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BETWEEN RUSSELL AND HILBERT: BEHMANN ON THE FOUNDATIONS OF MATHEMATICS

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Abstract. After giving a brief overview of the renewal of interest in logic and the foundations of mathematics in Göttingen in the period 1914-1921, I give a detailed presentation of the approach to the foundations of mathematics found in Behmann's doctoral dissertation of 1918, *Die Antinomie der transfiniten Zahl und ihre Auflösung durch die Theorie von Russell und Whitehead*. The dissertation was written under the guidance of David Hilbert and was primarily intended to give a clear exposition of the solution to the antinomies as found in Principia Mathematica. In the process of explaining the theory of Principia, Behmann also presented an original approach to the foundations of mathematics which saw in sense perception of concrete individuals the Archimedean point for a secure foundation of mathematical knowledge. The last part of the paper points out an important numbers of connections between Behmann's work and Hilbert's foundational thought.

§1. Logic and Foundations of Mathematics in Göttingen from 1910 to 1921. Recent work on Hilbert's program has focused, among other things, on the development of logic in Hilbert's school and on the philosophical underpinnings of the program. Sieg [30] and Moore [26] have investigated the development of first-order logic in Hilbert's 1917–18 lectures, Zach [37] has given an in-depth analysis of the propositional calculus in Hilbert's school from 1918 to 1928, and Mancosu [25] has investigated the philosophical context of Hilbert's approach to the foundations of mathematics. The *Habilitationsschrift* by Bernays [8] and Hilbert's 1917–1918 lectures [19] represent the starting point of these important developments. However, these lectures were not the product of a sudden reawakening of interest in logic and the

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foundations of mathematics within the Göttingen mathematical community. Rather, they were prepared by previous work in foundational issues which began around 1914. Central to these efforts was, in addition to Hilbert's charismatic role, the work of Heinrich Behmann and, starting in 1918, that of Bernays. Another factor which should not be underestimated is the presence in Göttingen in the early twenties of two Russian mathematicians, Schönfinkel and Boskovitz, who were also engaged in foundational work. Schönfinkel is well known for his article "Über die Bausteine der mathematischen Logik," all of whose contents go back to a talk he gave in Göttingen in 1920.[1] Moreover, the paper by Bernays and Schönfinkel [13] goes back to work by Schönfinkel in 1922 (see [13], p. 350). We know little about Boskovitz but he is, together with Behmann, thanked in the second edition of *Principia Mathematica* [36] (henceforth PM) for his help in correcting various oversights found in the first edition.[2] It is during the period from 1918 to 1922 that the major ideas and some results about metatheoretical investigations, such as completeness and decidability, are formulated and established with the appropriate degree of logical clarity.

If we look at the development of logic and the foundations of mathematics in Hilbert's thinking and in Göttingen in general in the early part of the 1910s we find very little of interest. At the time Hilbert seems to have been very busy with problems in the foundations of physics. Moreover, the Göttingen Mathematical Society does not list in its very active Colloquium any talks on the foundations of logic and mathematics in the period 1910–1913. The situation changes suddenly in 1914.[3] A list of the lectures given in logic and foundations of mathematics at the Colloquium between 1914 and 1921, as reported yearly in the *Jahresbericht der Deutschen Mathematiker-Vereiningung*, gives an idea of the increasing interest in logic and foundations in Göttingen from 1914 onwards.

February 17, 1914, D. Hilbert, Axiome der ganzen Zahlen

December 1, 1914, H. Behmann, Über mathematische Logik [1]

December 18, 1914, F. Berstein and K. Grelling, Über mathematische Logik. Ergänzungen und genauere Ausführungen zum Referat vom 1. Dezember

February 16, 1915, F. Bernstein, Über J. Königs "Neue Grundlagen der Logik, Arithmetik und Mengenlehre"

November 7, 1916, E. Zermelo, Über einige neuere Ergebnisse in der Theorie der Wohlordnung

November 14, 1916, H. Behmann, Die Russell-Whiteheadsche Theorie und die Paradoxien [2]

July 3, 1917, H. Behmann, Die Russell-Whiteheadsche Theorie und die Grundlagen der Arithmetik [2]

July 10, 1917, H. Behmann, Die Russell-Whiteheadsche Theorie und die Grundlagen der Arithmetik (Schluß) [2]

July 17, 1917, F. Bernstein, Geschichte der Mengenlehre

July 24, 1917, F. Bernstein, Geschichte der Mengenlehre (Schluß)

July 31, 1917, D. Hilbert, Referat über seine Vorlesungen über Mengenlehre [17]

November 20, 1917, P. Bernays, Weyl über die Grundlagen der Analysis [7]

November 27, 1917, D. Hilbert, Über axiomatisches Denken [20]

December 7, 1920, M. Schönfinkel, Elemente der Logik [29]

February 1 and 8, 1921, R. Courant and P. Bernays, Über die neuen arithmetischen Theorien von Weyl und Brouwer

February 21, 22, D. Hilbert, Eine neue Grundlegung des Zahlbegriffes [21]

May 10, 1921, H. Behmann, Das Entscheidungsproblem der mathematischen Logik [4]

December 6, 1921, P. Bernays and M. Schönfinkel, Das Entscheidungsproblem im Logikkalkül

Even a cursory look at the above list of talks should convince the reader that Behmann was a central player in the development of logic and the the foundations of mathematics in Hilbert's circle. I will try to spell out how knowledge of *Principia Mathematica* (henceforth PM) was acquired in Göttingen, thereby shedding light on the important problem of the relationship between Russell's logicism and Hilbert's program. After presenting a short account of Behmann's early career I will proceed to investigate his role as trait d'union between Russell's logicism and Hilbert's program.[4]

§**2. Heinrich Behmann's early career.** Heinrich Behmann was born in 1891. He attended the Realgymnasium in Vegesack (Bremen), graduated in 1909 and subsequently enrolled at the University of Tübingen, where he spent two semesters to study mathematics and physics. Later he moved to Leipzig, where he spent three semesters, and eventually to Göttingen in 1911. In 1914, Behmann volunteered for military duty. He was wounded in Poland in May 1915, which required a prolonged stay in a hospital. He resumed his studies in Göttingen in 1916. On June 5, 1918 he successfully defended his dissertation *Die Antinomie der transfiniten Zahl und ihre Auflösung durch die Theorie von Russell und Whitehead.* His advisor was David Hilbert. During his studies he had attended courses by, among others, Hilbert, Landau, Nelson, Reinach, and Weyl. From 1919 to 1921 he worked on his *Habilitationsschrift* [4]. On July 9, 1921 he completed the Habilitation and obtained the venia legendi. From 1921 to 1925 he was active as Privatdozent at the institute for mathematics in Göttingen, where Hilbert and Bernays were also active. Behmann left Göttingen in 1925 following a call to the University of Wittenberg (Halle).[5]

§3. Behmann's 1914 lecture on *Principia Mathematica* in Göttingen. We have seen from the above list of lectures that Behmann spoke on Russell's system in 1916 and 1917. However, his lecture from December 1, 1914 "Über mathematische Logik" was also advertised in the *Jahresberichte* as an introduction to the system of PM ("Der Vortrag bringt einige neue Gedanken aus den in den 'Principia mathematica' von Russell und Whitehead niedergelegten logischen Untersuchungen"). One would of course like to know what aspects of PM were emphasized in this first public introduction of Russell's work to the mathematical community in Göttingen. Fortunately, our curiosity can be satisfied. The lecture is extant in its entirety in the Behmann Nachlaß although it does not appear in the list of the documents contained in the Nachlaß compiled by Haas and Stemmler [15]. I will here stress only two aspects of this lecture that seem to be quite relevant for later developments in Hilbert's program. Let me also remark that on December 18, 1914 Bernstein and Grelling gave a follow-up talk about PM at the Colloquium in Göttingen.

Behmann began the lecture by stressing the fact that the term "Mathematische Logik" is ambiguous and can be used to characterize two different traditions. The first tradition consists in a general construction of logic by mathematical means ("Mathematik der Logik") and is associated with the names of Boole, Schröder, and partly Peano. The second tradition ("Logik der Mathematik") analyzes the role played by logic in the construction of mathematics. Behmann mentions Bolzano, Frege, and Russell as representative of this direction of work. Although PM belongs squarely in the second tradition it also accounts for results developed in the first tradition. Thus, for Behmann, PM is the first unified account of these two traditions.

Behmann moves on to introduce the main concepts of individual, proposition and propositional function. As I will come back to these issues when analyzing Behmann's dissertation let me simply remark that individuals are defined as "everything which in reality is directly given to us, thus not, say, things like sets or similar conceptual entities [Gedankenbilde]" He then goes on to discuss the concept of propositional function, the vicious circle principle, and type theory. The word "type" is translated as "Stufe" and Behmann asserts that he does this in reference to Frege's use of concepts of first and second order. After discussing the complications due to the ramified constructions Behmann wonders whether the solution proposed by PM is not too drastic. It is true, he says, that the paradoxes are avoided but at the same time many propositions, which seem perfectly innocuous, are in this way excluded and declared meaningless, e.g. "All propositions are either true or false." Concerning the paradoxes, whose solution "must be seen as a vital question for logic and especially for arithmetic," Behmann goes on to describe Russell's solution with the slogan "There are no classes at all." It is clear that Behmann has read more than PM, as he speaks of several unsuccessful previous attempts by Russell to solve the paradoxes and then mentions that the slogan given above captures in a nutshell what in the literature is called the "no-class theory." In this connection Poincaré is also quoted.

There are now two points in the remaining part of the lecture which deserve emphasis. The first concerns an explication of the no-class theory in terms of ideal elements in mathematics. Behmann uses the example of postulation of points at infinity in projective geometry. After expounding the conceptual difficulties encountered by the student presented with such concepts for the first time Behmann goes on to dispel the puzzlement as follows:

> Nevertheless, the correct solution of all these geometrical inconsistencies, once it is given, is not difficult to comprehend. To wit, it simply states that the ideal elements are not objects of geometry in the proper sense, but are first of all only words. As such they are parts of ways of speaking which one uses in geometry to bring a certain class of propositions into a form which is as simple as possible.[6]

Behmann then asserts that one only needs to generalize the above reflections on projective geometry to arrive at the no-class theory. As terms purporting to refer to points at infinity can always be eliminated from a sentence in which they occur, likewise can words like "class," "totality" etc. be eliminated from sentences in which they occur in favour of equivalent sentences in which these words do not occur.

It will be obvious to the reader that the role of ideal elements in mathematics, so much emphasized by Hilbert in his foundational work in the 1920s, is here brought to the fore with greatest clarity in connection to an explication of the main theses of Principia. There is no need to follow in detail Behmann's explication of how the strategy is supposed to work. More important is to stress another point of connection to Hilbert's work that emerges from this talk. This relates the axiom of reducibility—which claims that any class definable by a propositional function can be defined by a propositional function of the lowest level (i.e., a predicative function)—to completeness axioms in Hilbert's sense:

> The axiom thus states that in this one order there already are enough functions to define all possible classes; it can therefore be viewed as a kind of completeness axiom for the predicative functions.[7]

We will see that Hilbert was particularly interested in this intepretation of the axiom of reducibility.

Behmann gave three more talks on the system of Principia between 1916 and 1917. We do not have the texts of these talks but we can conjecture that they were preparatory work to his 1918 dissertation to which we now turn.

§4. Behmann's dissertation. Behmann's dissertation *Die Antinomie der transfiniten Zahl und ihre Auflösung durch die Theorie von Russell und*

Whitehead was written in 1918. The original is still preserved at the Niedersächsische Universitätsbibliothek in Göttingen. In 1922 Behmann published an account of the main contents of the dissertation in the *Jahrbuch der Mathematisch-Naturwissenschaftlichen Fakultät* with the same title [5]. The original dissertation is 351 pages long whereas the article is only 10 pages long. It is clear that much of the contents of the dissertation could thus only be mentioned in the article and much was glossed over.

The dissertation aims in the first place at giving an introduction to the system of PM. However, as not all of PM could be treated, Behmann selected the topic of the foundations of the natural numbers including the developments of Cantorian set theory. The topic is developed by using the solution of the antinomies of transfinite numbers—by means of the system developed by Russell and Whitehead—as a unifying theme.

We have seen that interest in the system of PM was alive in Göttingen since 1914 and one should consider that Hilbert himself was quite attracted by the logicist solution to the foundations of mathematics and for a while seems to have considered the logicist solution valid. Moore [26, p. 85] and Sieg [30, pp. 3, 11] have no doubts that Hilbert defended a logicist point of view in 1917–18 although he later became disenchanted with Russell's solution (see [20, p. 412, trans. p. 1113] for a strong praise of Russell's achievements). During this period Hilbert supervised Behmann's dissertation and also encouraged Bernays to write a *Habilitationsschrift* on topics related to the logical system of PM [8]. However, whereas Bernays' 1918 Habilitation centers on logic, Behmann's works skips over the logical parts of *Principia* to focus instead on the mathematical developments more immediately related to the construction of arithmetic and of Cantorian set theory. Behmann also took, at least up to 1922, the logicist solution to the foundations of mathematics to be successful.[8]

Behmann's dissertation remained unpublished due to the difficult situation of the post-war period. A letter from Behmann to Russell dated August 8, 1922, gives an account of the reasons that prompted Behmann to work on PM. He had discovered his passion for logic through PM but the difficulty of the exposition had led him to the idea of writing an introduction to it.[9]

Then Hilbert suggested a more focused approach to the topic:

> Prof. Hilbert then proposed to me, as a theme of dissertation, not to treat the whole work as such, but rather to make clear the particular way by which the plainly most serious among the antinomies of the Theory of Aggregates, that concerning the transfinite cardinal and ordinal number, is avoided by the logical theory of the Principia Mathematica. So I tested the theme in the way suggested by Prof. Hilbert, adding critical comments wherever I believed it necessary or desirable.

> But unhappily, on account of the want of paper, which was a consequence of the War and the Revolution strongly felt in our public life, there was no possibility of having a book printed "having a character extremely scientific and addressed to a small class of readers"—as the publisher remarked—, so that I am very sorry to not to be able to send you a copy of it.

Russell did actually get to see a copy of the work, through the Göttingen library, and judged it "a valuable and important piece of work."[10]

In order to give an overview of the contents of the dissertation it might be useful to list the contents of the six chapters that make it up. The first chapter (pp. 1–43) considers the formulation of the problem and the way to its solution. The second chapter (pp. 44–89) treats sentences and functions up to the first order. The third chapter (pp. 90–147) is devoted to functional logic in general. The fourth chapter (pp. 148–225) treats of the logic of classes. Chapter five (pp. 226–283) develops cardinal arithmetic. Finally chapter six (pp. 284–351) discusses the general results and more general philosophical questions. Of course, given the length of the work it is not possible to give a detailed account of its contents. In [5], Behmann emphasized the technical differences between his own exposition and that given in PM. In my analysis I will not emphasize the technical reconstruction but rather the philosophical outlook defended by Behmann in this period. I believe that Behmann's dissertation sheds additional light on the approach to the foundations of mathematics held by Hilbert and his co-workers during this period.

§**5. Abstract terms and context principle.** Behmann begins his dissertation by posing the problem of certainty in science. The development of arithmetic and the discovery of Cantorian set theory have shown that also within arithmetic there is room for doubt. Together with the mathematical fruitfulness came also a number of contradictions which could not be solved by the available logic and mathematics. However, the freedom originated by the Cantorian approach to mathematics should not be limited on accounts of the doubts originated by the paradoxes.

These antinomies, according to Behmann, are of different sorts (p. 3) and affect mathematics in different ways. The paradox of number [Paradox der Zahl] lies at the heart of mathematics. Rather than rejecting Cantorian mathematics one ought to analyze the mathematics of the finite and in particular the concept of number (p. 4). A solution in this direction has been given by symbolic logic (or mathematical logic) and in particular by PM. Given the complexity of PM, Behmann proposes to give an exposition of the system by developing only what is necessary to show how the paradox of number can be solved in this theory (p. 6). In the cardinal form the paradox is nothing else but Cantor's paradox of the cardinality of the universal set. In its ordinal form it resembles the Burali-Forti paradox. Behmann's emphasis is on the cardinal version of the paradox, and this is motivated by pedagogical reasons. At this point Behmann introduces the relevant set-theoretic notions. In this connection he announces that although he is not dealing with an axiomatic project he will use the axiom of choice as sparingly as possible. After presenting various cardinal and ordinal forms of the antinomies Behmann declares that the goal of the investigation is to free mathematics from the antinomies and that this should not be done by first constructing a pure logic but rather he will try to ground arithmetic so that the logical forms allowable will become clear through this foundational work (pp. 31–32). §5 (pp. 33–39) contains some of the most interesting methodological reflections. Antinomies arise, says Behmann, when two sentences stand in contradiction. In arithmetic—and Behmann explicitly includes also transfinite set theory under this name—such sentences contain concepts such as number and set. However, the concepts themselves cannot give rise to antinomies or contradictions. Contradictions only arise when the concepts are used in sentences. This leads Behmann to state a form of context principle:

> As a result, it would be a mistake from the start to try to investigate the abstract concepts and relations of arithmetic in themselves alone without bearing in mind the fact that all such terms mean something only in the context of the *sentence*, and that even the best-constructed concepts are of no use to us and cannot protect us from contradiction until we know how to *apply* them properly.[11]

This goes hand in hand with giving the concept of sentence pride of place in the foundational investigation:

> ... thus, as the latter consideration shows, we must now, from the outset, place the concept of *proposition* in the center of all our investigations. This means we may never regard concepts occurring in propositions—like "set," "number," "statement," "space," "force," etc.—in isolation, as independent logical objects, but always in the context of the propositions in which they can meaningfully appear.[12]

Behmann does not use the term 'context principle', he speaks of *Erkenntnis-zusammenhang* (epistemic context). The project now consists in developing a consistent system of sentences in which arithmetic can be developed.[13]

The last section of chapter 1 presents the six assumptions that will be used in the remaining part of the work (p. 40):

1. Axiom of individuals;
2. Axiom of true and false sentences;
3. Axiom of negation and disjunction;
4. Axiom of the variation of the constants and generalization;
5. Axiom of individual identity;

6. Axiom of reducibility.

In the next section I will spell out how the first four axioms are appealed to in Behmann's reconstruction.

§6. Individuals, propositional functions, and classes. It is customary to develop arithmetic by taking for granted the system of natural numbers. Given the possibility of the emergence of the antinomies this approach is, according to Behmann, barred for foundational research. Moreover, procedures involving abstract objects are also not to be allowed:

> Since, due to the nature of our object, we cannot even presuppose the simplest logical concepts in their traditional use as given, we must thus initially mistrust *all* abstract thinking. But since we need some kind of starting point, it must be such that our abstract thinking is not yet involved, whose existence is not—unlike the existence of numbers—based only on the possibility of thought, and thus cannot be falsified by thought. The only things satisfying this requirement, however, are now those that can be recognized directly without the aid of thought, i.e., solely through sense perception, which thus provide thought with the prerequisite material for its possibility. (So far, almost all false metaphysics suffered from the effort of wanting to create the objects of thought through thinking. As everywhere, however, here too the Archimedean point necessarily lies outside.) Thus, our *initial assumption* is that it is permissible to *take as given objects of experienced reality* (individuals) *whose existence precedes all thought.* (With this, of course, we consider ourselves justified in completely ignoring any possible distortion through perception.) We thus regard these individuals as already given and directly at the disposal of our thought.[14]

It is essential to remark that for Behmann reliance on empirical reality provides a consistency proof for the system:

> Since—as we may here undoubtedly assume—the objective world, i.e. the totality of the individuals with all their properties and relations, certainly constitues a consistent domain in the end, then clearly *every* proposition that has only individuals as objects, and which thus does not presuppose the existence of other things, must either correspond to or contradict the actual case, and thus fulfill the principles of contradiction and of the excluded middle.[15]

What is essential for Behmann is that these individuals can be considered as existent and they serve as primitive material to our thinking (p. 46). Moreover, Behmann asserts (p. 47) that these individuals have to be considered as a fixed domain that cannot be altered.[16] By means of properties grasped empirically [empirisch aufzufassenden Eigenschaften und Wechselbeziehungen der Individuen] Behmann defines the elementary propositions [einfache Aussagen] as those in which a direct perception is expressed.[17]

In order to be able to say something about these individuals it is necessary to use "elementary propositions." This leads to the second assumption: it must be possible to make consistent elementary propositions concerning individuals. More precisely,elementary propositions are characterized by containing only individuals and their objective properties. By contrast, abstract concepts such as "proposition," "property," "set," "number," and "existence" cannot occur in them. Moreover, universal and existential quantification cannot occur in them because that would contain an implicit appeal to sets. In Behmann's characterization an elementary proposition is such that by varying the name of individuals we always get a meaningful sentence. This is not true were we to allow abstract objects ("the sentence is green" is simply meaningless). The elementary propositions are, as soon as they are about specific individuals, either true or false. However, Behmann says that one should not say that propositions exist but only that the individuals with their empirical properties exist (p. 49). Talk of the collection of all elementary propositions cannot lead, according to Behmann, to antinomies as a proposition that refers to this collection is not elementary (p. 50).

In §8 Behmann states that the elementary propositions are closed under negation and disjunction. Implication and conjunction are defined in the usual way. We can also consider an elementary proposition without thinking of any specific individual. This leads to generalization: from fa to $f\hat{a}$, and to the notion of propositional function $f\hat{a}$. We are at this point still not justified in treating these propositional functions as objects of discourse. The fourth assumption thus leads to assert that given any elementary proposition it must be allowed to vary at will the individuals in order to construct the associated elementary propositional function. Moreover, it must be allowed to create new propositional functions or propositions by means of existential and universal generalization. The propositional functions, resp. propositions, thus obtained are called first-order functions or first-order propositions (p. 60). It is important to remark that propositions about propositional functions are only apparently about them and should be paraphrased as being about the individuals. Behmann clarifies that parts of sentences such as "no object," "proposition," "set," "function," "numbers" should be interpreted only as formal components of propositions. Let us simply give two quotes as paradigms for the many passages in which Behmann insists that all entities not reducible to the individuals are not existent.[18] On p. 106, by making use of the comparison with the differential calculus, Behmann reasserts that propositional functions are not real:

> In the same sense [as the differential dx], however, propositional functions, like all abstracta, are not real objects, but create only the

> deceptive appearance of object-hood by providing certain resting points for abstract thought moving at a great distance from its real, concrete objects.[19]

When we move to classes we find the same strategy of deontologization of anything which is not an individual or an empirical fact, a strategy which is of course the Russellian "no-class theory"[20]:

> The classes—and by the same token, incidentally, the numbers—are thus, as we earlier hinted, nothing else than figures of speech extremely useful for ease and clarity in presenting arithmetic, but can nonetheless become quite problematical as soon as one takes them seriously and, in violation of their nature, takes them for the names of objects.[21]

It should be clear from the above quotes that the general philosophical project defended by Behmann consists in reducing appeal to abstract entities to a formal construction with no ontological import. When we are clear about this it also becomes possible, within certain contexts, to treat the abstract objects *as if* they were individuals.[22]

I will now move to the last, and most philosophical, part of the dissertation.

§7. The philosophical foundations of arithmetic.

7.1. Experience of concrete objects as the foundation of arithmetic. In the last chapter of the dissertation Behmann comes back to his approach to the foundations of arithmetic described in the earlier part of the dissertation and engages in a number of interesting philosophical reflections. The system presented should of course recover the full power of ordinary arithmetic but his account of numbers in terms of the role played by numerals in a sentence should not be interpreted as formalism:

> the primary advantage of the theory proposed here is that our arithmetic in its entire development is in no way merely a playing with signs, but has from the beginning an objective content.[23]

In particular, Behmann stresses that whereas formalism cannot account for the applicability of mathematics to reality, in his approach this is taken care of by the very construction of the system. The request that all of mathematics refers in the last analysis to an empirical reality can of course raise issues concerning the problem of whether we can have only arithmetical statements to which an experience could correspond. On this view, not only can any statement involve only finitely many entities but indeed there must be a certain finite bound on the number of entities in any possible experience. The only reply given by Behmann is that accepting this limitation would make it impossible to obtain a complete and simple theory. In any case, when we generalize beyond experience by means of axioms we do indeed lose the possibility of checking every statement directly but, on the other hand, every single theorem is such that every possible experience must conform to it.

Concerning the appeal to concrete objects as the foundations of arithmetic Behmann provides a historical sketch that shows how the Russellian position is the natural development of a sequence of events in the foundations of mathematics. He mentions Dedekind's reduction of the irrationals to cuts of rationals and Kronecker's arithmetical standpoint. Kronecker criticized contemporary analysis and did not realize that the concept of number could be reduced to that of set. This latter step was taken by Frege. However, Frege's development led to contradictions and it was Russell who eliminated the notion of set in favour of that of concrete individuals.[24] The antinomies show that sets and relations, propositions, and properties cannot be the starting point:

> Since all abstract things—at least to the degree that they can enter into any relationship with arithmetic—are thus already excluded, all that remains are the concrete objects of experienced reality.[25]

Behmann's position thus ends up emphasizing the non-existence of the natural numbers except as a purely formal part of sentences:

> Arithmetic in truth does not have its own objects at all, but assumes rather no other object-hood than the other sciences do (which need not be imagined as real, but as at least possible), that is that of experience. Since we may have every right to assume that every reasonable statement about this reality conforms to the law of contradiction, the consistency in arithmetic is thus fundamentally ensured.[26]

However, the attempt to provide objectivity for arithmetic is often at odds with statements which sound like a straightforward instrumentalist position:

> we grasp the numbers as mere formal components of propositions ... In this sense, the numbers, too, are nothing other than a means of easily deriving and expressing specific bits of knowledge; but aside from this purpose, in themselves, they are nothing at all.[27]

Of course, when engaged in practical arithmetical work we do not need to go back to the empirical intuitions that form the objective content of our reflections. But the relation of abstract thought to concrete experience is best analyzed in terms of the problem of the aprioricity of arithmetic.

7.2. The aprioricity of arithmetic. In §51 of the dissertation Behmann tries to escape the almost inevitable conclusion that arithmetic becomes in this way a posteriori knowledge. He asks himself: how can one reconcile the aprioricity of arithmetic with such emphasis on its natural connection to experience? For Behmann the question becomes equivalent to that of whether the essence of a priori knowledge consists in having no relationship to experience and in not presupposing the latter as a condition of possibility for its significance. In other words, if we make experience an essential condition of possibility of arithmetic how can we preserve its aprioricity? Behmann holds that mathematical knowledge is synthetic a priori. According to him the essential point is that "synthetic a priori judgments are only possible in that they bring to expression the form of a possible experience" (p. 339). His claim is that the essential feature of a priori knowledge consists not in having actual experiences of a certain sort or other but rather that every possible experience must accord to it. To strengthen his position he also gives a long list of quotes from Kant and Schopenhauer to the effect that a priori knowledge is essentially tied to the possibility of experience. Consider for instance the following quotation by Schopenhauer:

> Since, as has been shown, the concepts borrow their material from intuition, and since as a result the entire construction of our world of thought is based on the world of intuition, we must thus be able, though perhaps via intermediary steps, to trace each concept back to the intuitions from which it itself directly derives or back to the concepts from which it is abstracted: i.e. we must be able to exhibit it with intuitions that relate to the abstractions as examples. These intuitions thus provide the true content of all our thought, and wherever they are lacking we have not had true concepts in mind, but mere words.[28]

Behmann goes as far as claiming that Schopenhauer had provided with his insights the philosophical foundation for the type-theoretical project presented in PM. Of course, Behmann is aware of the rather paradoxical fact that he has given an account of Russell's construction in terms of synthetic a priori knowledge, whereas Russell himself was, like the majority of logicist writers, quite anti-Kantian. One final thing should be emphasized about Behmann's attempt to put together empirical content and aprioricity of mathematics. The account makes no mention of pure intuition which for Kant was essential for claimimg the synthetic a priori nature of mathematics. In Behmann's case the synthetic part is given by empirical experience. However, for aprioricity his claims only amount to the fact that accord with experience and applicability are not in conflict with the aprioricity of mathematics. Unfortunately, what is lacking in Behmann is a positive account to the effect that mathematics is not empirical and indeed synthetic a priori.

7.3. Mathematical objects as fictions: Relationship to Vaihinger's philosophy of the As If. It could be considered remarkable, in light of the above claims, that when actually developing number theory and analysis one needs concern oneself with experience and the real meaning of numerical formulas and inferences as little as one would with the "internal construction of a calculator which has proved to be reliable."[29] Mathematics strikes us normally as a field of a priori [*immanente*] truths and this calls for an explanation. The concept which is here required, according to Behmann, is that of fiction. He claims that it would make no sense to go back to the experiential content of arithmetical equations every time. Rather, it is a need of thought economy [Denkersparnis] to interpret mathematical statements (e.g., $5 + 7 = 12$) as expression of a relation between fictional objects. This justifies the normal approach of mathematicians in starting with the number system and leaving the analysis of the real meaning of the arithmetical formulas to the business of the foundations of arithmetic. It is this point of view of economy of thought that also explains the formalist and platonist tendencies of many mathematicians and philosophers. The first reduces numbers to their symbols and the second hypostatizes their existence. Both of them are however unjustified as a foundational account of arithmetic, although practically, according to Behmann, they can be accounted for by this need for thought economy. When we admit these fictional objects for thought-convenience we proceed as in practical life when we use checks, which are a fictional substitute for a real sum of money. The concept of fiction was of course much discussed during this period. One of the major philosophical developments of the decade had been the publication of Hans Vaihinger's *Philosophie des Als Ob* [31]. However, the book had not been received favourably by mathematicians. Behmann found the dismissal of the book to be explained by a number of blunders and infelicities in Vaihinger's discussion of issues pertaining to mathematics. However, the fact that in mathematical practice mathematicians have always made use of fictions calls, according to Behmann, for a reevaluation of Vaihinger's ideas in the foundations of mathematics. He thus accepts Vaihinger's idea that numbers are fiction, although with an important specification:

> Note: it is not the number that is a fiction—in the sense explained earlier, the number is rather a merely formal component of arithmetical propositions—but rather: we *avail* ourselves of a fiction when we (unnecessarily) attribute to it an object-hood not appropriate to it.[30]

The mistake in Vaihinger's position, according to Behmann, had been that of considering fictions as necessary parts of arithmetical statements and inferences. Finally, Behmann agrees with Vaihinger that a fiction according to its essence contains in itself a contradiction. In fact, Behmann says, as soon as one goes beyond a certain realm of thought in which the fiction is applicable, one ends up with contradictions.

In conclusion, Behmann's approach to the foundations of mathematics can be summarized as follows. Empirical perception grounds the objectivity and consistency of mathematics. Number statements have to be considered as formal objects which acquire a meaning only by being traced back to their empirical source. However, mathematics is a priori and certain. Moreover, the practical need for thought economy justifies a philosophy of the As If for the practicing mathematician. One is allowed to hypostatize the numbers and treat them as fictions, although in reality, i.e. independently of their role as formal components of sentences, the numbers do not exist.

§8. Behmann's approach to the foundations and its relationship to Hilbert's foundational views. We have already mentioned that Hilbert was attracted to the logicist solution for a short period. However, his new approach to the foundations of mathematics in the early twenties seems to mark a dramatic new beginning. Sieg [30, p. 12] laments the gap in historical understanding concerning the Russellian influence on Hilbert. It seems to me that through Behmann's dissertation and his earlier lectures it is possible to see some elements of continuity between the logicist approach and Hilbert's program as it was presented in the early twenties. I claim that on some issues Hilbert was at this stage still in agreement with Russell's and Behmann's general outlook. But let me begin by first pointing out continuities with the earlier foundational work by Hilbert. Hilbert gave a positive evaluation of Behmann's achievements. This is found in manuscript in the *Promotion* files for 1918 at the University of Göttingen:

> Göttingen, 1 February 1918,
>
> The original aim and intent of the treatise submitted by Behmann is to introduce the world of thought of the discipline nowadays known as symbolic logic, which, in the hands of a number of eminent mathematicians and logicians, has become an important part of epistemology, and which has received its most mature treatment and presentation in recent years in the large-scale work "Principia Mathematica" by Russell and Whitehead. While symbolic logic for a long time seemed nothing more than a superficial formalistic development of Aristotle's theory of inference figures, Russell was the first to achieve definite successes in applying symbolic logic to difficult epistemological questions. At the very top of Russell's theory—as highest axiom of thought—stands the so-called axiom of reducibility. This axiom, including the related theory of types of Russell, poses extraordinary difficulties for its comprehension. To remove these, Behmann turns to the axiom of completeness I have introduced in arithmetic, which is not only logically similar to the axiom of reducibility, but is even internally materially related to it. By bringing out these connections, Behmann succeeds not only in clarifying the axiom of reducibility, but in surpassing Russell in applying Russell's theory to a particular deep problem—the solution of the antinomy of transfinite number. His result is essentially: all transfinite axiomatics is by its nature something incomplete, the set-theoretical concepts of Cantor are however strictly admissible. Every meaningfully posed set-theoretical problem retains meaning and is therefore capable of a solution. The presentation takes pains to presuppose

> no specific mathematical knowledge; it is thus accessible also to the non-mathematician. I hope to find a publisher for Behmann's work since—it seems to me—it fills a present mathematical and philosophical need. I propose to accept just 2 printed sheets as dissertation.
>
> Grade
>
> Hilbert[31]

We see that Hilbert strongly emphasizes the similarities between his arithmetical completeness axiom and the axiom of reducibility. Indeed, there is a note in Hilbert's hand in the margin of Behmann's official copy of the dissertation where, in connection with Behmann's discussion of the axiom of reducibility, we read "Mein arithmetisches 'Vollständigkeitsaxiom' citiren!" (see p. 157).

Let us now move on to the connection between Behmann's dissertation and Hilbert's mature program of the 1920s.[32] I have already emphasized that the issue of ideal elements is raised quite clearly by Behmann in 1914 and treated at length in his dissertation of 1918. Hilbert's detailed treatment of ideal elements is first found in the 1919–1920 lectures "Natur und mathematisches Erkennen" [23]. I should emphasize that I am not making here a priority claim on behalf of Behmann. Rather, I am simply interested in pointing out that within Hilbert's circle extensive discussion of ideal elements seems to have begun in connection to a discussion of Principia. This is of course consistent with the presence in Hilbert's early writings of the notion of ideal element, which was clearly central to nineteenth-century mathematics (e.g., Dedekind). What seems to me more original in Behmann, and strictly tied to Russell's no-class theory, is the connection between the context-principle and ideal elements. We find traces of this idea in a review of Müller's work *Der Gegenstand der Arithmetik* written by Bernays in 1923:

> In fact, the mathematician knows that in his science an especially fruitful and continually applied procedure consists in the introduction of "ideal elements" that are introduced purely formally as subjects of judgments, and that, however, when detached from the statements in which they occur formally, are nothing at all. [12, p. 521]

Once again, I am not concerned with priority issues but rather with highlighting, through Behmann's work, ideas which seem to have been shared by Hilbert and his immediate co-workers.

Let us now consider Behmann's emphasis on the importance of individuals as the only objective grounding of the edifice of PM. Behmann sees in this the only way to appeal to concrete objects without relying on abstract objects. In Hilbert we also find the idea that in order to overcome the dangers represented by abstract operations or thinking we must have extra-logical objects given to us before all thought:

> As we saw, abstract operation with general concept-scopes and contents has proved to be inadequate and uncertain. Instead, as a precondition for the application of logical inferences and for the activation of logical operations, something must already be given in representation [in der Vorstellung]: certain extra-logical discrete objects, which exist intuitively as immediate experience before all thought. If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession, (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else.[33]

I see here, in the acceptance of a class of concrete individuals as given before abstract thought, an element of continuity between Russell's (and Behmann's) conception and Hilbert's mature foundational thought. Notice that for Hilbert, just as for Russell and Behmann, the individuals are not susceptible of further analysis. Hilbert goes on to say:

> Because I take this standpoint, the objects [Gegenstände] of number theory are for me—in direct contrast to Dedekind and Frege—the signs themselves, whose shape [Gestalt] can be generally and certainly recognized by us—independently of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product. The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding and communication—is this: In the beginning was the sign.

At this point we see in what one of the novelties of Hilbert's approach consists. Rather than appealing to the totality of Russellian individuals he simply singles out a subclass of them: the signs.[34] On this basis he then goes on to provide his account of mathematics. Notice that even in Hilbert's case we have a process of de-ontologization of mathematics quite analogous to that found in Russell and emphasized in Behmann. One can provide a foundation of mathematics without making the problematic assumption that the numbers—conceived as independently existing entities—exist as abstract objects. Of course, there is also a major difference. Hilbert puts great emphasis on the surveyability of the objects and in general on the combinatorial (finitistic) operations that we must be able to carry out on them. This demand, which is the true core of Hilbert's finitism, is absent in Behmann's account of the individuals. There is one last issue I would like to raise in connection to the nature of Hilbert's finitism and a possible relation to Behmann's work. We have seen that for Behmann the individuals are given through empirical perception. What is Hilbert's position on this issue? This is a complicated issue. In [25, pp. 168–171] I have argued that the intuition of concrete objects postulated in Hilbert in the early 20s as a precondition for logical thought should be understood as empirical perception and not as a form of Kantian a priori intuition. I also argued that it was only in the late twenties that Hilbert and Bernays started appealing to a form of pure intuition as the grounding for finitism. Of course, this is not the place to rehearse the evidence for my claim, which relies of a detailed textual reading of Hilbert's and Bernays' texts from the early twenties.[35] If my analysis of the nature of finitist intuition in the early twenties is correct then this would point to an interesting convergence of views on the nature of intuition between Hilbert and his student Behmann.

There is of course much more that could be said about specific connections between Behmann's work and Hilbert's program. My intention here was only to prepare the ground for further analyses. For instance, one could develop the connection between Behmann and Hilbert in a different direction. In 1922 Behmann wrote an important *Habilitationsschrift* on the decision problem (Entscheidungsproblem). This was connected to ideas formulated in Hilbert's Paris talk in 1900. After the formalization of logic given in PM the project could finally be tackled at a technical level. And Behmann is, together with Bernays and Schönfinkel, one of the most engaged in looking for a solution to the problem, referred in the twenties as "the main problem of mathematical logic." The history of the attempts to solve the decision problem in the 1920s still awaits a devoted historian but we know the cast of actors: Behmann, Hilbert, Schönfinkel, Bernays, Herbrand, Ramsey, Ackermann. It is in a lecture by Behmann "Entscheidungsproblem und Algebra der Logik," delivered in Göttingen on May 10, 1921, that the main ideas for this set of investigations are first spelled out in Hilbert's school. Behmann introduces the term "Entscheidungsproblem" and speaks about it as an "*übermathematisches Problem*" [3, p. 2].[36] Although a detailed analysis of the technical results is here out of the question I would like to quote from this lecture to indicate how important themes which are associated with Hilbert's school seem to occur here for the first time and foreshadow important developments such as the analysis of decidability by means of computability theory:

> As is well known, symbolic logic can be *axiomatized*, that is, it can be reduced to a system of relatively few basic formulas and basic rules, so that *proving theorems* appears as a *mere calculating procedure.* One has merely to write new formulas next to given ones, where the rules specify what can be written in every case. Proving has assumed the *character of a game,* so to speak. It is just like in chess, where one transforms *the given position into a new one* by moving one of one's own pieces and removing one of the opponent's pieces if appropriate, and where moving and removing must be allowed by the rules of the game.[37]

> But this comparison makes it blatantly clear that the *standpoint of symbolic logic* just outlined *cannot suffice* for our problem. For that standpoint tells us, like the rules of chess, *only what one may do but not what one should do.* In both cases, what one should do remains a matter of *inventive* thinking and fortunate *combination.* We require much more: not just every single allowed operation, but the *process of calculation* itself is specified by rules, in other words, we require the *elimination of thinking in favor of mechanical calculation.* When a logical mathematical proposition is presented, the procedure we ask for must give complete instructions on how *to ascertain by a deterministic computation after finitely many steps* whether the given proposition is true or false. I would like to call the problem formulated above the *general decision problem.*
>
> It is of fundamental importance for the character of this problem that *only mechanical calculations* according to given instructions, without any thought activity in the stricter sense, are admitted as tools for the proof. One could, if one wanted to, speak of *mechanical* or *machinelike* thought (Perhaps one could later let the procedure be carried out by a machine).[38]

Behmann then went on to describe his work as a reevaluation and a unification under a new viewpoint of the tradition of the algebra of logic (Boole, De Morgan, Schröder, Peirce) emphasizing at the same time that none of the mathematicians active in that tradition had formulated the *Entscheidungsproblem* although they had as one of their chief goals the construction of algorithms in algebra for the solution of equations. However, Schröder's notation is considered completely unusable [vollständig unbrauchbar] and thus Behmann frames his presentation in the notation of Russell and Whitehead with some modifications.

In conclusion, we have seen that Behmann's work contains several ideas which are connected to Hilbert's foundational views. Of course, one would need to delve deeper into the connection between Behmann and Hilbert and the role Behmann played in reevaluating the tradition of the algebra of logic within Hilbert's school. However, this could not be achieved within the scope of this paper. I will have reached my goal if I have convinced the reader that Behmann's work up to 1921 contains a wide variety of aspects which are central for a proper understanding of the context within which Hilbert's program developed and that they deserve the attention of the historian of logic and the foundations of mathematics.

NOTES

1. The talk was delivered on December 7, 1920, and was entitled "Elemente der Logik." The results were published in [29].

2. The list of corrections was sent to Russell by Behmann. The first mention of Boskovitz's corrections is found in a letter from Behmann to Russell dated August 23, 1923 (Behmann Archive, Erlangen). Behmann sent a list of corrections, including those of Boskovitz with a letter dated September 19, 1923 (Russell Archive, McMaster University, 110205e). Indeed, a letter from Boskovitz to Russell—dated July 3, 1923—containing the mentioned corrections is preserved in the Russell archive at McMaster University, call number 110240a. The latter letter contains also side remarks by Behmann concerning the validity of Boskovitz's list of errata for PM.

After his stay in Göttingen Schönfinkel moved to Moscow. As for Boskovitz he ended up in Budapest (see Behmann to Scholz, November 3, 1927, Behmann Archive, Erlangen). Behmann praised Boskovitch in the letter just quoted as one of the very few people with a deep mastery of PM.

3. I do not mean to imply that Hilbert had lost all interest in mathematical logic. Indeed, according to Behmann, this was not the case: "So weit ich mich erinnere, habe ich das Verfahren der Normalform . . . durch Hilbert, der sich ja vor dem Kriege selbständig mit dem Problem der symbolischen Darstellung der Logik, und zwar insbesondere der Aussagenlogik, beschäftigte, kennen gelernt" (Behmann to Scholz, December 29, 1927, Behmann Archive, Erlangen). The normal form procedure is found in [18]. Sieg [30, p. 8] gives additional evidence from Hilbert's lecture course given in 1910 "The quadrature of the circle and related problems." Peckhaus [28] also provides very relevant information for the period in question. However, Behmann does not appear in Peckhaus' account.

4. Sieg [30, pp. 37–38] has edited the postcard exchange between Russell and Hilbert concerning a possible invitation for Russell to visit Göttingen. In particular Hilbert wrote to Russell on April 12, 1916 that "we have been discussing in the Mathematical Society your theory of knowledge already for a long time and . . . we had intended, just before the outbreak of the war, to invite you to Göttingen, so that you could give a sequence of lectures on your solution to the problem of the paradoxes" (quoted in [30], p.12) The visit, however, never materialized. I have found further unpublished details on this invitation for Russell in the acts of the Universitätskuratorium Göttingen. See, in particular, document 5a of Hilbert XVI, IV. Mention of the discussion of Russell's work in Göttingen is also found in a letter from Hugo Dingler to Hilbert dated January 2, 1915 (Dingler Nachlaß, Aschaffenburg) where Dingler, in reply to a previous letter by Hilbert, expresses his regret for having missed the discussion on Russell's work (Hilbert Nachlaß, Staats- und Universitätsbibliothek, Göttingen, Cod. Ms. D. Hilbert 74). In Hilbert's original letter to Dingler (December 26, 1914, Dingler Nachlaß), Hilbert mentions Paul Hertz, Bernstein, and Grelling as his closest associates on foundational issues.

5. The information is taken from the brief biography appended to the dissertation and from [15]. Further details about Behmann's academic career are also found in the acts of the Universitätskuratorium of the University of Göttingen.

6. "Trotz alledem bietet die richtige Auflösung all dieser geometrischen Widersprüche, sobald sie einmal gegeben ist, dem Verständnis keine erhebliche Schwierigkeit dar. Sie besagt nämlich einfach, daß die idealen Elemente nicht im eigentlichen Sinne Gegenstände der Geometrie sind, sondern zunächst einmal nur Worte und als solche Teile von Ausdrucksweisen, deren man sich in der Geometrie bedient, um eine gewisse Klasse von Sätzen auf eine möglichst einfache Form zu bringen." (p. 10)

7. "Das Axiom sagt also aus, daß in dieser einen Stufe schon genügend viele Funktionen vorhanden sind, um alle überhaupt möglichen Klassen zu definieren; es läst sich daher auch als eine Art Vollständigkeitsaxiom für die prädikativen Funktionen auffassen." (p. 14)

8. In 1922 he writes: "Mag man zu der auf den ersten Anblick gewiss befrendenden Russell-Whiteheadschen 'Theorie der logischen Typen' stehen, wie man will, man wird nicht bestreiten können, daß es ihr in der Tat gelungen ist, nicht nur die Mengenlehre [. . .] sondern

die gesamte Logik, von der hier die Arithmetik im weitesten Sinne nur als ein Ausschnitt erscheint, auf eine Grundlage zu stellen, mit der sich jedenfalls vollkommen sicher arbeiten läßt und die, wie der Erfolgt lehrt, trotz der in ihr liegenden unvermeidlichen Beschränkungen das stolze Werk der Begründung der Arithmetik aus bloßer Logik in einer—grundsätzlich gesprochen—völlig strengen und bewundernswert geschlossenen Form tatsächlich geleistet hat—was selbstverständlich nicht ausschließt, daß dieses Werk sogar in wesentlichen Punkten der Vervollkommnung fähig und bedürftig ist." [5, p. 56] Behmann, however, expressed doubts whether the axiom of reducibility is to be considered as an axiom and spoke of it as a postulate (p. 196) showing that it fails in certain restricted domains. For Hilbert's similar position on the axiom of reducibility see Sieg's [30, p. 17], which also contains additional information on the development of Hilbert's views regarding Russell's logicism.

9. "It was, in fact, that work of yours [PM] that first gave me a view of that wonderful province of human knowledge which ancient Aristotelian Logic has nowadays become by the use of an adequate symbolism. But, I daresay, it might be said of your work just as well what H. Weyl said of his own book, that "it offers the fruit of knowledge in a hard shell," requiring indeed a considerable amount of labour in order to be accustomed to that particular manner of thinking, equally different from that of common life and that of college logic and philosophy, which is absolutely necessary for a rigorous treatment of the topic.

Several years ago, I had therefore resolved to write something like an introduction or commentary to that work, providing a way by which the unavoidable difficulties of understanding are separately treated and, in consequence of it, may be clearly grasped and overcome soon by the unacquainted reader in order that the Principia Mathematica might become as well known as both the work and the topic deserve." Behmann to Russell, August 8, 1922 (English in original). Although I originally used the letter preserved in the Behmann archive in Erlangen, there is also a copy in the Russell archive at McMaster University (1368).

Some of the ideas of the dissertation can also be found in the text for a talk delivered in 1927 in Kiel entitled "Die Russelsche Theorie der logischen Typen." In this connection see also Hilbert's correspondence with Russell (quoted in [30], p. 12 and appendix) which shows that Russell's theory of knowledge was being widely discussed in Göttingen earlier than 1916. Sieg [30, p. 32] asks what was being read by Russell at the time. Behmann quotes the following works in his dissertation: Principia Mathematica, Mathematical Logic as Based on the Theory of Types (1908), Principles of Mathematics (1903), The Problems of Philosophy (1912). I have found no additional texts by Russell explicitly quoted during the period between 1914 and 1921 in the writings (published or unpublished) of Hilbert, Behmann, or Bernays. Bernays, who only returned to Göttingen in 1918, wrote to Russell that he began studying Russell's *Principia* only in 1917/18 after Hilbert's 1917/18 lecture course. See [9].

10. Russell to the librarian of the Göttingen University Library, July 2, 1924, Behmann Archive, Erlangen.

11. "Es wäre infolgedessen von vornherein verfehlt, wenn wir uns anschicken wollten, die abstrakten Begriffe und Relationen der Arithmetik für sich allein zu untersuchen, ohne dabei des Umstandes zu gedenken, dass alle derartigen Begriffe immer erst im Zusammenhange des *Satzes* etwas bedeuten und dass auch die bestgebildeten Begriffe uns nichts nützen und uns auch nicht vor Widersprüchen bewahren, solange wir nicht wissen, wie wir sie richtig zu *verwenden* haben." (p. 34)

12. "so müssen wir nun auch, wie uns die letzte Ueberlegung zeigte, den Begriff der *Aussage* von vornherein in den Mittelpunkt unserer gesamten Untersuchungen stellen. Das bedeutet, wir dürfen die in Aussagen auftretenden Begriffe—wie "Menge," "Zahl," "Aussage," "Raum," "Kraft" u.s.f.—niemals für sich allein—als selbstständige logische Gegenstände—betrachten, sondern immer nur im Zusammenhang derjenigen Aussagen, in denen sie sinnvoll auftreten können." (p. 34)

13. “Wir bauen nach und nach ein gewisses festes *System von Aussagen* auf, das wir so einrichten, dass keine zwei in ihm enthaltenen oder—was übrigens auf dasselbe hinauskommt—durch einwandfreie Schlüsse aus ihm ableitbaren Aussagen, sofern sie als richtig erkannt sind, jemals einander widersprechen, dass andererseits aber gewiss all arithmetisch brauchbaren (richtigen und falschen) Aussagen in ihm vorkommen.” (p. 35)

14. “Da wir zufolge der Natur unseres Gegenstandes nicht einmal die einfachsten logischen Begriffe in ihrem überkommenen Gebrauch voraussetzen dürfen, müssen wir infolgedessen zunächst *allem* abstrakten Denken gegenüber misstrauisch sein. Weil wir aber doch irgend einen Ausgangspunkt haben müssen, so darf dies nur ein solcher sein, bei dem unser abstraktes Denken noch nicht beteiligt ist, dessen Existenz also nicht, wie z.B. die der Zahlen, erst auf der Möglichkeit des Denkens beruht und der darum auch nicht durch es verfälscht sein kann. Die einzige Dinge, die dieser Forderung genügen, sind nun aber diejenigen, die ohne Zuhilfenahme des Denkvermögens, d.h. also durch die sinnliche Wahrnehmung allein, unmittelbar erkannt werden, die somit dem Denken erst den zu seiner Möglichkeit unbedingt notwendigen Stoff geben. (Fast alle falsche Metaphysik krankte bisher an dem Bestreben, Gegenstände des Denkens selbst denkend erzeugen zu wollen. Wie überall, muss aber auch hier der Archimedische Punkt notwendig ausserhalb liegen.) Infolgedessen is unsere *erste Annahme* die, dass es gestattet sei, von *Gegenständen der Erfahrungswirklichkeit* (individuals) *als dem vor allem Denken Existierenden auszugehen* (Damit halten wir uns natürlich für berechtigt, von einer etwa möglichen Verfälschung durch die Wahrnehmung vollständig abzusehen.) Diese Individuen betrachten wir somit als fertig vorliegend und für unser Denken unmittelbar verfügbar.” (pp. 44–45)

15. “Denn da die objektive Welt, d.h. die Gesamtheit der Individuen mit all ihren Eigenschaften und Beziehungen, wie wir hier unbedenklich voraussetzen dürfen, im letzen Grunde gewiss einen widerspruchslosen Bereich bildet, so muss offenbar *jede* Aussage, die nur Individuen zu Gegenständen hat, die also keine anderen Dinge als existierend voraussetzt, dem wirklichen Tatbestand entweder entsprechen oder widersprechen und somit den Sätzen vom Widerspruch und vom ausgeschlossenen Dritten genügen.” (p. 107)

16. A few quotes from PM on individuals. “For this purpose, we will use such letters as a, b, c, x, y, z, w, to denote objects which are neither propositions nor functions. Such objects we shall call *individuals*. Such objects will be constituents of propositions or functions, and will be *genuine* constituents, in the sense that they do not disappear on analysis, as (for example) classes do, or phrases of the form ‘the so-and-so.’ ” [36, p. 51]

“The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their *uses* are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such symbols are so defined that, when the definiens is substituted for the definiendum, there no longer remains any symbol which could be supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals.” [36, p. 72]

“We may explain an individual as something which exists on its own account” [36, p. 162]

17. “Wir wollen nun eine Aussage, in der eine unmittelbare Wahrnehmung zum Ausdruck kommt oder doch wenigstens vorgestellt wird, als ‘*einfache Aussage*’ bezeichnen.” (p. 47)

18. Explaining why certain propositions, such as “A propositional function fx cannot be considered as an object in the system of propositions constructed,” do not imply that propositional functions are objects. Behmann states that:

> Es steht uns jedoch frei , den obigen Satz folgendermassen zu deuten “Das Zeichen $f\hat{x}$—also ein konkretes Individuum—darf nicht so betrachtet werden, als ob ihm [...] ein eigentlicher Gegenstand entspräche, der mithin als Gegenstand der Aussage gelten dürfte, d.h. es hat nur als ein künstlich abspaltbarer Bestandteil eines umfassenderen Zeichens zu gelten.” (p. 64)

In chapter 3 Behmann goes on to the construction of propositional functions and of propositions of second order. These are obtained by variation of the propositional and functional components occurring in proposition of zero and first order (p. 95). Generalizations to higher order can be obtained in the obvious way. In PM we read concerning propositions:

> Owing to the plurality of objects of a single judgment, it follows that what we call a "proposition" (in the sense in which this is distinguished from the phrase expressing it) is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol. [36, p. 44]

19. "In gleichem Sinne [as the differential dx] sind nun aber auch die Aussagefunktionen genau wie alle Abstrakta keine echten Gegenstände, sondern sie erwecken nur, indem sie dem in grösserer Entfernung von seinen eigentlichen konkreten Gegenständen sich bewegenden abstrakten Denken gewisse Ruhepunkte bieten, den trügerischen Schein einer Gegenständlichkeit." (p. 106)

20. Behmann appeals explicitly to Russell's "no-class" theory and then says:

> Das bedeutet, dass wir die Klassen nur als formale Bestandteile der sie betreffenden Aussagen anerkennen, deren eigentliche Aussagegegenstände hingegen einzig die in den Klassen enthaltenen—oder auch nicht enthaltenen—Individuen sind. (p. 165)

In PM the idea is expressed as follows:

> The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their *uses* are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such symbols are so defined that, when the definiens is substituted for the definiendum, there no longer remains any symbol which could be supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals. [36, p. 72]

21. "Die Klassen—und gleicherweise übrigens die Zahlen—sind dann, wie wir schon früher andeuteten, gar nichts weiter als Redensarten, die für die bequeme und übersichtliche Darstellung der Arithmetik von äusserstem Nutzen sind, aber gleichwohl recht bedenklich werden können, sobald man sie ernst nimmt und ihrer Natur zuwider für Namen von Gegenständen hält." (p. 159)

22. "Es liegt nun recht nahe, zu vermuten—und lässt sich, wie in den Principia Mathematica gezeigt wird, auch streng beweisen, sobald man alle für die mathematische Logik benötigten Axiome wirklich heranzieht—dass man mit den Individuenklassen beim praktischen Schliessen genau so verfahren kann, als ob sie Individuen wären, solange man sich nur hütet, sie mit den wirklichen Individuen geradezu in eine Reihe zu stellen. Denn die wesentliche Eigenschaft, die der Bereich der Individuenklassen—aber nicht der Bereich aller Klassen überhaupt—mit dem Individuenbereich gemeinsam hat, ist ja die der Konsistenz. Und wir dürfen die Elemente eines konsistenten abstrakten Bereiches, solange wir ihn gegen alle anderen Bereiche scharf abgrenzen, d.h. die Typenfestsetzungen streng innehalten,—so z.B. auch die des Bereiches der natürlichen oder der reellen Zahlen—innerhalb gewisser Grenzen so betrachten, *als ob* sie dem Denken ursprünglich gegebene Dinge wären." (pp. 187–88)

23. "Gegenüber jenen Theorien der (endlichen oder unendlichen) Arithmetik, die die blosse Möglichkeit des symbolischen Formalismus—wir könnten auch sagen: des schriftlichen Rechnens als Grundlage haben, liegt der Hauptvorzug der hier vertretenen Theorie offensichtlich darin, dass unsere Arithmetik ihrer ganzen Entstehung nach keinesfalls bloss ein Spiel mit Zeichen ist, sondern von vornherein einen objektiven Inhalt hat." (pp. 313–314)

24. "Die letzte Wirklichkeit, auf die sich die der Menge noch zurückführen liess, konnte natürlich einzig die der zu zählenden *Dinge* sein. Aber von welcher Art werden nun diese

Dinge sein müssen, um dem Denken als sicherer, nicht schon von Widersprüchen bedrohter Ausgangspunkt dienen zu können?" (p. 335)

The individuals provide arithmetic with its objective content: "Bereits an früherer Stelle hatten wir es ja ausgesprochen, dass die Arithmetik nur dadurch als Wissenschaft—d.h. als ein System objectiv gültiger Erkenntnisse—Berechtigung hat, dass sie einen von aussen her gegebenen Stoff vorfindet, den sie in ihren Untersuchungen nach einer gewissen Seite zu ergründen hat. Ohne einen solchen Stoff—als welcher bekanntlich die Individuen dienen—wäre sie in der Tat gerade so unmöglich, wie die Astronomie, wenn es keine Gestirne gäbe." (p. 153)

25. "Da somit alle abstrakten Dinge—wenigstens soweit sie überhaupt zur Arithmetik in Beziehung treten können—bereits ausgeschlossen sind, bleiben einzig noch die konkreten Gegenstände der Erfahrungswirklichkeit." (p. 335)

26. "Die Anschauung, zu der man von hier aus gelangt, ist nun die in dieser Abhandlung zugrunde gelegte und ausführlich erörtete, dass die Arithmetik in Wahrheit gar nicht ihre eigenen Gegenstände hat, sondern keine andere Gegenständlichkeit voraussetzt als andere Wissenschaften auch, nämlich die der Erfahrung (die zwar nicht unbedingt als wirklich, aber doch wenigstens als möglich vorgestellt werden muss). Da wir jede vernünftige Aussage über diese Wirklichkeit mit gutem Recht als dem Satz vom Widerspruch gemäss voraussetzen dürfen, ist damit auch die Widerspruchslosigkeit der Arithmetik grundsätzlich gewärleistet." (p. 336)

27. "Die Zahlen fassen wir als blosse formale Bestandteile von Aussagen ... In diesem Sinne sind auch die Zahlen nichts weiter als Mittel, um gewisse Erkenntnisse bequem abzuleiten und auszusprechen, aber abgesehen von dieser Bestimmung und für sich allein sind sie gar nichts." (p. 337)

28. "Da nun, wie gezeigt worden, die Begriffe ihren Stoff von der anschauenden Erkenntnis entlehnen, und daher das ganze Gebäude unserer Gedankenwelt auf der Welt der Anschauungen ruht; so müssen wir von jedem Begriff, wenn auch durch Mittelstufen, zurückgehen können auf die Anschauungen, aus denen er unmittelbar selbst, oder aus denen die Begriffe, deren Abstraktion er wieder ist, abgezogen worden: d.h. wir müssen ihn mit Anschauungen, die zu den Abstraktionen im Verhältnis des Beispiels stehen, belegen können. Diese Anschauungen also liefern den realen Gehalt alles unseres Denkens, und überall, wo sie fehlen, haben wir nicht Begriffe, sondern blosse Worte im Kopfe gehabt," quoted in [2, pp. 341–342]

29. "Gegenüber dieser letzten Feststellung bleibt allerdings die Tatsache bemerkenswert, dass wir uns beim wirklichen Aufbau der Arithmetik—wir meinen hier nicht nur die Zahlentheorie, sondern auch die Analysis im allgemeinen—um diese ihr wesensnotwendige Beziehung zur Erfahrung und damit um den wahren Inhalt der Zahlformeln und Schlussweisen kaum in höherem Masse zu kümmern pflegen als etwa um den als zuverlässig erprobten inneren Bau einer Rechenmachine, uns nichtsdestoweniger aber innerhalb dieser Gebiete vor Widersprüchen mit Recht für völlig geschützt halten." (p. 343)

30. "Wohlgemerkt: nicht die Zahl ist eine Fiktion—sie ist vielmehr in dem früher erklärten Sinne ein bloss formaler Bestandteil arithmetischer Aussagen—sondern wir *bedienen* uns einer Fiktion, wenn wir, was an sich ja keineswegs nötig ist, ihr—nicht aus Rücksichten der Denknotwendigkeit, sondern der Denkbequemlichkeit—eine ihr in Wahrheit nicht zukommende Gegenständlichkeit im tatsächlichen Denken dennoch zuerkennen, d.h. die für echte Gegenstände geltenden logischen Gesetze unbedenklich, bez. nachdem wir uns der Zulässigkeit dieses Verfahrens innerhalb des fraglichen Bereiches versichert haben, auch auf sie anwenden." (p. 346)

31. "Göttingen, den 1. Februar 1918,

Die eigentliche Aufgabe und Absicht der vorliegenden Abhandlung von Behmann ist es, in die Gedankenwelt derjenigen Disziplin einzuführen, die man als symbolische Logik bezeichnet, die unter den Händen einer Reihe bedeutender Mathematiker und Logiker ein wichtiger Bestandteil der Erkenntnistheorie geworden ist und schliesslich in den letzten Jahren in dem gross-angelegten Werke "Principia Mathematica" von Russell und Whitehead die reifste Bearbeitung und Darstellung erfahren hat. Während lange Zeit hindurch die symbolische Logik nichts anderes als eine formalistische äusserliche Weiterbildung der Aristotelischen Schlussfigurentheorie zu sein schien, gelang es Russell, mit der Anwendung der symbolischen Logik auf die schwierigsten erkenntnistheoretischen Fragen zum ersten Male sichere Erfolge zu erziehen. Obenan in der Russelschen Theorie steht—als oberstes Denkaxiom—das sogenannte Reduzierbarkeitsaxiom. Dieses Axiom, einschliesslich der damit verbundenen Typentheorie von Russell bietet dem Verständnis ausserordetliche Schwierigkeiten. Um diese zu beseitigen zieht Behmann das von mir in die Arithmetik eingeführte Vollständigkeitsaxiom heran, welches nicht nur seinem logischen Charakter nach mit dem Reduzierbarkeitsaxiom gleichartig ist, sondern auch sachliche innere Zusammenhänge mit jenem zeigt. Indem Behmann diese Zusammenhänge herausarbeitet, gelingt es ihm nicht nur jenes Reduzierbarkeits-axiom klar zu fassen, sondern auch die Anwendung der Russelsche[n] Theorie auf ein spezielles tiefliegendes Problem—die Auflösung der Antinomie der transfiniten Zahl—über Russell hinaus durchzuführen. Sein Resultat bleibt im Wesentlichen: alle transfinite Axiomatik ist ihrer Natur nach etwas unabgeschlossenes, aber die mengenth[eoretische] Begriffe von Cantor sind streng zulässig. Jedes an sich vernünftlich gestellte mengentheoretische Problem behält Bedeutung und ist daher einer Lösung fähig.

Bei der Darstellung ist besonderer Wert darauf gelegt, dass dabei keinerlei spezifisch mathematische Kenntnisse voraussetzt werden; diesselbe ist daher auch dem Nicht-Mathematiker verständlich. Ich hoffe für die Behemmanische Schrift einen Verleger zu finden, da sie—wie mir scheint—einem mathematisch-philosophischen Bedürfnis in der Gegenwart Rechnung trägt. Ich beantrage, bereits 2 Druckbogen als Dissertation gelten zu lassen.

Praedikat

Hilbert"

UAG, Phil. Fak. Promotionen: B vol. VI (1917–), Universitätsarchiv Göttingen

32. For an introduction to Hilbert's program see [25] and the references given there.

33. [21], p. 134 of the translation. This position is repeated verbatim in many later publications by Hilbert.

34. See note 18 for Behmann's interpretation of signs as concrete individuals.

35. Consider, for instance, the emphasis on relying on primitive forms of cognition as a starting point for the construction of arithmetic. "An appeal to an intuitive grasp of the number series as well as to the multiplicity of magnitudes is certainly to be considered. But this could certainly not be a question of an intuition in the primitive sense; for, certainly no infinite multiplicities are given to us in the primitive intuitive mode of representation. And even though it might be quite rash to contest any farther-reaching kind of intuitive evidence from the outset, we will nevertheless make allowance for that tendency of exact science which aims as far as possible to eliminate the finer organs of cognition [Organe der Erkenntnis] and to rely only on the most primitive means of cognition." [10, trans., p. 163]. Compare [11, p. 226]: "Hilbert's theory does not exclude the possibility of a philosophical attitude that conceives the numbers as existing, nonsensible objects ... Nevertheless the aim of Hilbert's theory is to make such an attitude dispensable for the foundations of the exact sciences." In this connection it might be worthwhile to investigate the possible influence of Russell's theory of knowledge on Hilbert and Bernays.

36. The terminology is undoubtedly Behmann's. He writes to Russell in 1922: "It was what I call the Problem of Decision, formulated in the said paragraph [6, §1] that induced me to study the logical work of Schröder. And I soon recognized that, in order to solve my particular problem, it was necessary first to settle the main problem of Schröder's Calculus of Regions, his so-called Problem of Elimination ... Indeed, the chief merit of the said problem [The Decision Problem] is, I daresay, due to the fact that it is a problem of fundamental importance

on its own account, and, unlike the applications of earlier Algebra of Logic, not at all imagined for the purpose of symbolic treatment, whereas, on the other hand, the only means of any account for its solution are exactly those of symbolic logic" [6]. In this connection see also Behmann to Scholz (December 27, 1927, Behmann Archive, Erlangen) where, in addition to reiterating his originality in the formulation of the Entscheidungsproblem, Behmann also provides a historical overview of the early history of the Entscheidungsproblem. One should not confuse Behmann's introduction of the Entscheidungsproblem either with the Entscheidungsproblem formulated, in connection to Grelling's paradox, in Hessenberg [16] or with the Kroneckerian requirement of decidability for the introduction of concepts in mathematics (for the latter see Pasch [27]). Sieg [30, p. 19] tentatively claims that the first occurrence of the word Entscheidungsproblem is found in lectures of Hilbert from 1922–23. The assertion should be corrected in light of the above.

37. The metaphor of the chess game to speak about formalized mathematics had great fortune in the twenties. However, it is usually attributed to Weyl who formulated it in print in 1924 [33, 34].

38. " Bekanntlich läßt sich die symbolische Logik *axiomatisieren,* d. h. auf ein System verhältnismäßig weniger Grundformeln und Grundregeln zurückführen, sodaß auch das *Beweisen von Sätzen* nunmehr als ein *bloßes Rechenverfahren* erscheint. Man braucht nur noch zu gegebenen Formeln neue hinzuschreiben, wobei durch Regeln bereits festgelegt ist, was man jeweils hinschreiben darf. Das Beweisen hat sozusagen den *Charakter eines Spieles* angenommen. Es ist etwa wie beim Schachspiel, wo man durch Verschieben eines der eigenen Steine, gegebenenfalls mit Wegnahme einer gegnerischen, die *jeweils vorliegende Stellung in eine neue* verwandelt, wobei nun das Verschieben und das Wegnehmen durch die Regeln des Spieles erlaubt sein muß.

Aber gerade dieser Vergleich zeigt uns auch in krasser Weise, daß uns der eben geschilderte *Standpunkt der symbolischen Logik* für unser Problem *noch keineswegs genügen* kann. Denn dieser sagt uns wie die Regeln des Schachspiels *nur, was man tun darf,* und *nicht, was man tun soll.* Dies bleibt in dem einen wie in der anderen Falle eine Sache des erfinderischen *Nachdenkens,* der glücklichen *Kombination.* Wir verlangen aber weit mehr: daß nicht etwa nur die erlaubten Operationen im einzelnen, sondern auch der *Gang der Rechnung selbst* durch Regeln festgelegt sein soll, m.a.W. eine *Ausschaltung des Nachdenkens zugunsten des mechanischen Rechnens.* Ist irgend eine logische mathematische Aussage vorgelegt, so soll das verlangte Verfahren eine vollständige Anweisung geben, wie man *durch eine ganz zwangläufige Rechnung nach endlich vielen Schritten* ermitteln kann, ob die gegebene Aussage *richtig oder falsch* ist. Das oben formulierte Problem möchte ich *das allgemeine Entscheidungsproblem* nennen.

Für das Wesen des Problems ist von grundsätzlicher Bedeutung, daß als Hilfsmittel des Beweises *nur das ganz mechanische Rechnen* nach einer gegebenen Vorschrift , ohne irgendwelche Denktätigkeit im engeren Sinne, zugelassen wird. Man könnte hier, wenn man will, von *mechanischem* oder *machinenmässigem* Denken reden. (Vielleicht kann man es später sogar durch eine Machine ausführen lassen)." [3, pp. 5–6]

REFERENCES

[1] Heinrich Behmann, *Über mathematische Logik*, unpublished manuscript, dated December 1, 1914. Behmann Archive, Erlangen.

[2] ———, *Die Antinomie der transfiniten Zahl und ihre Auflösung durch die Theorie von Russell und Whitehead*, ***Dissertation***, Universität Göttingen, 1918, 352 pp.

[3] ———, *Entscheidungsproblem und Algebra der Logik*, unpublished manuscript, dated May 10, 1921. Behmann Archive, Erlangen.

[4] ———, *Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem*, ***Mathematische Annalen***, vol. 86 (1922), pp. 163–229.

[5] ———, *Die Antinomie der transfiniten Zahl und ihre Auflösung durch die Theorie von Russell und Whitehead*, ***Jahrbuch der Mathematisch-Naturwissenschaftlichen Fakultät in Göttingen***, vol. 23 (1922), pp. 55–64.

[6] ———, *Letter to Bertrand Russell, August 8, 1922*, Behmann Archive, Erlangen, and Russell Archive, McMaster University, 1368, 1922.

[7] ———, *Weyls Kritik der Analysis*, Hilbert Nachlaß, Niedersächsiche Staats- und Universitätsbibliothek, Göttingen, Cod. Ms. Hilbert 685, Nr. 3, Bl. 13–20, 1917?

[8] ———, *Beiträge zur axiomatischen Behandlung des Logik-Kalküls*, ***Habilitationsschrift***, Universität Göttingen, 1918, Bernays Nachlaß, WHS, Bibliothek, ETH Zürich, Hs 973.192.

[9] Paul Bernays, *Letter to Bertrand Russell, 1920*, Russell Archive, McMaster University, 110208b.

[10] ———, *Über Hilberts Gedanken zur Grundlegung der Arithmetik*, ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 31 (1922), pp. 10–19, English translation in [24], pp. 215–222.

[11] ———, *Erwiderung auf die Note von Herrn Aloys Müller: Über Zahlen als Zeichen*, ***Mathematische Annalen***, vol. 90 (1923), pp. 159–63, English translation in [24], pp. 223–226.

[12] ———, *Review of: Aloys Müller, "Der Gegenstand der Mathematik"*, ***Die Naturwissenschaften***, vol. 11 (1923), pp. 520–22.

[13] Paul Bernays and Moses Schönfinkel, *Zum Entscheidungsproblem der mathematischen Logik*, ***Mathematische Annalen***, vol. 99 (1928), pp. 342–372.

[14] William Bragg Ewald (editor), ***From Kant to Hilbert. A source book in the foundations of mathematics***, vol. 2, Oxford University Press, Oxford, 1996.

[15] Gerrit Haas and Elke Stemmler, *Der Nachlaß Heinrich Behmanns (1891–1970): Gesamtverzeichnis*, ***Aachener Schriften zur Wissenschaftstheorie, Logik und Logikgeschichte***, vol. 1 (1981), pp. 1–39.

[16] Gerhard Hessenberg, *Grundbegriffe der Mengenlehre*, ***Abhandlungen der Fries'schen Schule (Neue Folge)***, vol. 1 (1906), pp. 479–706.

[17] David Hilbert, *Mengenlehre*, Lecture notes by Margarethe Loeb. Sommer-Semester 1917. Unpublished manuscript. Bibliothek, Mathematisches Institut, Universität Göttingen.

[18] ———, *Logische Principien des mathematischen Denkens*, Vorlesung, Sommer-Semester 1905. Lecture notes by Ernst Hellinger. Unpublished manuscript. Bibliothek, Mathematisches Institut, Universität Göttingen.

[19] ———, *Prinzipien der Mathematik*, Lecture notes by Paul Bernays. Winter-Semester 1917–18. Unpublished typescript. Bibliothek, Mathematisches Institut, Universität Göttingen.

[20] ———, *Axiomatisches Denken*, ***Mathematische Annalen***, vol. 78 (1918), pp. 405–15, Lecture given at the Swiss Society of Mathematicians, 11 September 1917. Reprinted in [22], pp. 146–56. English translation in [14], pp. 1105–1115.

[21] ———, *Neubegründung der Mathematik: Erste Mitteilung*, ***Abhandlungen aus dem Seminar der Hamburgischen Universität***, vol. 1 (1922), pp. 157–77, Reprinted with notes by Bernays in [22], pp. 157–177. English translation in [14], pp. 1115–1134.

[22] ———, ***Gesammelte Abhandlungen***, vol. 3, Springer, Berlin, 1935.

[23] ———, ***Natur und mathematisches Erkennen: Vorlesungen, gehalten 1919–1920 in Göttingen***, Birkhäuser, Basel, 1992.

[24] Paolo Mancosu (editor), ***From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s***, Oxford University Press, Oxford, 1998.

[25] ———, *Hilbert and Bernays on metamathematics*, In *From Brouwer to Hilbert* [24], pp. 149–188.

[26] GREGORY H. MOORE, *Hilbert and the emergence of modern mathematical logic*, ***Theoria (Segunda Época)***, vol. 12 (1997), pp. 65–90.

[27] MORITZ PASCH, *Die Forderung der Entscheidbarkeit*, ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 27 (1918), pp. 228–232.

[28] VOLKER PECKHAUS, ***Hilbertprogramm und Kritische Philosophie***, Vandenhoeck und Ruprecht, Göttingen, 1990.

[29] MOSES SCHÖNFINKEL, *Über die Bausteine der mathematischen Logik*, ***Mathematische Annalen***, vol. 92 (1924), pp. 305–316, English translation in [32], pp. 355–366.

[30] WILFRIED SIEG, *Hilbert's programs: 1917–1922*, this BULLETIN, vol. 5 (1999), no. 1, pp. 1–44.

[31] HANS VAIHINGER, ***Die Philosophie des Als Ob***, Felix Meiner, Leipzig, 1911.

[32] Jean van Heijenoort (editor), ***From Frege to Gödel. A source book in mathematical logic, 1897–1931***, Harvard University Press, Cambridge, Mass., 1967.

[33] HERMANN WEYL, *Randbemerkungen zu Hauptproblemen der Mathematik*, ***Mathematische Zeitschrift***, vol. 20 (1924), pp. 131–50, Reprinted in: [35], pp. 433–52.

[34] ———, *Die heutige Erkenntnislage in der Mathematik*, ***Symposion***, vol. 1 (1925), pp. 1–23, Reprinted in: [35], pp. 511–42. English translation in: [24], pp. 123–42.

[35] ———, ***Gesammelte Abhandlungen***, vol. 2, Springer, Berlin, 1968.

[36] ALFRED NORTH WHITEHEAD and BERTRAND RUSSELL, ***Principia mathematica***, vol. 1, Cambridge University Press, Cambridge, 1910, Quotes from the 1978 reprint of the second edition, 1927.

[37] RICHARD ZACH, *Completeness before Post: Bernays, Hilbert, and the development of propositional logic*, this BULLETIN, vol. 5 (1999), no. 3 (this issue), pp. 331–366.

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