Bernays points out, the consistency of the calculus, of course, follows from its soundness. Lest the reader—recall that the intended readership includes Hilbert and his colleagues among the Göttingen faculty—have reservations about the "logical interpetation," Bernays points out that the interpretation of the variables by truth values is of no consequence, the same results could be obtained by an arithmetical interpretation using 0 and 1.

The semantic completeness of the calculus is proved in §3, along the lines of the footnote in [33] mentioned above. What may be pointed out here is that the formulation of syntactic completeness given by Bernays is slightly different from the lectures and independent of the presence of a negation sign: it is impossible to add an unprovable formula to the axioms without thus making all formulas provable. Bernays sketches the proof of syntactic completeness along the lines of Hilbert's lectures, but leaves out the details of the derivations.

Bernays also addresses the question of decidability. Decidability was not addressed at all in the lecture notes, even though Hilbert had posed it as one of the fundamental problems in the investigation of the calculus of logic. In his talk in Zürich in 1917, he said that an axiomatization of logic cannot be satisfactory until the question of decidability by a finite number of operations is understood and solved [34, p. 413]. Bernays gives this solution for the propositional calculus by observing that

> [t]his consideration does not only contain the proof for the completeness of our axiom system, but also provides a uniform method by which one can decide after finitely many applications of the axioms whether an expression of the calculus is a provable formula or not. To decide this, one need only determine a normal form of the expression in question and see whether at least one variable occurs negated and unnegated as a factor in each simple product. If this is the case, then the expression considered is a provable formula, otherwise it is not. The calculus therefore can be completely trivialized.[31]

2.6. A brief comparison with Post's thesis. Emil L. Post's dissertation of 1920 [65] is the locus classicus for all of the basic metatheoretical results about the propositional calculus.[32] It contains an explicit account of the truth table method, and the fundamental theorem that a formula is provable from the axioms of *Principia*[33] if and only if it defines a truth function which is always equal to '+' (true). From the fundamental theorem, Post deduces a number of consequences. Among them are, for instance, that the truth table method provides a decision procedure for derivability in the propositional calculus and that the addition of any unprovable formula yields an inconsistent system (inconsistency is understood here alternatively as proving both a formula and its negation, and as proving every formula). Post uses the term "closed" for systems which are such that the addition of an unprovable formula makes all formulas provable (p. 177).

Post's paper contains a number of other contributions. These are, on the one hand, a discussion of truth-functional completeness, and on the other, the introduction of many-valued logics. We will see later that Bernays's approach to proving independence of the axioms involved something very much like many-valued logics. It might also be pointed out that some of the discussion of truth-functional completeness can also be found in Bernays. On pp. 16–19 of [5], Bernays makes a number of remarks which are relevant here. For instance, there we find the claim that "all relationships between truth and falsity of propositions can be expressed using conjunction ('and'), disjunction (exlusive 'or') and negation, and thus also using the symbolism of our calculus."[34] Another remark concerns the equivalence of formulas in propositional logic. Two formulas are defined to be equivalent if $\alpha \sim \beta$ is provable ('$\sim$' is the *Principia* notation for the biconditional; Hilbert uses '$=$'). By the completeness theorem, this is the case if and only if $\alpha \sim \beta$ is valid. From this, Bernays shows that any formula is equivalent to one containing only negation and disjunction, or only negation and conjunction, or only negation and implication, and that corresponding claims for negation and equivalence or conjunction and disjunction do not hold. What we do not find, however, is a *proof* that every truth function can be represented by, say, negation and disjunction. A proof of this can be found in lecture notes to a course by Hilbert given in 1920 [35, pp. 18–19], the same year that Post submitted his dissertation.

The discussion of the fragment without negation leads Bernays to pose the question of whether there might be an axiom system in which all and only the provable negation-free propositional formulas are derivable. He claims that this can in fact be done, but does not give an axiomatization. We shall return to this question in §5.

§3. Hilbert or Bernays? It is well known that Bernays played an important role in the development of Hilbert's program in the 1920s, and that he wrote the monumental *Grundlagen der Mathematik* [46, 47] essentially alone, of course using Hilbert's ideas.[35] Mancosu [57, p. 175] stresses Bernays's contributions to the program in giving "more explicit discussion of the central philosophical topics surrounding Hilbert's program," and in clarifying Hilbert's views. Of course, there are also Bernays's published contributions to the program, for instance the work on the Entscheidungsproblem with Schönfinkel [11], and the investigations of the propositional calculus in the

Habilitationsschrift. Through contact with his colleagues in Göttingen, Bernays had great influence on technical developments, and his contributions and suggestions are acknowledged not only by Hilbert himself. I would like to argue here that Bernays was in fact instrumental already for the technical advances made in 1917–18, and that the development of propositional and first-order logic in [33] is at least as much due to Bernays as it is to Hilbert. Moore [60] and Sieg [69] point out that these advances are not only the formulation of calculi for propositional and first-order logic, but in particular the investigation of meta-logical questions about these calculi: consistency, completeness, decidability. These are the questions that Hilbert [34] emphasized as important questions to be answered for the calculus of logic. Their solution is in large part due to Bernays.

The winter term of 1917–18 was Bernays's first semester as Hilbert's assistant in Göttingen. Bernays characterized his duties as assistant as follows: "So I was [Hilbert's] assistant. That job was not like what assistants usually do here [in Zürich], helping the students with exercises and such. I had nothing to do with that. On the one hand, we discussed foundational questions, and on the other I helped with the preparation of his lectures and prepared lecture notes."[36] Bernays held an appointment as *außerplanmäßiger Assistent*, which meant that he did not have a regular position which carried a salary, but that he relied on stipends. Hilbert urged Bernays to obtain the *venia legendi* so that he would be able to teach courses. Bernays submitted his application for the *Habilitation* on 9 July 1918, it included the *Habilitationsschrift* [5].[37] Bernays gave his *Probevorlesung* on 23 December 1918, and the dean of the Faculty of Philosophy granted the *venia legendi* on 14 January 1919.[38] The *Habilitationsschrift* contains page references to the lecture notes for the 1917–18 lectures, so the lecture notes must have been finished by the time Bernays submitted the thesis in July 1918. The winter term lasted from 1 October 1917 to 2 February 1918—approximately 15 weeks of classes. The course on *Prinzipien der Mathematik* was given on Thursdays, 9–11 am. Bernays's own shorthand notes, which he took during the lecture, survive in his *Nachlaß* in Zürich.[39] Bernays marked the end of each lecture with a horizontal line, and thus a comparison with the lecture notes makes it possible to ascertain which parts of the lecture were given when. Approximately, we find the following: The first seven lectures correspond the Part A on geometry. The first version of the propositional calculus is developed in the next three lectures, corresponding to pp. 63–80 of [33]. The predicate and class calculi are discussed in the next two lectures, corresponding to pp. 81–129. Already it is remarkable that the typewritten notes contain a lot of material that is not contained in Bernays's notes, e.g., the extended discussion of syllogisms on pp. 99–105. The last three lectures cover the following: the axioms and rules of the restricted function calculus (corresponding to pp. 129–135); application to inferences with a singular premise (pp. 180–181), the extended function calculus, definition of identity, number, sets (pp. 188–194), paradoxes (pp. 209–218); and paradoxes continued. We see that key parts of the lecture notes were apparently not covered in the lecture: the sections on derivations of theorems and rules in the propositional calculus (pp. 140–179) including consistency and completeness are completely missing from Bernays's notes, as is the last section on type theory (pp. 219–245); the sections on the extended function calculus, set theory and the paradoxes were only briefly sketched. In total, 117 pages—almost the latter half of the lecture notes—correspond to three two-hour lectures, 8 pages of shorthand notes out of 55. Not surprisingly, while the typescript keeps very closely to the structure of the lectures for the first one hundred pages or so (half-empty pages where a lecture ended, references to subjects discussed "the last time"), these last 117 read more like a monograph than like lecture notes.

The documentary record thus strongly suggests the following: The important results on the propositional and the restricted function calculus were obtained after the lectures were given, approximately in the period February–May 1918, when Bernays elaborated his notes to the lecture. The *Habilitationsschrift* was written after the lecture notes were completed, in the Spring of 1918. Some additional circumstantial evidence can be adduced for the thesis that the additional parts of the lecture notes, including the important results, are due in large part to Bernays. For one, the completeness proof is referred to a number of times by members of the Hilbert school. Bernays mentions it in the introduction of the *Habilitationsschrift*, where he states that proofs of consistency and completeness can be found in the 1917–18 lecture notes, before he gives the proof itself. We have seen that these proofs were not given in the actual lectures, and so these remarks must be understood as merely pointing the reader to the details of the normal form theorem (which was not proved in the *Habilitationsschrift*) rather than crediting Hilbert with the results. The published version [6] does not mention Hilbert's lectures at all. Behmann [3] presents the decision procedure based on the completeness proof and refers in this connection only to [5], although Behmann is certainly aware of the 1917–18 lectures (they are quoted on p. 165, and he almost certainly took the class). In the notes to a course on mathematical logic given in Göttingen in the summer term 1922, Behmann writes:

> These questions [of independence] concerning the axiomatics of elementary propositions were treated a few years ago by the Göttingen mathematician Bernays (Habilitationsschrift, unfortunately not published), and, one may well say, given a complete and satisfactory answer. [...] Bernays also rigorously proved completeness, i.e., has shown that every universally valid elementary proposition can indeed be derived from the basic formulas according to the basic rules.[40]

The most convincing piece of evidence may be the following remark by Bernays:

> My knowledge [of logic] was very incomplete at the time, in 1917. Before Hilbert took up the [investigation of the foundations of mathematics] directly again, which he had started much earlier [in [32]], he did not immediately lecture on that, but he gave a course on mathematical logic. And I was in charge of writing up [ausarbeiten] that lecture course, and I did this in such a way that I avoided free variables. I had looked at Russell a little bit, and first I found it too broad and did not like it in all respects, but in particular I did not understand what it means to say "for all x, $F(x)$, then $F(y)$ follows." In fact, the application of free variables is something technical. These are two ways to represent generality. One has generality on the one hand through bound variables and on the other through free variables. There is no such difference in natural langauge. So I avoided free variables at first. This is a possible way of approach, and later others have also done it this way. So that was a lecture course, which was written up, and then was filed in the library of the Mathematical Institute.[41]

Bernays's testimony here clearly indicates that the formulation of the quantifier axioms in [33] is due to him. It explains the particular form of these axioms, and why they differ so much from the corresponding postulates of *Principia Mathematica* and from later presentations (e.g., in [43],[42] which is otherwise based closely on the lecture notes from 1917–18). We may infer from this that the extent of Bernays's influence on the formulation and presentation of the results in the lecture notes from 1917–18 goes far beyond merely typing up what Hilbert said. Some of the results may or may not be due to Bernays. For instance, it is possible that Hilbert simply did not have enough time to present the completeness proof, but told Bernays to include it in the typescript. Given the amount of material that was not covered in the lecture, and the character of Hilbert and Bernays's working relationship, it is clear that a large amount of that material must have been worked out by Bernays alone. The fact that something as central as the formulation of the quantifier axioms is due to Bernays shows that it is very likely that he was the author of the parts of the lecture notes not covered in the lecture itself, and even that much of the material that was covered is in fact due to him. Be that as it may, the insights and results that are certainly due to Bernays—a clear syntax-semantics distinction, formulation of semantic completeness, independence results—are important enough to earn Bernays a prominent place in the history of the subject.

Why did Bernays not claim the results as his? A possible explanation may be his pronounced modesty.[43] (By the same token, if the results were exclusively Hilbert's, Bernays would have made a point of noting that when he presented the proofs, e.g., in [6] and [11].) Also recall the then prevalent tendency described by Bernays in a quote in §1 above, not to take mathematical logic seriously—at the time, he may well have thought of the results as not worth mentioning.

§4. Dependence and independence. Consistency and independence are the requirements that Hilbert laid down for axiom systems of mathematics time and again. Consistency was established—but the "contributions to the axiomatic treatment" of propositional logic could not be complete without a proof that the axioms investigated are independent. In fact, however, the axiom system for the propositional calculus, slightly modified from the postulates in (*1) of *Principia Mathematica*, is not independent. Axiom 4 is provable from the other axioms. Bernays devotes §4 of the *Habilitationsschrift* to give the derivation, and also the inter-derivability of the original axioms of *Principia* ($2'$) and ($4'$) with the modified versions (2) and (4) in presence of the other axioms. Together the derivations also establish the dependence of (*1.5) from the other propositional postulates in *Principia*.

Independence is of course more challenging. The method Bernays uses is not new, but it is applied masterfully. Hilbert had already used arithmetical interpretations in [31] to show that some axioms are independent of the others. The idea was the same as that originally used to show the independence of the parallel postulate in Euclidean geometry: To show that an axiom α is independent, give a model in which all axioms but α are true, the inference rules are sound, but α is false. Schröder was the first to apply that method to logic. §12 of his *Algebra of Logic* [68] gives a proof that one direction of the distributive law is independent of the axioms of logic introduced up to that point. The interpretation he gives is that of the "calculus of algorithms," developed in detail in Appendix 4. Bernays combines Schröder's idea with Hilbert's arithmetical interpretation and the idea of the consistency proof for the first propositional calculus in [33] (interpreting the variables as ranging over a certain finite number of propositions, and defining the connectives by tables). He gives six "systems" to show that each of the five axioms (and a number of other formulas) is independent of the others. The systems are, in effect, finite matrices. He introduces the method as follows:

> In each of the following independence proofs, the calculus will be reduced to a finite system (a finite group in the wider sense of the word [footnote: that is, without assuming the associative law or the unique invertability of composition]), where for each element a composition ("symbolic product") and a "negation" is defined. The reduction is given by letting the variables of the calculus refer to elements of the system as their values. The "correct formulas" are characterized in each case as those formulas which only assume values from a certain subsystem T for arbitrary values of the variables occurring in it.[44]

In the published version [6], the elements of the subsystem T are called *ausgezeichnete Werte*—designated values. The term is commonly used today.

I shall not go into the details of the derivations and independence proofs.[45] Let me just say that Bernays's method was of some importance in the investigation of alternative logics. For instance, Heyting [27] used it to prove the independence of his axiom system for intuitionistic logic and Gödel [26] was influenced by it when he defined a sequence of sentences F_n so that each F_n is independent of intuitionistic propositional calculus together with all F_i, $i > n$.[46] The many-valued logics Gödel used to show this are now called Gödel logics. It may be debated whether Bernays's systems can properly be called many-valued *logics,* but they certainly had the distinction of being useful in proving independence results in logic, an achievement considered important.

§5. Axioms and rules. In the final section of his *Habilitationsschrift*, Bernays considers the question of whether some of the axioms of the propositional calculus may be replaced by rules. This seems like a natural question, given the relationship between inference and implication: For instance, axiom 5 suggests the following rule of inference: (Recall that $\alpha\beta$ is Hilbert's notation for the disjunction of α and β. See §2.3 for a list of the axioms and rules.)

$$\frac{\alpha \to \beta}{\gamma\alpha \to \gamma\beta}\,\mathrm{c}$$

which Bernays used earlier as a derived rule. Indeed, axiom 5 is in turn derivable using this rule and the other axioms and rules. Bernays considers a number of possible rules

$$\frac{\begin{matrix}\alpha \to \beta\\ \beta \to \gamma\end{matrix}}{\alpha \to \gamma}\,\mathrm{d} \qquad \frac{\alpha\alpha}{\alpha}\,\mathrm{r}_1 \qquad \frac{\alpha}{\alpha\beta}\,\mathrm{r}_2 \qquad \frac{\alpha\beta}{\beta\alpha}\,\mathrm{r}_3 \qquad \frac{\alpha(\beta\gamma)}{(\alpha\beta)\gamma}\,\mathrm{r}_4$$

$$\frac{\varphi(\alpha\alpha)}{\varphi(\alpha)}\,\mathrm{R}_1 \qquad \frac{\varphi(\alpha\beta)}{\varphi(\beta\alpha)}\,\mathrm{R}_3$$

and shows that the following sets of axioms and rules are equivalent (and hence, complete for propositional logic):

1. Axioms: 1, 2, 3, 5; rules: a, b
2. Axioms: 1, 2, 3; rules: a, b, c
3. Axioms: 2, 3; rules: a, b, c, r_1
4. Axioms: 2; rules: a, b, c, r_1, R_3
5. Axioms: $\overline{X}X$[47]; rules: a, b, c, r_1, r_2, r_3, r_4

Bernays also shows, using the same method as before, that these axiom systems are independent, and also the following independence results:[48]

6. Rule c is independent of axioms: 1, 2, 3; rules: a, b, d (showing that in (2), rule c cannot in turn be replaced by d);
7. Rule r_2 is independent of axioms: 1, 3, 5; rules: a, b, (thus showing that in (1) and (2), axiom 2 cannot be replaced by rule r_2);
8. Rule r_3 is independent of axioms: 1, 2; rules: a, b, c (showing similarly, that in (1) and (2), rule r_3 cannot replace axiom 3);
9. Rule R_3 is independent of axioms: $\overline{X}X$, 3; rules: a, b (showing that R_3 is stronger than r_3, since 3 is provable from R_3 and $\overline{X}X$);
10. Rule R_1 is independent of axioms: $\overline{X}X$, 1; rules: a, b (showing that R_1 is stronger than r_1, since 1 is provable from $\overline{X}X$ and R_1);
11. Axiom 2 is independent of axioms: $\overline{X}X$, 1, 3, 5; rules: a, b, and
12. Axiom 2 is independent of axioms: $\overline{X}X$; rules: a, b, c, r_1, R_3 (showing that in (5), $\overline{X}X$ together with r_2 is weaker than axiom 2).

The detailed study exhibits, in particular, a sensitivity to the special status of rules like R_3, where subformulas have to be substituted. The discussion foreshadows developments of formal language theory in the 1960s. Bernays also mentions that a rule (corresponding to axiom 1), allowing inference of $\varphi(\alpha)$ from $\varphi(\alpha\beta)$ would be incorrect (and hence, "there is no such generalization of r_1").

Bernays's discussion of axioms and rules, together with his discussion of expressibility in the "Supplementary remarks to §2–3" (discussed above at the end of §2.6), shows his acute sensitivity for subtle questions regarding logical calculi. His remarks are quite opposed to the then-prevalent tendency (e.g., Sheffer and Nicod) to find systems with fewer and fewer axioms, and foreshadow investigations of relative strength of various axioms and rules of inference, e.g., of Lewis's modal systems, or more recently of the various systems of substructural logics.

At the end of the "Supplementary remarks," Bernays isolates the positive fragment of propositional logic (i.e., the provable formulas not containing negation; here $+$ and $\rightarrow$ are considered primitives) and claimed that he had an axiomatization of it. He did not give an axiom system, but stated that it is possible to choose a finite number of provable sentences as axioms so that completeness follows by a method exactly analogous to the proof given in §3. The remark suggests that Bernays was aware that the completeness proof is actually a proof schema, in the following sense. Whenever a system of axioms is given, one only has to verify that all the equivalences necessary to transform a formula into conjunctive normal form are theorems of that system. Then completeness follows just as it does for the axioms of *Principia*.

In his next set of lectures on the "logical calculus" given in the Winter semester of 1920,[49] Hilbert makes use of the fact that these equivalences are the important prerequisite for completeness. The propositional calculus we find there is markedly different from the one in [33] and [5], but the influences are clearly visible. The connectives are all primitive, not defined, this time. The sole axiom is $\overline{X}X$, and the rules of inference are:

$$\frac{X}{XY}\,\text{b2} \qquad \frac{\begin{array}{c}X\\Y\end{array}}{X+Y}\,\text{b3}$$

plus the rule (b4), stating: "Every formula resulting from a correct formula by transformation is correct." "Transformation" is meant as transformation according to the equivalences needed for normal forms: commutativity, associativity, de Morgan's laws, $\overline{\overline{X}}$ and X, and the definitions of $\rightarrow$ and $=$ (biconditional). These transformations work in both directions, and also on subformulas of formulas (as did R_1 and R_3 above).[50] One equivalence corresponding to modus ponens must be added, it is: $(X+\overline{X})\,Y$ is intersubstitutable with Y.

Anyone familiar with the work done on propositional logic elsewhere might be puzzled by this seemingly unwieldy axiom system. It would seem that the system in [35] is a step backward from the elegance and simplicity of the *Principia* axioms. Adjustments, if they are to be made at all, it would seem, should go in the direction of even more simplicity, reducing the number of primitives (as Sheffer did) and the number of axioms (as in the work of Nicod and later Łukasiewicz). Hilbert is motivated by different concerns. He was not only interested in the simplicity of his axioms, but in their efficiency. Decidability, in particular, supersedes considerations of independence and elegance. The presentation in [35] is designed to provide a decision procedure which is not only efficient, but also more intuitive to use for a mathematician trained in algebraic methods. Bernays's study of inference rules made clear, on the other hand, that such an approach can in principle be reduced to the axiomatics of *Principia*. One may ask whether the truth table method is not just as efficient a decision procedure. As any computer scientist working in automated theorem proving knows, truth tables are the worst possible decision procedure for propositional logic—exponential not only in the worst case, but in *every* case. In a similar vein, the subsequent work on the decision problem is not strictly axiomatic, but uses transformation rules and normal forms. The rationale is formulated by Behmann:

> The form of presentation will not be axiomatic, rather, the needs of practical calculation shall be in the foreground. The aim is thus not to reduce everything to a number (as small as possible) of logically independent formulas and rules; on the contrary, I will give as many rules with as wide an application as possible, as I consider appropriate to the practical need. The logical dependence of rules will not concern us, insofar as they are merely of independent practical importance. [. . .] Of course, this is not to say that an axiomatic

> development is of no value, nor does the approach taken here preempt such a development. I just found it advisable not to burden an investigation whose aim is in large part the exhibition of new results with such requirements, as can later be met easily by a systematic treatment of the entire field.[51]

Such a systematic treatment, of course, was necessary if Hilbert's ideas regarding his logic and foundation of mathematics were to find followers. Starting in [37] and [38], Hilbert presents the logical calculus not in the form of *Principia*, but by grouping the axioms governing the different connectives. In [37], we find the "axioms of logical consequence," in [38], "axioms of negation." The first occurrence of axioms for conjunction and disjunction seems to be in a class taught jointly by Hilbert and Bernays during Winter 1922–23, and in print in Ackermann's dissertation [2]. The project of replacing the artificial axioms of *Principia* with more intuitive axioms grouped by the connectives they govern, and the related idea of considering subsystems such as the positive fragment, is Bernays's. In 1918, he had already noted that one could refrain from taking $+$ and $\rightarrow$ as defined symbols and consider the problem of finding a complete axiom system for the positive fragment. The notes to the lecture course from 1922–23 [44, p. 17] indicate that the material in question was presented by Bernays. In 1923, he gives a talk entitled "The role of negation in propositional logic:"

> In axiomatizing the propositional calculus, the predominant tendency is to reduce the number of basic connectives and therewith the number of axioms. One can also, on the other hand, sharply distinguish the various connectives; in particular, it would be of interest to investigate the role of negation.[52]

The emphasis of separating negation from the other connectives is of course necessitated by Hilbert's considerations on finitism as well.[53]

Full presentations of the axioms of propositional logic are also to be found in [39], and in slightly modified form in a course on logic taught by Bernays in 1929–30. The axiom system we find there is almost exactly the one later included in [46].

I. $A \rightarrow (B \rightarrow A)$
$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
$(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

II. $A \,\&\, B \rightarrow A$
$A \,\&\, B \rightarrow B$
$(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \,\&\, C))$

III. $A \to A \vee B$
$B \to A \vee B$
$(B \to A) \to ((C \to A) \to (B \vee C \to A))$

IV. $(A \sim B) \to (A \to B)$
$(A \sim B) \to (B \to A)$
$(A \to B) \to ((B \to A) \to (A \sim B))$

V. $(A \to B) \to (\overline{B} \to \overline{A})$
$(A \to \overline{A}) \to \overline{A}$
$A \to \overline{\overline{A}}$
$\overline{\overline{A}} \to A$[54]

The algebraic perspective, evident only a few years earlier by the adoption of associativity, commutativity, and distributivity as axioms in some way or other, is completely lacking here. On the other hand, the influence of Frege is palpable in groups I, IV, and V. In [7], Bernays claims that the axioms in groups I–IV provide an axiomatization of the positive fragment, and raises the question of a decision procedure. This is where he first follows up on his his claim in [5] that such an axiomatization is possible.

§6. **Lasting influences.** Let me now summarize the advances made by Bernays and Hilbert and try to put them in the historical context of the development of mathematical logic and the foundations of mathematics. The most important of these contributions are certainly the distinction between syntax and semantics, the formulation of syntactic and semantic completeness, the proof of completeness for the propositional calculus, and the proof of decidability.

The history of the the concept(s) of completeness of an axiomatic system has yet to be written. The need for such a history, however, is apparent; completeness is the most fundamental property—alongside consistency—that an axiom system can have, and proofs of completeness and incompleteness of some kind or another count among the most celebrated results of mathematical logic. One need only mention the names of Gödel and Tarski in this connection to illustrate its importance. Although I cannot undertake the task of providing this history here, I want to indicate some of its milestones, since the work of Hilbert and Bernays I have been discussing is probably among the most important.

As we have seen, one of the roots of completeness as a property of axiom systems is the completeness axiom that Hilbert introduced in [29]. The axiom was not present in the first edition of *Foundations of Geometry*, but was included in the French translation of 1900, and then in the second German edition of 1903.[55] In the lectures from 1905 and again in "Axiomatic thought" [34] the axiom was formulated as the requirement that the addition of entities (numbers) to a model of the axioms would result in inconsistencies. In 1906, writing in Göttingen, Johannes Mollerup discusses Hilbert's axiomatization of the reals, and—without explicitly criticizing Hilbert on this issue—shifts the focus from completeness as something to be stipulated to something to be proven. He writes: "So we have two requirements for an axiom system, namely *first* an arithmetical requirement of consistency, and *second* a set-theoretic requirement of completeness" [59, p. 237]. König [51, p. 209] also criticizes Hilbert's use of the completeness axiom, stating that "the 'completeness axiom' is an intuition we should come to have of a completed thought-system; 'completeness' is an assumption which cannot even be formulated as an 'axiom' in our synthesis; just as the assumption of consistency cannot be so formulated."[56]

Hilbert did not address or acknowledge these criticism explicitly, and the completeness axiom survives in subsequent editions of the *Foundations of Geometry*. In the lectures from 1917–18, however, completeness is first formulated as a property of the propositional calculus in the form: whenever a hitherto non-derivable formula is added to the system, the system becomes inconsistent. The shift from talking about adding elements to talking about adding formulas (new axioms) may be explained in one of two ways: Possibly Hilbert and Bernays agreed that completeness should not be formulated as an axiom, but should be a property which one should prove about the system, and then the formulation corresponding to Post completeness seems to be the straightforward adaptation. On the other hand, if we take into account that the "elements" described by an axiom for propositional logic *are* propositions, then Post completeness says about propositions exactly the same thing that the completeness axiom says about the reals. Such an interpretation of propositions as the "things" that an axiom system is about is actually hinted at by Hilbert [31, pp. 257–58], and is supported by Hilbert's comparison of Russell's axiom of reducibility to the completeness axiom in 1918 (reported in [58, §8]).

By 1921 at least, Hilbert is well aware of the difference between the requirement expressed by the completeness axiom and completeness of axiom system in the syntactic sense, which is equivalent to the requirement that it proves or refutes every formula of the language.[57] The latter requirement is obviously closely related to Hilbert's "no ignorabimus," the conviction that every well-posed mathematical question can be answered positively or negatively. Where and how does the shift from the completeness axiom to the question of completeness *of* the axioms occur? Was it the recognition that in the context of logic the two amount to the same, and that syntactic completeness also makes the informal question of completeness ("We will have to require that all other facts of the area in question are consequences of the axioms."[58]) more precise? I cannot give an answer to this interesting and important question here. The issues are complicated enough to warrant their own extended treatment.[59]

The question of semantic completeness arises only when one makes a clear distinction between syntax and semantics. In 1917, Hilbert is still heavily influenced by Russell and Whitehead's *Principia*, and the influence is clearly visible in the lecture notes from 1917–18. But already there, Hilbert brings his view of axiomatics to bear: Derivation rules are formulated with more care, the expressions of the system are defined recursively, and we find metatheoretical results stated and proved which Russell and Whitehead considered misplaced because they could not be formulated within the system. But, as Sieg [69] points out, the axiom systems still come with a built-in interpretation, as it were. Bernays [5] makes the division between syntax and semantics complete.[60] The axioms and rules are stated purely formally—the study of the axiom system would be idle, were it not possible to give a "logical interpretation." This logical interpretation is precisely truth-value semantics for the propositional calculus. Now the semantical concept of completeness arises naturally: every valid formula is formally derivable.

The main application of the completeness proof, besides establishing that the propositional calculus provides an adequate formalization of the domain of propositional logic, is that of its decidability. The decision problem is vaguely formulated in [29] and [34], but in Bernays's *Habilitationsschrift* we find a model example of what a decision procedure looks like. This procedure serves as the model for subsequent attacks on the *Entscheidungsproblem*. For these attacks, however, axiomatics is put aside in favor of semantic methods. Behmann seems to be the first to state the decision problem explicitly:

> A general [set of] instructions shall be exhibited, according to which the *correctness or falsity* of an arbitrary given claim, which can be formulated with logical means, can be decided after a finite number of steps; this aim shall be realized at least within the bounds—which are to be determined exactly—within which its realization is in fact possible.[61]

The decision problem was, of course, another great problem on which Hilbert's students were working fervently in the 1920s. We have seen how the early work by Bernays and Hilbert in 1917–18 provides a paradigm for the solution. A decision procedure should be a determinate method to answer, in a finite number of steps, whether a logical formula is provable. But one should not forget that Bernays's decision procedure not only provides a model for what *kind* of result was to be proved, but *how* it should be proved. The method of transformation to normal forms, which was used by Behmann, Schönfinkel, and ultimately Gödel, can be traced back to Bernays's *Habilitationsschrift* [5] and Hilbert's 1905 lectures. With the semantic completeness and the work of [3], a shift towards semantic methods occurred, which was foreshadowed by semantic procedures for deciding validity and equivalence of propositional formulas in [35].

It is not until 1928 that completeness resurfaces. At the Congress of Mathematicians in Bologna, Hilbert poses the syntactical completeness of arithmetic and the semantic completeness of first order logic as problems of the foundations of mathematics [40, 41].[62] (In the 1917–18 lectures, it was already conjectured that the function calculus was not Post complete. This was subsequently proved by Ackermann.) Completeness of first-order logic is also posed as a question in the book with Ackermann [43]. The question is solved a year later in Gödel's dissertation [25].[63]

The metalogical investigations of Bernays on independence and axioms versus rules in 1918 laid the groundwork for several later developments. On the one hand, they provided a rigorous justification for the "algebraic" methods of manipulating formulas (e.g., of "applying" the law of associativity to subformulas) that were used as the official formulation of propositional logic until about 1923. At that time the strictures of Hilbert's developing finitism made it clear that distinctions must be made between the unproblematic connectives (disjunction, conjunction, and in particular the conditional, or "consequence"), and the problematic part, namely negation and the quantifiers. Here, too, Bernays's investigations helped satisfy these strictures by separating the axioms for the unproblematic notions from those for the problematic ones. In the notation of *Prinicipia*, this would not have been possible: there, the unproblematic notion of consequence was even defined in terms of negation (and disjunction).

The development of clear and intuitive axioms for propositional logic, and the investigations of the extent to which axioms can be replaced by rules undoubtedly also had great influence on Gentzen's development of natural deduction and the sequent calculus. Bernays was still teaching in Göttingen at the time when Gentzen was preparing his thesis [23], and in all likelihood was in close contact with him. Bernays was working closely with Paul Hertz throughout the 1920s, and Hertz's work on axiom systems is commonly acknowledged to be one of Gentzen's main sources.[64] The picture is far from complete, however, and it seems well worth filling in the details. In the course of this, in particular in a reexamination of Hertz's work on logic, it may well be that further important contributions by Bernays may come to light.

NOTES

1. Hilbert issued the invitation in September 1917 at the occasion of Hilbert's talk on axiomatic thought in Zürich. Reid [66, p. 151] reports that the invitation was made in the Spring of 1917. Bernays, however, reported the former version in [9], in an interview on 25

July 1977 [10], and even in a letter to Reid (27 November 1968, Bernays Nachlaß, WHS, ETH Zürich, Hs 975.3775). As far as I can see, there is no evidence for Reid's version.

2. See also [1].

3. Pierce, Wittgenstein, and Post are commonly credited with the truth-table method of determining propositional validity; Post for the completeness of propositional calculus; and Pierce, Post, and Łukasiewicz for the invention of many-valued logics. The method of using many-valued matrices for independence proofs was also discovered independently by Łukasiewicz and Tarski.

4. "[Sie] hatte zwar durchaus mathematischen Character, aber so die damalige Auffassung war die, dass man diese Grundlagenuntersuchungen, die an die mathematische Logik anknüpften, dass man die mathematisch nicht für voll genommen hat, nicht wahr, ja, das ist ja so ganz nett, das ist so halb spielerisch, nicht wahr, [...] und ich war auch so in der Tendenz [...] und habe das sozusagen auch nicht so ganz für voll genommen, und da [...] hatte ich keinen solchen Eifer, das rechtzeitig zu publizieren, und das ist erst sehr viel später, und doch eigentlich nicht ganz vollständig, sondern bloss mit gewissen Partien herausgekommen [...] so ist das, ist Manches zum Beispiel in den Darstellungen der Entwicklung der mathematischen Logik ist das zum Teil nicht, nicht wahr, entsprechend zum Ausdruck gekommen, was ich da in dieser Arbeit hatte." Interview, 25 July 1977 [10]; also reported by Specker [70]. All translations are mine except where English translations are noted in the bibliography. It might be interesting to list some historical accounts and how they treat Bernays. Jørgensen [49], who in other respects provides a very comprehensive account of the developments in symbolic logic up to 1930, mentions neither Post nor Bernays in connection with completeness or independence results. Kneale and Kneale [50] treat Bernays's independence proofs in depth and give his completeness proof, but credit it to Post. Bocheński [13] mentions Post in connection with the decision procedure for propositional logic, but does not mention Bernays. Church [14] cites Bernays's results on dependence and independence, but does not mention him in connection with consistency, completeness, or decidabilty. Surma [71] makes no mention of Bernays at all.

5. "Die Paradoxien, die wir im voranstehenden kennen gelernt haben, zeigen zur Genüge, dass eine Prüfung und Neuaufführung der Grundlagen der Mathematik und Logik unbedingt nötig ist." [31], p. 215

6. A marginal note on p. 224 instructs: "write more simply = 'equal' ".

7. [31], pp. 225–228.

8. "We may think of 0—if we want to proceed intuitively—as the proposition which 'expresses nothing' and which therefore is the ideally correct one; we may call every proposition identical to 0 a *correct* [richtige] or maybe better *non-contradictory* [widerspruchslose] proposition ... " [31], p. 226

9. "Es müßte nun untersucht werden, wie weit die Axiome von einander unabhängig sind [...] Das Wichtigste aber wäre hier der Nachweis, dass die 12 Axiome sich nicht widersprechen, d.h. daß man aus ihnen durch die festgelegten Proceße [sic] keine Aussage herleiten kann, die den Axiomen widerspricht, $X + \overline{X} \equiv 0$ etwa. Das sind alles hier nur Andeutungen, die noch keineswegs vollkommen durchgeführt sind, und man hat in Einzelheiten noch sehr viel freie Hand; überhaupt liefert dieser ganze AbschnittAbschnitt vorläufig eigentlich mehr Materialien zu einer endgültigen Lösung der interessierenden Fragen, als eine endgültige Lösung von ihnen." [31], pp. 230–31

10. The notation $X \mid Y$ was introduced in [32], this is changed to $X \to Y$ in a marginal note on p. 236. The influence of Frege is obvious here: "b follows from a" is motivated as excluding the second of the four possibilities: $a + b$, $a + \overline{b}$, $\overline{a} + b$, $\overline{a} + \overline{b}$, compare *Begriffsschrift*, §5.

11. When proving a similar normal form theorem for the calculus in [33, p. 149], the fact that normal forms are not unique is pointed out in a footnote. Even if the procedure

outlined by Hilbert were deterministic and would thus produce unique normal forms for every formula, different formulas may still have different normal forms, a fact which will become important below.

12. “Eine Aussage Y folgt aus einer andern X dann und nur dann, wenn sie von der From $A \cdot X$ ist, wo A irgend eine Aussage ist. Schliessen heisst richtige Aussagen mit irgend welchen Aussagen multiplizieren.” [31], p. 246.

13. “Ich will hier noch auf eine, wohl die wichtigste Anwendung der Normalform einer Aussage und ihrer Eindeutigkeit hinweisen. Wir wollen—und darauf müssen wir und zunächst beschränken—eine endliche Anzahl von Aussagen $a, b, c \ldots$ (Axiome über die behandelten Dinge oder Eigennamen) zu Grund legen. Dann kann es überhaupt nur endlich viele Aussagen darüber (d.h. aus diesen Grundaussagen zusammengesetzte Aussagen) geben; denn jede läßt sich auf eine Summe von Produkten im wesentlichen eindeutig bringen, wo in jedem Summand dieselbe Grundaussage nur in der ersten Dimension erscheinen und dasselbe Produkt auch nur einmal als Summand auftreten kann. Jede richtige Aussage muß aus der Summe der Axiome $a + b + \cdots$ durch einen gewissen Multiplikator A folgen (Beweis), und für dieses A gibt es nach dem gesagten auch nur endlich viele Formen. So ergibt sich hier, daß für jeden Satz nur *endlich viele Beweismöglichkeiten* existieren, und wir haben damit in dem vorliegenden primitivsten Falle das alte Problem gelöst, daß jedes richtige Resultat sich durch einen *endlichen Beweis* erzielen lassen muß. Dies Problem war eigentlich der Ausgangspunkt aller meiner Untersuchungen auf unserem Gebiete und die Erledigung dieses Problemes im allerallgemeinsten Falle der Beweis, daß es in der Mathematik kein “Ignorabimus” geben kann, muß auch das letzte Ziel bleiben.” [31], pp. 248–9.

14. This correction is made by Hilbert later in the lectures (p. 257), see also [62, pp. 70–72].

15. Of course it would be enough to know that there are only finitely many normal forms, and we can check all of these. But Hilbert does not have a deterministic and finite procedure to produce all these.

16. I do not mean to suggest that Hilbert was not interested in the foundations of mathematics during this period. Sieg [69] has pointed out that Hilbert lectured a number of times on foundations of mathematics and physics during that time. These lectures, however, contain far less of logical interest than those of 1905 or those after 1917; most of them were courses on “elementary mathematics from a higher standpoint,” a topic on which Klein had also often lectured. Even though Hilbert may not himself have worked much on the subject, there is a lot of activity in foundations of mathematics in Göttingen at the time, as the list of lectures in the Mathematical Society published in the *Jahresberichte der Deutschen Mathematiker-Vereinigung* shows. Mancosu [58] gives a survey of the developments going on in the early 1910s. He stresses in particular the role of Heinrich Behmann in introducing the mathematicians in Göttingen to the *Principia Mathematica*.

17. [33], call number 6817a.44a

18. Heinrich Behmann was completing his dissertation entitled “The antinomy of transfinite number and its solution by Russell and Whitehead” [*Die Antinomie der transfiniten Zahl und ihre Auflösung durch Russell und Whitehead*] under Hilbert in the Spring of 1918 (see [58]); it would be interesting to compare it with the presentation of the paradoxes and type theory in the 1917–18 lectures.

19. “Man beschränke den Bereich der Aussagen, indem man überhaupt nur die beiden Aussagen 0 und 1 zulässt, und deute dementsprechend die Gleichungen als eigentliche Identitäten. Ferner definiere man Summe und Produkt durch die 8 Gleichungen […] welche dadurch characterisiert sind, dass sie in richtige arithmetische Gleichungen übergehen, sofern man die symbolische Summe durch den Maximalwert der Summanden und das symbolische Produkt durch den Minimalwert der Faktoren ersetzt. Als Gegenteil der Aussage 0 erkläre man die Aussage 1 und als Gegenteil von 1 die Aussage 0.

Diese Definitionen führen jedenfalls zu keinem Widerspruch, da in jeder von ihnen ein neues Zeichen erklärt wird. Andererseits kann man durch endlich viele Versuche feststellen, dass bei den getroffenen Festsetzungen allen Axiomen I–XII Genüge geleistet wird. Diese Axiome können daher gleichfalls keinen Widerspruch ergeben. So lässt sich für unseren Kalkül die Frage der Widerspruchslosigkeit vollkommen zur Entscheidung bringen." [33], p. 70

20. Where Hilbert [33] uses '$=$', Bernays [5] uses the Russellian '$\sim$'.

21. The use of substitution is indicated at the beginning of *2. A substitution rule was explicitly included in the system of Russell [67], and Russell also acknowledged its necessity later (e.g., in the introduction to the second edition of *Principia*). For a discussion of the origin of the propositional calculus of *Principia* and the tacit inference rules used there, see O'Leary [61].

22. This becomes clear from Bernays [5], who makes a point of distinguishing between correct and provable formulas, in order "to avoid a circle." In [35, p. 8], we read: "It is now the first task of logic to find those combinations of propositions, which are always, i.e., without regard for the content of the basic propositions, *correct*."

23. This rule is tacitly used in *Principia*, but Russell's view that logic is universal prevented him from formulating it as a rule. Replacement "can be proved in each separate case, but not generally [...]" [74, p. 115].

24. [33], p. 144. There is actually a gap in the proof. Hilbert argues that since multiple substituitions can be reduced to successive single substitutions, only the cases where $\varphi(\alpha)$ is $\overline{\alpha}$, $\alpha\gamma$ and $\gamma\alpha$ need to be considered. Somewhere, however, induction has to play a role. What should be done is to prove that whenever $\alpha \to \beta$ and $\beta \to \alpha$ is provable then so are $\overline{\alpha} \to \overline{\beta}$, $\overline{\beta} \to \overline{\alpha}$, $\alpha\gamma \to \gamma\alpha$ and $\gamma\alpha \to \gamma\beta$, and then argue by induction on the depth of the occurrence of α in ϕ. Compare in this regard Post's [65, p. 170] proof of essentially the same result; his proof uses induction on the complexity of formulas.

25. "Dieses System von Axiomen wäre als widerspruchsvoll zu bezeichen, falls sich daraus zwei Formeln ableiten liessen, die zueinander in der Beziehung des Gegemteils stehen." [33], p. 150.

26. "Wenden wir uns nun zu der Frage der *Vollständigkeit*. Wir wollen das vorgelegte Axiomen-System vollständig nennen, falls durch die Hinzufügung einer bisher nicht ableitbaren Formel zu dem System der Grundformeln stets ein widerspruchsvolles Axiomensystem entsteht." [33], p. 152

27. "Dieses letzte Axiom trägt einen durchaus allgemeinen Character und ist in jedem Axiomensystem irgendwelcher Art in gewisser Form anzufügen; hier ist es, wie wir sehen werden, von ganz besonderer Bedeutung. Das Zahlensystem soll nach ihm so beschaffen sein, daß bei jeder Anfügung neuer Elemente Widersprüche auftreten, was für Festsetzungen man auch über sie treffe; lassen sich Dinge angeben die sich widerspruchslos anfügen lassen, so müssen sie dem Systeme in Wahrheit schon angehören." [31], p. 17

28. The possibility for such a move was of course already implicit in Hilbert's earlier writings on the foundations of geometry.

29. "Das aufgestellte Axiomen-System könnte kein besonderes Interesse beanspruchen, wenn es nicht einer bedeutsamen inhaltlichen Interpretation fähig wäre.

Eine solche Interpretation ergibt sich auf folgende Art:

Die Variablen fasse man als Symbole für *Aussagen* (Sätze) auf.

Als characteristische Eigenschaft der Aussagen soll angesehen werden, dass sie entweder wahr oder falsch und nicht beides zugleich sind.

Das symbolische Produkt deute man als die Verknüpfung zweier Aussagen durch 'oder,' wobei diese Verknüpfung nicht im Sinne der eigentlichen Disjunktion zu verstehen ist, welche das Zusammenbestehen der beiden Aussagen ausschliesst, sondern vielmer derart, dass 'X

oder Y' dann und nur dann zutrifft (d.h. wahr ist), wenn mindestens eine der beiden Aussagen X, Y zutrifft." [5], pp. 3–4.

30. "Die Bedeutsamkeit unseres Axiomen-Systems für die Logik beruht nun auf folgender Tatsache: Versteht man unter einer 'beweisbaren' Formel eine solche, die sich gemäss den Axiomen als richtige Formel erweisen lässt [footnote: Den Begriff der beweisbaren Formel neben dem der richtigen Formel (welcher nicht vollständig abgegrenzt ist) einzuführen, erscheint mir zur Vermeidung eines Zirkels als notwendig.], und unter einer 'allgemeingültigen' Formel eine solche, die im Sinne der angegebenen Deutung bei beliebiger Wahl der für die variablen einzusetztenden Aussagen (also für beliebige 'Werte' der Variablen) stets eine wahre Aussage ergibt, so gilt der Satz:

Jede beweisbare Formel ist eine allgemeingültige Formel und umgekehrt.

Was zunächst die erste Hälfte dieser Behauptung betrifft, so lässt sie sich folgendermassen begründen: Man verifiziert zuerst, dass sämtliche Grundformeln allgemeingültige Formeln sind. Hierzu hat man nur endlich viele Fälle auszuprobieren, denn die Ausdrücke des Kalküls sind alle von der Art, dass bei der logischen Interpretation ihre Wahrheit und Falschheit eindeutig bestimmt ist, wenn von jeder der für die Variablen einzusetzenden Aussagen feststeht, ob sie wahr oder falsch ist, während im übrigen der Inhalt dieser Aussagen gleichgültig ist, sodass man als Wert der Variablen anstatt der Aussagen nur Wahrheit und Falschheit zu betrachten braucht." [5], p. 6.

31. "Diese Betrachtung enthält nicht allein den Beweis für die Vollständigkeit unseres Axiomen-Systems, sondern sie liefert uns überdies noch ein einheitliches Verfahren, durch welches man bei jedem Ausdruck des Kalküls nach endlich vielen Anwendungen der Axiome entscheiden kann, ob er eine beweisbare Formel ist oder nicht. Zum Zweck dieser Entscheidung braucht man nur für den betreffenden Ausdruck eine Normalform zu bestimmen und nachzusehen, ob darin bei jedem der einfachen Produkte mindestens eine Variable sowohl unüberstrichen wie überstrichen als Glied vorkommt. Trifft dies zu, so ist der untersuchte Ausdruck eine beweisbare Formel, andernfalls ist er es nicht. Der Kalkül lässt sich demnach vollkommen trivialisieren." [5], pp. 15–16.

32. For biographical information on Post and his influences, see [17]. Davis [18] points out that some of the clarifications that Hilbert and Bernays achieved, e.g., the distinction between syntax and semantics, correct and provable formulas, and between theorems about the calculus and theorems in the calculus, were also seen by Lewis [53], who strongly influenced Post. In fact, the last of the distinctions just mentioned is emphasized by Post.

33. Recall that the axioms investigated by Hilbert and Bernays are not precisely the axioms of *Principia*. While Hilbert and Bernays augment the axiom system with an unrestricted substitution rule, Post's substitution rule allows only substitution of formulas containing one connective.

34. "In der Tat lassen sich ja alle Beziehungen zwischen Wahrheit und Falschheit von Aussagen mit Hülfe der Konjunktion ('und'), der Disjunktion (ausschliessendes 'oder') und der Negation, also auch durch die Symbolik unseres Kalküls zum Ausdruck bringen, und sofern solche Beziehungen für beliebige Aussagen gelten, müssen die ihnen entsprechenden symbolischen Ausdrücke in dem definierten Sinne allgemeingültige Formeln sein." [5], p. 16.

35. "Hilbert hat ja da [an den *Grundlagen der Mathematik*] eigentlich nicht mitgearbeitet, was da benutzt wurde waren sehr viele Gedanken von Hilbert, aber an der Ausgestaltung hat er eigentlich nicht mitgearbeitet, auch schon eigentlich beim ersten Band nicht und beim zweiten schon gar nicht." Interview, 27 August 1977 [10].

36. "[A]lso da war ich damals [Hilberts] Assistent [. . .] [die] Assistentenbeschäftigung, das ist nicht so eine Beschäftigung, wie sie hier [in Zürich] im allgemeinen die Assistenten haben, die den Studenten helfen bei den Übungen, damit hatte ich gar nichts zu tun, sondern das war ganz privatim bei Hilbert, also daß wir einerseits diskutierten über die grundsätzlichen Fragen

und dann auch, daß ich ihm für seine Vorlesungen zum Teil half, bei den Vorbereitungen mithalf und Ausarbeitungen machte.” Interview, 25 July 1977 [10].

37. Habilitationsakte Paul Bernays, Gemeinsames Prüfungsamt der mathematisch-naturwissenschaftlichen Fakultäten, Universität Göttingen.

38. Bernays Papers, ETH Library/WHS, Hs 976.3.

39. Bernays, “Kolleg von Hilbert über Grundlagen. Zu meiner Göttinger Habilitationsschrift.” Unpublished manuscript. Bernays Nachlaß, WHS,ETH Zürich, Hs 973.184

40. “Diese Fragen, die die Axiomatik der elementaren Verknüpfungsaussagen betreffen, sind vor wenigen Jahren von dem Göttinger Mathemathiker Bernays behandelt worden (Habilitationsschrift, leider nicht gedruckt) und man kann wohl sagen zu einem vollständigen und befriedigendem Abschluß gebracht worden. [. . .] Bernays hat die Vollständigkeit ebenfalls streng bewiesen, also gezeigt, daß jede allgemeingültige elementare Verknüpfungsaussage tatsächlich aus den Grundformeln nach den Grundregeln abgeleitet werden kann [. . .]. ” [4], p. 97.

41. “Damals zum Beispiel ja, auch meine eigenen Kentnisse waren da noch sehr sehr unvollständing, als ich da zunächst, zum Beispiel, 1917, damals. Wie gesagt, zuerst bevor Hilbert direkt an seine Sache wieder ging, die er ja schon viel früher angefangen hatte, da war, da hat er noch nicht gleich darüber gelesen, sondern er hat über mathematische Logik gelesen, eine Vorlesung. Und die hab ich auch ausgearbeitet und die hab ich, nicht wahr, und zwar in solcher Weise, daß ich die freien, das hab ich Ihnen glaub ich erzählt, daß ich die freien Variablen vermieden habe. Bei Russell hatte ich mir so ein bißchen einiges angekuckt aber erstens war mir das überhaupt zu breit diese Art der Behandlung, sagte mir nicht in jeder Hinsicht zu, aber insbesondere hab ich das nicht recht verstanden, was das heisst für alle x, $F(x)$ dann folgt $F(y)$. Tatsächlich ist ja auch die Anwendung der freien Variablen, das ist etwas Technisches, nicht wahr. Es sind eigentlich zwei Arten der Darstellung der Allgemeinheit. Man hat die Allgemeinheit eben einerseits durch die gebunde Variable und andererseits durch die freie Variable. Solch einen Unterschied gibt es nicht in der gewöhnlichen Sprache, nicht wahr. Nun hab ich also da zunächst die freien Variablen vermieden. Das ist ein mögliches Verfahren, das ist auch später wieder von anderen manchmal gemacht worden so. Das ist also eine Vorlesung, die ist ausgearbeitet worden und hat auch nachher da im Hilbertschen, da im Lesezimmer vom Institut gestanden.” Interview, 27 August 1977 [10]

42. The use of free variables was also avoided in lectures on the *Logik-Kalkül* [35]. Free variables are first used in Hilbert’s talks of 1922 [37, 38] and in lectures taught by Hilbert and Bernays in 1922–23 [45, 44].

43. See, e.g., Lauener’s [52] testimony.

44. “Es wird also bei jedem der folgenden Unabhängigkeits-Beweise der Kalkül auf ein endliches System (eine endliche Gruppe im weiteren Sinne des Wortes [footnote in text: Das heisst ohne Voraussetzung des assoziativen Gesetzes und der eindeutigen Umkehrbarkeit der Komposition]) zurückgeführt, für dessen Elemente eine Komposition (‘symbolisches Produkt’) und eine ‘Negation’ definiert ist, und diese Zurückführung findet in der Weise statt, dass die Variablen des Kalküls auf die Elemente jenes Systems als ihre Werte bezogen werden. Die ‘richtigen Formeln’ sollen jedesmal dadurch characterisiert sein, dass sie für beliebige Werte der vorkommenden Variablen nur Werte eines gewissen Teilsystems T annehmen.” [6], pp. 27–28.

45. The interested reader may consult [50], pp. 689–694, and, of course, [6]. The method was discovered independently by Łukasiewicz [54], who announced results simlar to those of Bernays. Let me remark in passing that Bernays’s first system defines Łukasiewicz’s 3-valued implication.

46. Gödel [26] quotes the independence proofs given in [39].

47. $\overline{X}X$, of course, is the principle of the excluded middle, and is synonymous in the system with $X \rightarrow X$.

48. These results extend the method of the previous sections insofar as the independence of rules is also proved. To do this, it is shown that an instance of the premise(s) of a rule always takes designated values, but the corresponding instance of the conclusion does not. This extension of the matrix method for proving independence was later rediscovered by Huntington [48].

49. According to the *Verzeichnis der Vorlesungen* for the semester, the course was announced under the title "Formal logic and its epistemological value [Formale Logik und ihr erkenntnistheoretischer Wert]." The term lasted 5 January 1920–31 March 1920. Lecture notes by Bernays survive at the library of the Institute of Mathematics at the University of Göttigen [35].

50. This is not stated explicitly, but is evident from the derivation on p. 11.

51. [3], p. 167.

52. "Bei der Axiomatisierung des Aussagenkalküls herrscht die Tendenz vor, die Anzahl der Grundverknüpfungen und damit die der Axiome zu reduzieren. Man kann aber andererseits auch die Rolle der verschiedenen Verknüpfungen scharf voneinander sondern; insbesondere ist es von Interesse, die Rolle der Negation zu untersuchen." Talk given in the Mathematical Society at Göttingen, 20 February 1923, as reported in *Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung*, vol. 32 (1922), p. 22.

53. Compare, e.g., the logical axioms in [37] and [38]. In the latter paper, Hilbert notes: "In [37] I had still avoided [the negation sign]; as it turned out, the sign for 'not' can be used in the present, slightly modified presentation of my theory without danger." [38, p. 152] He could not have avoided the negation sign if the whole calculus was based on it.

54. Paul Bernays, notes to "Mathematische Logik," lecture course held Winter semester 1929–30, Universität Göttingen. Unpublished shorthand manuscript. Bernays Nachlaß, WHS, ETH Zürich, Hs 973.212. The signs '&' and '$\vee$' were is first used as signs for conjunction and disjunction in [45]. The third axiom of group I and the second axiom of group V are missing from the system given in [46]. The first (*Simp*), third (*Comm*), and fourth axiom (*Syll*) axioms of group I are investigated in the published version of the *Habilitationsschrift* [6], but not in the original version [5].

55. For a discussion of the history of Hilbert's *Foundations of Geometry*, and in particular of the completeness axiom, see Toepell [72, pp. 254–256] and Birkhoff and Bennett [12].

56. "Ebenso ist das 'Vollständigkeitsaxiom' eine an dem fertigen Denkbereiche zu erlangende Anschauung; die 'Vollständigkeit' ist ein Postulat, das in unserer Synthese überhaupt nicht als 'Axiom' des Denkbereichs gefaßt werden kann; ebensowenig wie das Postulat der Widerspruchslosigkeit." [51], p. 209. König's book was known in Göttingen: Felix Bernstein reported on it in the Mathematical Society on 16 February 1915.

57. See [36], pp. 18–19, where both the distinction and the equivalence are pointed out.

58. "Weiters interessiert uns die *Vollständigkeit* des Axiomensystems. Wir werden verlangen müssen, dass alle übrigen Thatsachen des vorgelegten Wissensbereiches Folgerungen aus den Axiomen sind," [31], p. 13.

59. I would like to just mention as two more possible influences the work of the American postulate theorists (Huntington, Veblen) on categoricity [15, 16], and the exchange between Husserl and Hilbert on completeness in 1901, recently analyzed by Majer [55].

60. The pivotal role that Bernays [5] played in the shift from syntactic to semantic completeness is stressed by Moore [60].

61. "Es soll eine ganz bestimmte allgemeine Vorschrift angegeben werden, die über die Richtigkeit oder Falschheit einer beliebig vorgelegten mit rein logischen Mitteln darstellbaren Behauptung nach einer endlichen Anzahl von Schritten zu entscheiden gestattet, oder zum mindesten dieses Ziel innerhalb derjenigen — genau festzulegenden — Grenzen verwirklicht

werden, innerhalb deren seine Verwirklichung tatsächlich möglich ist." [3], p. 166, emphasis mine. This was Behmann's *Habilitationsschrift*, he received his *venia legendi* in July 1921. Behmann spoke on his results to the mathematical society in Göttingen on 10 May 1921, the talk was entitled "Das Entscheidungsproblem der mathematischen Logik" (*Jahresberichte der Deutschen Mathematiker-Vereinigung, 2. Abteilung*, vol. 30 (1921), p. 47). The manuscript of the talk survives in the Behmann Papers in Erlangen. This seems to be the first documented use of the expression "Entscheidungsproblem." Behmann had requested leave from his teaching duties in late September 1920 to work on his Habilitation, the problem was probably formulated in its full generality sometime in early to mid-1920.

62. The notion of syntactic completeness of a theory is closely related to what we now call "complete theories.," i.e., theories which either prove or refute every sentence of the language. [41] proposes the proof of syntactic completeness of arithmetic as a finitistic analog of the proof of completeness in the sense of categoricity. That completeness and categoricity are not the same was realized only with Skolem's discovery of nonstandard models.

63. See [60] and [20] for a discussion of Gödel's motivations and influences.

64. See in this regard the introduction to [24], and [19].

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