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TARSKI'S SYSTEM OF GEOMETRY

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Abstract. This paper is an edited form of a letter written by the two authors (in the name of Tarski) to Wolfram Schwabhäuser around 1978. It contains extended remarks about Tarski's system of foundations for Euclidean geometry, in particular its distinctive features, its historical evolution, the history of specific axioms, the questions of independence of axioms and primitive notions, and versions of the system suitable for the development of 1-dimensional geometry.

In his 1926–27 lectures at the University of Warsaw, Alfred Tarski gave an axiomatic development of elementary Euclidean geometry, the part of plane Euclidean geometry that is not based upon set-theoretical notions, or, in other words, the part that can be developed within the framework of first-order logic. He proved, around 1930, that his system of geometry admits elimination of quantifiers: every formula is provably equivalent (on the basis of the axioms) to a Boolean combination of basic formulas. From this theorem he drew several fundamental corollaries. First, the theory is complete: every assertion is either provable or refutable. Second, the theory is decidable—there is a mechanical procedure for determining whether or not any given assertion is provable. Third, there is a constructive consistency proof for the theory. Substantial simplifications in Tarski's axiom system and the development of geometry based on them were obtained by Tarski and his students during the period 1955-65. All of these various results were described in Tarski [41], [44], [45], and Gupta [5].

Aside from the importance of its metamathematical properties, Tarski's system of geometry merits attention because of the extreme elegance and simplicity of its set of axioms, especially in the final form that it achieved around 1965. Yet, until fairly recently, no systematic development of geometry based on his axioms existed. In the early 1960s Wanda Szmielew and Tarski began the project of preparing a treatise on the foundations of geometry developed within the framework of contemporary mathematical logic. A systematic development of Euclidean geometry based on Tarski's axioms was to constitute the first part of the treatise. The project made

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some progress: drafts of the first part of the treatise were written. However, the project was never completed. Over the years, Szmielew had gradually changed her views on the foundations of geometry, and had begun a development along different lines (see Szmielew [38] and the review Moszyńska [18]). Her untimely death put an end to all prospects for completion of the work.

Eventually, Wolfram Schwabhäuser did prepare such a treatise (in German), based in part on the draft of Szmielew and Tarski (see Section 7 for details). Around 1978 he asked Tarski to send him suggestions for material to be included in the monograph. It was at that time that Tarski and I wrote the following notes in the form of a very long letter (some 40 pages) from Tarski to Schwabhäuser.

The letter was never intended for print. Over the years, however, a number of people urged that it be published because of the insights and the historical information that it provides.

Putting the letter in a form suitable for publication has necessitated some editing. For example, the first section originally listed, without comment, all of the sentences that played a role in the discussion of the axiom sets for Tarski's system (almost all of the sentences were, at one time or another, included as axioms in at least one of the versions of Tarski's system). Since the letter was intended for Schwabhäuser only, there was no explanation of the notation and the formalism being used, and no explanation of the intuitive geometrical content of the sentences. Such explanations, accompanied by figures, have now been added. A similar remark concerns the definitions given in Sections 5 and 6. In the remaining parts of the letter the modifications undertaken have been minor: occasional changes of wording, deletion or subordination to footnotes of questions and remarks that were specifically directed at Schwabhäuser, rearrangement of a few paragraphs, and so on. Quite recently, some remarks suggested by the referee have been added as footnotes.[1] The asterisk symbol has been used to distinguish these remarks from those made in the original version of the paper.

The letter proper really begins with Section 2, which outlines the evolution of Tarski's set of axioms from the original 1926–27 version to the final versions used by Szmielew and Tarski in their unpublished manuscript and by Schwabhäuser-Szmielew-Tarski [29]. There follows, in Section 3, a discussion of the distinctive features of Tarski's approach to Euclidean geometry—the features which set it apart from other systems of Euclidean geometry that can be found in the literature. The fourth section contains some historical observations about the individual axioms and their use by previous authors. The historically important question of the independence of the axioms and of the primitive notions, in reference to Tarski's system,

[1] I am indebted to the referee and to Maria Moszyńska for several helpful suggestions.

is addressed in Section 5. Section 6 makes some observations about 1-dimensional geometry (which is excluded from the discussion in the earlier parts of the letter). Finally, in the last section the work of Wanda Szmielew and its relationship to the treatise Schwabhäuser-Szmielew-Tarski [29] is discussed.

§1. List of sentences involved in the discussion. In contrast to other systems of geometry (for example Hilbert's system for the geometry of space) in which points, lines, planes, etc., are all primitive "geometrical objects", in Tarski's system there is only one type of primitive geometrical object: points. In other words, all (first-order) variables $a, b, c, \ldots$ (denoted by lower case Roman letters) are assumed to range over points. There are two primitive geometrical (that is, non-logical) notions: the ternary relation B of "betweenness" and quaternary relation $\equiv$ of "equidistance" or "congruence of segments". We shall write $B(abc)$ to express that the relation of betweenness holds among the points a, b, and c; intuitively, this means that the point b lies on the line segment joining a and c. Similarly, we shall write $ab \equiv cd$ to express that the relation of equidistance holds among the points a, b, c, and d; intuitively, this means that the distance from a to b is the same as the distance from c to d, or, put another way, the line segment joining a and b is congruent to the line segment joining c and d.

The logical symbols of the language include equality, the sentential connectives of conjunction, disjunction, negation, implication, and the biconditional, which are denoted by

$$\wedge, \qquad \vee, \qquad \neg, \qquad \rightarrow, \qquad \leftrightarrow,$$

respectively, the universal quantifier $\forall$, and the existential quantifier $\exists$. We adhere to the standard conventions regarding omission of parentheses and the precedence of logical operators. For instance, conjunction has precedence over implication, so that the formula $\varphi \wedge \psi \rightarrow \delta$ is to be read as $(\varphi \wedge \psi) \rightarrow \delta$. The following sentences will be referred to in various parts of the discussion. In formulating these sentences we follow the standard practice of treating variables not within the scope of a quantifier as being universally quantified.

Ax. 1. Reflexivity Axiom for Equidistance

$$ab \equiv ba.$$

Ax. 2. Transitivity Axiom for Equidistance

$$ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs.$$

Ax. 3. Identity Axiom for Equidistance

$$ab \equiv cc \rightarrow a = b.$$

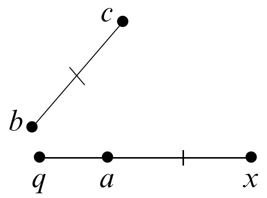

FIGURE 1. Axiom of Segment Construction.

Ax. 4. Axiom of Segment Construction

$$\exists x\ (B(qax) \wedge ax \equiv bc).$$

The intuitive content of this axiom is that, given any line segment bc, one can construct a line segment congruent to it, starting at any point a and going in the direction of any ray containing a. The ray is determined by the point a and a second point q, the endpoint of the ray. The other endpoint of the line segment to be constructed is just the point x whose existence is asserted. (See Figure 1.)

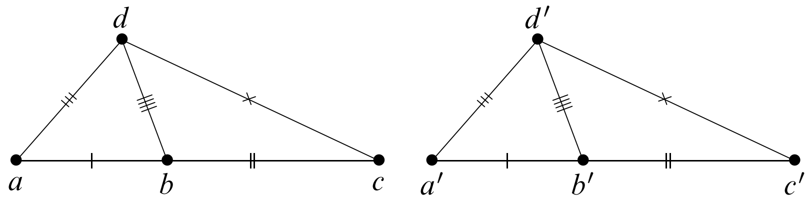

FIGURE 2. Five-Segment Axiom.

Ax. 5. Five-Segment Axiom

$$[a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \\ \wedge ad \equiv a'd' \wedge bd \equiv b'd'] \rightarrow cd \equiv c'd'.$$

The Five-Segment Axiom asserts (in the non-degenerate case) that, given two triangles $\triangle acd$ and $\triangle a'c'd'$, and given interior points b and b' of the sides ac and $a'c'$, from the congruences of certain corresponding pairs of line segments (indicated by hatch marks in Figure 2), one can conclude the congruence of another pair of corresponding line segments (indicated by cross-hatches in Figure 2). Thus, this axiom is similar in character to the well-known theorems of Euclidean geometry that allow one to conclude, from hypotheses about the congruence of certain corresponding sides and angles in two triangles, the congruence of other corresponding sides and angles.

Ax. 5_1. Variant of the Five-Segment Axiom

$$[a \neq b \wedge b \neq c \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \\ \wedge ad \equiv a'd' \wedge bd \equiv b'd'] \rightarrow cd \equiv c'd'.$$

This variant differs from the Five-Segment Axiom only in the presence of the additional inequality $b \neq c$ in the hypotheses.

Ax. 6. Identity Axiom for Betweenness

$$B(aba) \rightarrow a = b.$$

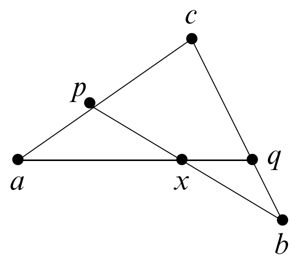

FIGURE 3. Inner Pasch Axiom.

Ax. 7. First (or Inner) form of the Pasch Axiom

$$B(apc) \wedge B(bqc) \rightarrow \exists x\ [B(pxb) \wedge B(qxa)].$$

The Pasch axiom in the plane asserts that a line intersecting a triangle in one of its sides, and not intersecting any of the vertices, must intersect one of the other two sides. In Ax. 7 the line bp that intersects the triangle $\triangle aqc$ in the extension of the side cq (expressed by the hypothesis $B(bqc)$) is assumed to intersect the side ac (the "outer" side of the triangle from the perspective of bp). This is expressed by the hypothesis $B(apc)$, which asserts that p is a point on the segment ac; the degenerate cases when p coincides with a or c are allowed. (Ax. 7 also allows the triangle itself to be degenerate.) The conclusion is that the line intersects the side aq in a point x; this is expressed by the assertion that $B(qxa)$. Aside from allowing degenerate cases, Ax. 7 asserts only one special case of the Pasch axiom, namely the case when b lies on the extension of the side cq in the direction from c to q. In this case, the line bp must intersect the side aq in some point x that is on the segment bp, which is expressed by the assertion $B(pxb)$. In other words, it intersects the "inner" side of the triangle (from the perspective of bp—see Figure 3). It is from this that the name "inner form" derives.

In the outer form of the Pasch Axiom, formulated as Ax. 7_1 below, the point b lies on the extension of the side cq in the direction from q to c, and the line is assumed to intersect the "inner" side of the triangle (from the perspective of bp). The conclusion is that it must intersect the side aq in some point x on the extension of the side bp; this is expressed by the assertion $B(bpx)$. In other words, it intersects the "outer" side of the triangle (from the perspective of bp—see Figure 4). This gives rise to the name "outer form".

Ax. 7_1. Second (or Outer) form of Pasch Axiom

$$B(apc) \wedge B(qcb) \to \exists x\ [B(axq) \wedge B(bpx)].$$

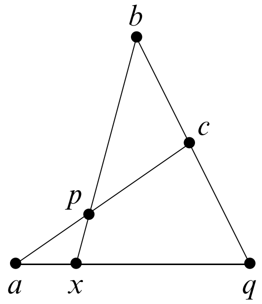

FIGURE 4. Outer Pasch Axiom.

Ax. 7_2. Variant of Axiom 7_1

$$B(apc) \wedge B(qcb) \to \exists x\ [B(axq) \wedge B(xpb)].$$

This variant differs from Ax. 7_1 only in the final betweenness assertion, which is reversed.

Ax. 7_3. Weak Pasch Axiom

$$B(atd) \wedge B(bdc) \to \exists x\ \exists y\ [B(axb) \wedge B(ayc) \wedge B(ytx)].$$

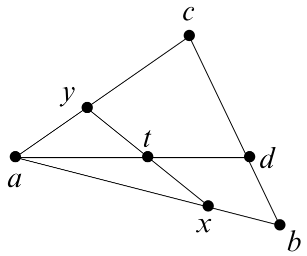

FIGURE 5. Weak Pasch Axiom.

Ax. $8^{(1)}$. Lower 1-Dimensional Axiom

$$\exists a\ \exists b\ (a \neq b).$$

Ax. $8^{(2)}$. Lower 2-Dimensional Axiom

$$\exists a\ \exists b\ \exists c\ [\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab)].$$

The Lower 2-Dimensional Axiom asserts that there exist three non-collinear points.

Ax. $8^{(n)}$. Lower n-Dimensional Axiom for $n = 3, 4, \ldots$

$$\exists a\, \exists b\, \exists c\, \exists p_1\, \exists p_2 \ldots \exists p_{n-1}$$
$$\Bigg[\bigwedge_{1 \le i < j < n} p_i \neq p_j \wedge \bigwedge_{i=2}^{n-1} ap_1 \equiv ap_i \wedge \bigwedge_{i=2}^{n-1} bp_1 \equiv bp_i$$
$$\wedge \bigwedge_{i=2}^{n-1} cp_1 \equiv cp_i \wedge [\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab)] \Bigg].$$

The Lower n-Dimensional Axiom for $n = 3, 4, \ldots$ asserts that there exist $n-1$ distinct points $p_1, p_2, \ldots, p_{n-1}$ and three points a, b, c such that each of the three points is equidistant from each of the $n-1$ points, but the three points are not collinear. In other words, the set of all points equidistant from each of $n-1$ distinct points $p_1, p_2, \ldots, p_{n-1}$ is not always a line. As we shall see in a moment, the Lower n-Dimensional Axiom (for every $n \geq 1$) is just the negation of the corresponding Upper $(n-1)$-Dimensional Axiom.

Ax. $9^{(0)}$. Upper 0-Dimensional Axiom

$$a = b.$$

Ax. $9^{(1)}$. Upper 1-Dimensional Axiom

$$B(abc) \vee B(bca) \vee B(cab).$$

The Upper 1-Dimensional Axiom says that any three points are collinear.

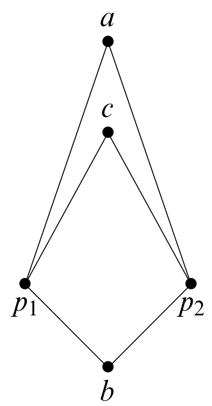

FIGURE 6. Upper 2-Dimensional Axiom.

Ax. $9^{(n)}$. Upper n-Dimensional Axiom for $n = 2, 3, \ldots$

$$\left[\bigwedge_{1\le i<j\le n} p_i \neq p_j \wedge \bigwedge_{i=2}^{n} ap_1 \equiv ap_i \wedge \bigwedge_{i=2}^{n} bp_1 \equiv bp_i \wedge \bigwedge_{i=2}^{n} cp_1 \equiv cp_i\right] \rightarrow [B(abc) \vee B(bca) \vee B(cab)].$$

The Upper n-Dimensional Axiom for $n = 2, 3, \ldots$ asserts that any three points a, b, c which are equidistant from each of n distinct points p_1, p_2, $\ldots$, p_n must be collinear (see Figures 6 and 7); in other words, the set of all points equidistant from each of n distinct points is a line. Notice that Ax. $9^{(n)}$ is not valid in the standard k-dimensional model of Euclidean geometry for $k > n$. For instance, in three dimensions $(k = 3)$ the set of points equidistant from two given distinct points $(n = 2)$ is a plane, not a line. Thus, Ax. $9^{(n)}$ asserts that the dimension of the geometry is at most n. Consequently, its negation—which is just the Lower $(n+1)$-Dimensional Axiom—asserts that the dimension is more than n. Notice that in formulating Ax. $9^{(n)}$ we have not stated that $p_1, \ldots, p_n$ are not collinear, not coplanar, etc., but only that they are different. This suffices.

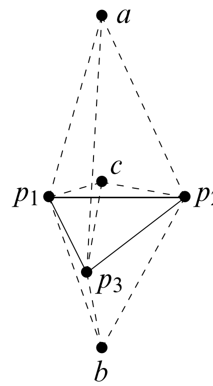

FIGURE 7. Upper 3-Dimensional Axiom.

Ax. $9_1{}^{(2)}$. Alternative form of Axiom $9^{(2)}$

$$\exists y \ \{([B(xya) \vee B(yax) \vee B(axy)] \wedge B(byc)) \vee ([B(xyb) \vee B(ybx) \vee B(bxy)] \wedge B(cya)) \vee ([B(xyc) \vee B(ycx) \vee B(cxy)] \wedge B(ayb))\}.$$

The alternative form of Ax. $9^{(2)}$ has the merit of being formulated in terms of the betweenness relation alone, without recourse to the notion of equidistance. In the non-degenerate case of three non-collinear points a, b, and c, it asserts that if from any point x we draw the lines to each of the three vertices of $\triangle abc$, then at least one of these lines intersects the side opposite the vertex. This is true in the plane because any point x lies in one of the seven regions determined by the lines through the vertices of the triangle; the region to which it belongs is determined by the relation of betweenness holding among the point x, one of the vertices of the triangle, and a point y on the side of the triangle opposite this vertex. The assertion is obviously false in dimensions greater than two.

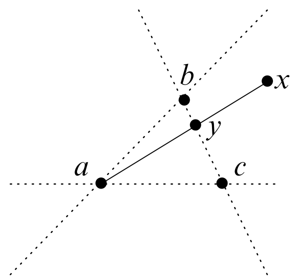

FIGURE 8. Alternative Form of Axiom $9^{(2)}$.

Ax. $9_2{}^{(2)}$. Variant of Ax. $9_1{}^{(2)}$

$$\exists y\ \{([B(xya) \vee B(yax) \vee B(axy)] \wedge B(byc)) \\ \vee ([B(xyb) \vee B(ybx)] \wedge B(cya)) \\ \vee ([B(xyc) \vee B(ycx)] \wedge B(ayb))\}.$$

In Ax. $9_1{}^{(2)}$ the case when x is in the interior of $\triangle abc$ is treated in all three disjuncts (in the clauses $B(axy)$, $B(bxy)$, and $B(cxy)$ respectively). It can safely be omitted from the second and third disjuncts. This leads directly to the variant of Ax $9_1{}^{(2)}$.

Ax. 10. First Form of Euclid's Axiom

$$B(adt) \wedge B(bdc) \wedge a \neq d \rightarrow \exists x\ \exists y\ [B(abx) \wedge B(acy) \wedge B(xty)].$$

The First Form of Euclid's Axiom says that through any point t in the interior of an angle $\angle bac$ there is a line—here, the line xy—that intersects both sides of the angle—here, in the points x and y (see Figure 9).

Ax. 10_1. Variant of Axiom 10

$$B(adt) \wedge B(bdc) \wedge a \neq d \rightarrow \exists x\ \exists y\ [B(abx) \wedge B(acy) \wedge B(ytx)].$$

The variant differs from Ax. 10 only in that the final betweenness assertion is reversed.

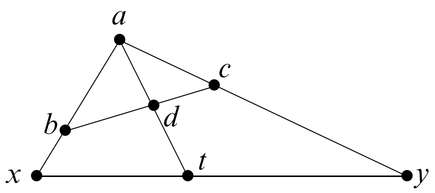

FIGURE 9. First Form of Euclid's Axiom.

Ax. 10_2. Second Form of Euclid's Axiom

$$B(abc) \vee B(bca) \vee B(cab) \vee \exists x\, [ax \equiv bx \wedge ax \equiv cx].$$

The second form of Euclid's Axiom says that in any (non-degenerate) triangle there is a point that is equidistant from each of the vertices. In other words, every triangle can be inscribed in some circle (see Figure 10).

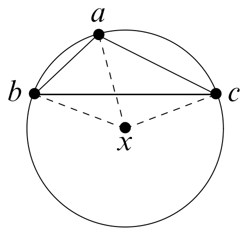

FIGURE 10. Second Form of Euclid's Axiom.

Ax. 10_3. Third Form of Euclid's Axiom

$$[B(abf) \wedge ab \equiv bf \wedge B(ade) \wedge ad \equiv de \\ \wedge B(bdc) \wedge bd \equiv dc] \rightarrow bc \equiv fe.$$

The third form of Euclid's Axiom says that the line connecting the midpoints of two sides of a triangle is half the length of the third side (see Figure 11). This is equivalent to the assertion that the sum of the interior angles of a triangle is equal to two right angles.

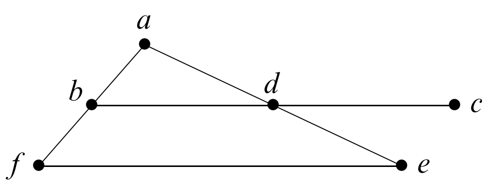

FIGURE 11. Third Form of Euclid's Axiom.

Ax. 11. Axiom of Continuity

$$\exists a\,\forall x\,\forall y\,[x \in X \wedge y \in Y \to B(axy)] \to \exists b\,\forall x\,\forall y\,[x \in X \wedge y \in Y \to B(xby)].$$

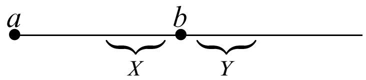

FIGURE 12. Axiom of Continuity.

The Axiom of Continuity asserts: any two sets X and Y such that the elements of X precede the elements of Y with respect to some point a (that is, $B(axy)$ whenever x is in X and y is in Y) are separated by a point b (see Figure 12). This axiom is not formulated within the framework of first-order logic, since the variables X and Y are second-order variables (that is, they are assumed to range over sets of points). However, in many applications we only need instances of the axiom in which the sets X and Y are definable by first-order formulas (possibly with the help of parameters). The collection of these first-order instances constitutes the (elementary) Axiom Schema of Continuty.

As. 11. Axiom Schema of Continuity

$$\exists a\,\forall x\,\forall y\,[\alpha \wedge \beta \to B(axy)] \to \exists b\,\forall x\,\forall y\,[\alpha \wedge \beta \to B(xby)],$$

where α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x.

Ax. 12. Reflexivity Axiom for Betweenness

$$B(abb).$$

Ax. 13. $a = b \to B(aba)$.

Ax. 14. Symmetry Axiom for Betweenness

$$B(abc) \to B(cba).$$

Ax. 15. Inner Transitivity Axiom for Betweenness

$$B(abd) \wedge B(bcd) \to B(abc).$$

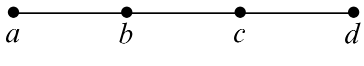

FIGURE 13. Transitivity Axioms for Betweenness.

Ax. 16. Outer Transitivity Axiom for Betweenness

$$B(abc) \wedge B(bcd) \wedge b \neq c \rightarrow B(abd).$$

Ax. 17. Inner Connectivity Axiom for Betweenness

$$B(abd) \wedge B(acd) \rightarrow [B(abc) \vee B(acb)].$$

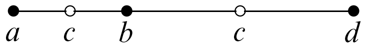

FIGURE 14. Inner Connectivity Axiom for Betweenness.

Ax. 18. Outer Connectivity Axiom for Betweenness

$$B(abc) \wedge B(abd) \wedge a \neq b \rightarrow [B(acd) \vee B(adc)].$$

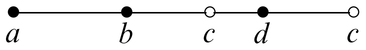

FIGURE 15. Outer Connectivity Axiom for Betweenness.

Ax. 19. $a = b \rightarrow ac \equiv bc.$

Ax. 20. Uniqueness Axiom for Triangle Construction

$$[ac \equiv ac' \wedge bc \equiv bc' \wedge B(adb) \wedge B(ad'b) \wedge B(cdx) \wedge B(c'd'x) \wedge d \neq x \wedge d' \neq x] \rightarrow c = c'.$$

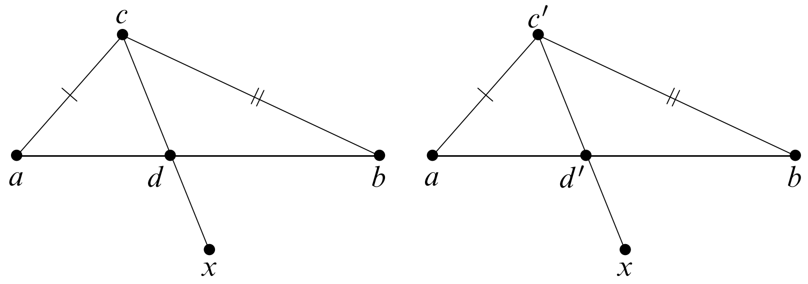

FIGURE 16. Uniqueness Axiom for Triangle Construction.

The Uniqueness Axiom for Triangle Construction asserts that at most one triangle can be constructed on a given segment, using a given side of the segment, with prescribed lengths for the other two sides. Specifically, it says: given a segment ab, there cannot be two distinct points c and c' on the same side of ab such that triangles $\triangle abc$ and $\triangle abc'$ are congruent (see Figure 16).

Degenerate triangles are allowed. The side of the segment ab on which the points c and c' lie is determined by a point x on the side opposite to c (and c') and a point d (respectively d') between a and b; the condition of being on the side opposite to x is expressed by the assertion $B(cdx)$ (respectively $B(c'd'x)$).

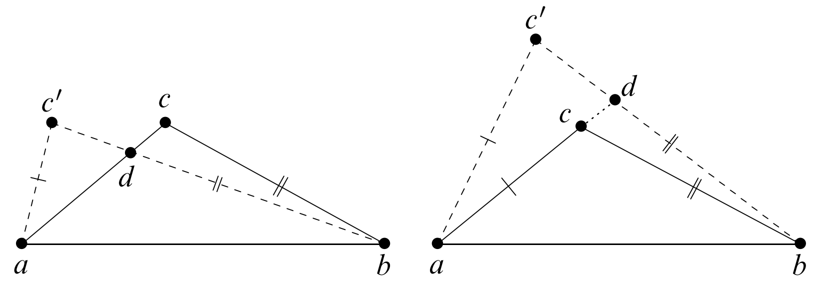

FIGURE 17. Variant of Axiom 20.

Ax. 20_1. Variant of Axiom 20

$$(a \neq b \wedge ac \equiv ac' \wedge bc \equiv bc' \wedge B(bdc') \wedge [B(adc) \vee B(acd)]) \rightarrow c = c'.$$

The variant of Ax. 20 uses a shorter, but more sophisticated, way of asserting that c and c' are on the same side of the segment ab (see Figure 17).

Ax. 21. Existence Axiom for Triangle Construction

$$ab \equiv a'b' \rightarrow \exists c\, \exists x\, (ac \equiv a'c' \wedge bc \equiv b'c' \wedge B(cxp) \wedge [B(abx) \vee B(bxa) \vee B(xab)]).$$

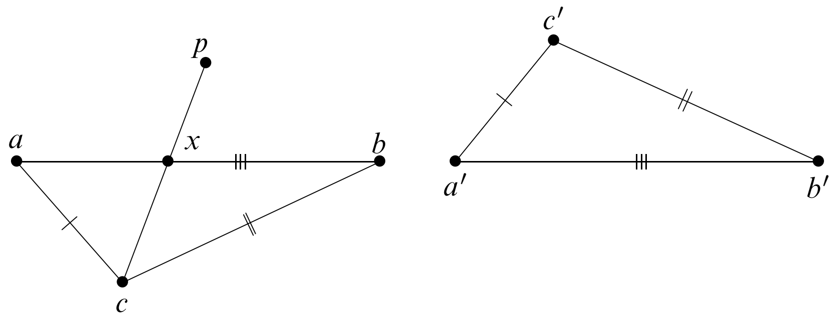

FIGURE 18. Existence Axiom for Triangle Construction.

The Existence Axiom for Triangle Construction asserts that, for any triangle $\triangle a'b'c'$ and any segment ab congruent to the side $a'b'$, there exists a point c on a specified side of ab such that triangles $\triangle abc$ and $\triangle a'b'c'$ are congruent (see Figure 18). As in Ax. 20, degenerate triangles are allowed.

The side of ab is specified as being the one opposite a point p; the point x collinear with a and b is used to express that p and c are on opposite sides of ab.

Ax. 22. Density Axiom for Betweenness

$$x \neq z \to \exists y\, [x \neq y \wedge z \neq y \wedge B(xyz)].$$

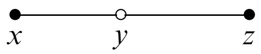

FIGURE 19. Density Axiom for Betweenness.

Ax. 23. $[B(xyz) \wedge B(x'y'z') \wedge xy \equiv x'y' \wedge yz \equiv y'z'] \to xz \equiv x'z'$.

FIGURE 20. Axiom 23.

Ax. 24. $B(xyz) \wedge B(x'y'z') \wedge xz \equiv x'z' \wedge yz \equiv y'z' \to xy \equiv x'y'$.

The picture for Ax. 24 is nearly the same as that for Ax. 23; see Figure 20.

§**2. Historical remarks concerning Tarski's system.** The axiom set for Euclidean geometry adopted in Schwabhäuser-Szmielew-Tarski [29] originates with Tarski. In its original form it was constructed in 1926–27 and presented in his course given that year at the University of Warsaw (see Tarski [45], footnote 34). It appeared in the paper Tarski [45], submitted for publication in 1940, but published only in 1967 in a restricted number of copies.[2]

All the axioms are formulated in terms of two primitive notions, the ternary relation of betweenness, B, and the quaternary relation of equidistance, $\equiv$, among points of a geometrical space. However, in the original axiom set the binary relation of equality between points, $=$, is also treated as a primitive geometrical notion—as opposed to all later versions of this set in which this notion is treated as the logical identity. The original set consists of twenty axioms, Ax. 1–Ax. 4, Ax. 5_1, Ax. 6, Ax. 7_2, Ax. $8^{(2)}$, Ax. ${9_1}^{(2)}$, Ax. 10, Ax. 12–Ax. 21, as well as all instances

[2]This paper (which is really a short monograph) is reproduced on pp. 289–346 of Tarski [46], volume 4.

of the axiom schema As. 11. Thus it is an axiom set for elementary 2-dimensional Euclidean geometry. The possibility of modifying the dimension axioms Ax. $8^{(2)}$ and Ax. $9_1^{(2)}$ in order to obtain an axiom set for n-dimensional geometry is briefly mentioned. (The case $n = 1$ will be disregarded in this section and in Sections 3–5.) The passage to an axiom set for the full (non-elementary) Euclidean geometry, by replacing all instances of the axiom schema As. 11 with Ax. 11, is not mentioned explicitly.

The next version of the axiom set appeared in Tarski [41]. Since $=$ is treated there as a logical notion, Ax. 13 and Ax. 19 are easily derivable from the remaining axioms, and therefore have been omitted. Ax. 20 is replaced by a somewhat more concise variant, Ax. 20_1; we do not analyze this modification since Ax. 20 is dropped entirely in subsequent versions.

A rather substantial simplification of the axiom set in Tarski [41] was obtained in 1956–57 as a result of joint efforts by Eva Kallin, Scott Taylor, and Tarski (see Tarski [44], p. 20, footnote). First, four axioms, Ax. 5_1, Ax. 7_2, Ax. $9_1^{(2)}$, and Ax. 10, have been respectively replaced by equivalent formulations Ax. 5, Ax. 7_1, Ax. $9^{(2)}$, and Ax. 10_1. In the case of Ax. $9_1^{(2)}$ the new formulation differs essentially from the old one, in both its form and its mathematical content. In the remaining three cases the differences are very slight. Some remarks in the later discussion will throw light on the purpose of all these modifications. Next, in the modified axiom set six axioms, Ax. 12, Ax. 14, Ax. 16, Ax. 17, Ax. 20_1, and Ax. 21, are shown to be derivable from the remaining ones, and hence are omitted. Thus we arrive at the set consisting of twelve axioms: Ax. 1–Ax. 6, Ax. 7_1, Ax. $8^{(2)}$, Ax. $9^{(2)}$, Ax. 10_1, Ax. 15, Ax. 18, and all instances of the old axiom schema As. 11. This axiom set was discussed by Tarski in his course on the foundations of geometry given at the University of California, Berkeley, during the academic year 1956–57. It appeared in print in Tarski [44]. It was pointed out there that, by enriching the logical framework of our system of geometry and by replacing the axiom schema As. 11 with the (second-order) sentence Ax. 11, we arrive at an axiom set for the full (non-elementary) 2-dimensional Euclidean geometry. Also, it was mentioned that, by replacing Ax. $8^{(2)}$ and Ax. $9^{(2)}$ in either of the two above axiom sets with their n-dimensional analogues ($n = 3, 4, \dots$), which are explicitly listed in Section 1 above as Ax. $8^{(n)}$ and Ax. $9^{(n)}$, axiom sets for n-dimensional geometry are obtained.

Some general metamathematical results, published at about the same time in Scott [30] and Szmielew [37], show that the dimension axioms Ax. $8^{(n)}$ and Ax. $9^{(n)}$, and Euclid's axiom Ax. 10_1 can be equivalently replaced by a great variety of sentences. The results will be discussed in Section 4, in connection with the axioms involved. It does not seem that these results lead to any formal simplification of the axiom sets discussed here.

The last simplifications so far obtained are due to Gupta [5], where it is shown that Ax. 6 and Ax. 18 can be derived from the remaining axioms in Tarski [44]. Contrary to what could be expected, the derivation is not quite easy, and actually in the case of Ax. 18 it is rather involved. The axiom sets thus reduced clearly consist of sentences Ax. 1–Ax. 5, Ax. 7_1, Ax. $8^{(n)}$ and Ax. $9^{(n)}$ for $n = 2, 3, \dots,$ Ax. 10_1, Ax. 15, and either all instances of the schema As. 11, or the single sentence Ax. 11. We shall denote this axiom set by $EG^{(n)}$ in the elementary case and $FG^{(n)}$ in the non-elementary case.

Some further results in Gupta [5] provide the possibility of constructing various equivalent variants of $EG^{(n)}$ and $FG^{(n)}$; again, however, this does not seem to lead to any formal improvements in $EG^{(n)}$ and $FG^{(n)}$. In particular, by results in *op. cit.*, pp. 12, 20, 40, 42–91, it turns out to be possible to replace Ax. 7_1 and Ax. 15 by Ax. 7 and Ax. 6 respectively.

Around 1965 Szmielew, in collaboration with Tarski, prepared a manuscript containing a full development of 2-dimensional Euclidean geometry based upon a variant of $EG^{(2)}$. As an axiom set she took the collection of sentences obtained from $EG^{(2)}$ by replacing Ax. 7_1, Ax. 10_1, and Ax. 15 with Ax. 7, Ax. 10_2, and Ax. 6 respectively. The equivalence of this variant with $EG^{(2)}$ follows immediately from the previously mentioned results in Gupta [5] and Szmielew [37]. As will be seen in Section 7, the manuscript just mentioned was used as a basis in preparing a substantial portion of the first part of Schwabhäuser-Szmielew-Tarski [29]. The axiom set upon which Part I of that work is founded differs from that of the manuscript only in that Ax. 10 (a slight variant of the original form of Euclid's axiom, Ax. 10_1, in $EG^{(n)}$), is used instead of Ax. 10_2; moreover, for $n > 2$ the lower and upper dimension axioms differ from those in $EG^{(n)}$—a modification which is made possible by the general result in Scott [30]. We shall denote the axiom set of Schwabhäuser-Szmielew-Tarski [29] by $EH^{(n)}$ in the elementary case and $FH^{(n)}$ in the non-elementary case.[3]

Obviously, there are a number of basic elementary laws which enter at a very early stage in the systematic development of geometry and which do not occur in both sets $EG^{(n)}$ and $EH^{(n)}$, or at least in one of them; such are for instance Ax. 12, Ax. 14, Ax. 16–Ax. 18, as well as Ax. 6 in $EG^{(n)}$ and Ax. 15 in $EH^{(n)}$. It seems that, in general, the derivation of these laws is somewhat simpler on the basis of $EH^{(n)}$ rather than $EG^{(n)}$. In particular, the involved derivation of Ax. 18 from Gupta [5] in $EG^{(n)}$ can be simplified

[3]It would be interesting to know whether Ax. 6 may equivalently be replaced by Ax. 15 in $EH^{(n)}$ and $FH^{(n)}$. If this were the case, then in all axiom sets discussed, $EG^{(n)}$, $FG^{(n)}$, $EH^{(n)}$, and $FH^{(n)}$, either of the axioms Ax. 6 and Ax. 15 could be equivalently replaced by the other for $n \geq 2$.

in $EH^{(n)}$. This may speak somewhat in favor of the selection of $EH^{(n)}$, as opposed to $EG^{(n)}$, as a basis for the development of Euclidean geometry.[4,5]

§3. Distinctive features of Tarski's system of foundations of geometry. Tarski's system of foundations of geometry has a number of distinctive features, in which it differs from most, if not all, systems of foundations of Euclidean geometry that are known from the literature. Of the earlier systems probably the two closest in spirit to the present one are those in Pieri [23] and Veblen [51].

One of the important features of Tarski's development is the clear distinction between the full geometry and its elementary part. By "elementary" we understand that portion of geometry which, loosely speaking, can be developed without the help of set-theoretic notions. In technical terms the system of elementary geometry, based upon one of the axiom sets $EG^{(n)}$ or $EH^{(n)}$, is developed entirely within the framework of first-order predicate logic. On the other hand, the system of full geometry, based upon $FG^{(n)}$ or $FH^{(n)}$, requires as a framework a system of higher-order logic, or else the first-order logic enriched by some fragment of (axiomatic) set theory. When speaking in these notes of models of elementary or full Euclidean geometry, we can

[4]I [Tarski] am not sure that the choice of $EH^{(n)}$ over $EG^{(n)}$ is quite justified. It seems to me that the derivation of Ax. 6 and Ax. 7 from $EG^{(2)}$ is not very involved. As regards the derivation of Ax. 7, it seems to me considerably simpler than that of Ax. 7_1, in $EH^{(2)}$, and, as opposed to what happens in Schwabhäuser-Szmielew-Tarski [29], it can be given at an early stage of the development, which is probably more natural. If you [Schwabhäuser] wish, I can send you the derivation of Ax. 7 which I have found in my old course notes.

[5]*There are various finitely axiomatized subsystems of n-dimensional elementary Euclidean geometry that have also played an important role in modern foundational research. One such is the theory of geometrical constructions that can be carried out in n-dimensional space using only a straightedge and compass. Tarski [44] observed that a set of axioms for this geometry can be obtained from $EG^{(n)}$ (or, equivalently, from $EH^{(n)}$) by replacing all instances of the Continuity Schema, As. 11, with a single sentence, the Circle Axiom. This sentence asserts that any segment which joins two points, one inside and one outside a given circle (with which the segment is coplanar), must intersect that circle. Let us denote the resulting set of axioms by $CG^{(n)}$. Its models are, up to isomorphisms, just the n-dimensional Cartesian spaces over Euclidean ordered fields—ordered fields in which every positive number has a square root.

Another such subsystem is the theory based on the set of axioms $PG^{(n)}$ obtained by deleting from $EG^{(n)}$ all instances of As. 11 (without adjoining the Circle Axiom). This geometry is weaker than that of straightedge and compass constructions: up to isomorphisms, its models are precisely the n-dimensional Cartesian spaces over Pythagorean ordered fields—ordered fields in which, for any elements a and b, there is an element c such that $a^2 + b^2 = c^2$.

It should be mentioned that the models of $EG^{(n)}$ are, up to isomorphisms, just the n-dimensional Cartesian spaces over real closed fields—Euclidean ordered fields in which every polynomial of odd degree with coefficients from the field has a root. This is the principal representation theorem in Tarski [44]. Of course the theory $FG^{(n)}$ has a unique model, up to isomorphisms: the n-dimensional Cartesian space over the field of real numbers.

restrict ourselves to standard models, i.e., to familiar Cartesian models over the field of real numbers. (This does not apply however to models involved in proofs of independence of axioms and primitive notions.)

Another distinctive feature of Tarski's system is the formal simplicity of the axioms upon which the development is based. As opposed to Tarski's system, in all the systems of geometry known from the literature, at least some—and sometimes even most—axioms are not formulated directly in terms of primitive notions, but contain also other notions, previously defined. It is evident that the formal complexity of such an axiom set becomes apparent only if the axioms are reformulated exclusively in terms of primitive notions by eliminating all defined ones. When referring below to axiom sets from the literature, we shall assume that such a reformulation has actually been carried out.

As is well known, the notion of simplicity is rather ambiguous, and is open to various interpretations. If we consider systems of full geometry which are based upon finite axiom sets, we can use as a measure of simplicity the most obvious criterion, namely the total length of the axiom set, i.e., the sum of the lengths of all its particular axioms. (When determining the length of an axiom we count all the occurrences of variables, as well as logical and non-logical constants, which appear in this axiom, but we disregard parentheses and commas. It makes little difference whether or not we count the initial universal quantifiers and immediately following variables which are not explicitly printed.)

If we compare in this sense the formal simplicity of $FG^{(n)}$ or $FH^{(n)}$ with axiom sets for n-dimensional geometry known from the literature, the conciseness of the former becomes apparent. For illustration consider the axiom set of Pieri [23] for the full 3-dimensional Euclidean geometry. It consists of 24 axioms. The only primitive notion used in the system is the ternary relation which holds among points a, b, c if, and only if, b and c are equidistant from a. Most of Pieri's axioms are formulated with the help of some defined notions. However, after each of them, with the exception of the two non-elementary axioms XXIII and XXIV, he gives a reformulation in which all of the defined notions have been eliminated. It now turns out that the length of just one of Pieri's 24 axioms, in fact XXI (a form of the Pasch axiom) is not much smaller than the total length of the set $FG^{(3)}$, and *a fortiori* of the slightly shorter set $FH^{(3)}$. If, in addition to XXI, we consider one of Pieri's shorter axioms, say XI, then the sum of the lengths of these two axioms proves to exceed the total length of $FG^{(3)}$.

It would be natural to conjecture that the conciseness of Tarski's axiom set has been achieved by including in this set certain geometrical laws of simple structure, intuitively clear content, and great deductive power, which had not previously been used in constructing sets of axioms. From the remarks in the next section, we shall see that this is not the case. On the contrary, nearly every one of Tarski's axioms, or a simple variant of it, has been used as an axiom in some earlier work.

It seems to us that two factors did contribute to the conciseness of the axiom set. The first was the selection of the betweenness and equidistance relations as the only two primitive notions. Both notions have a clear and simple geometrical meaning; the former represents the affine, the latter the metric, aspect of geometry. Moreover, the two notions have jointly a great expressive power in the sense that they permit us to formulate in a natural and concise way most of the basic laws and definitions involved in the development of geometry. It should be mentioned that the system of geometry outlined in Veblen [51] is based on just the same two primitive notions.

To describe the second factor, notice that laws of elementary geometry are traditionally formulated in such a way that limit cases, called also trivial or degenerate cases, are excluded by means of special restrictive premises appearing in the formulations of the laws. Omission of these restricting premises not only simplifies the structure of the laws, but frequently strengthens their deductive power as well, and, in particular, permits us to derive from them some more elementary laws, which otherwise would have to be formulated as separate axioms. For example, Ax. 7 is usually formulated with the restrictive assumption that a, c, b (or a, c, q) are not collinear, while Ax. 4 is usually provided with the premise $b \neq c$; since, however, the restrictions in $FH^{(n)}$ have been omitted, we can first immediately derive Ax. 12 from Ax. 3 and Ax. 4, then Ax. 14 from Ax. 6, Ax. 7, and Ax. 12, and finally Ax. 15 from Ax. 7 and Ax. 14. For the reasons stated above we have omitted all such restrictive premises in constructing our axiom set whenever such an omission has not affected the validity of the axiom involved.

Turning now to Tarski's system of elementary geometry, we notice that its axiom set, $EG^{(n)}$ or $EH^{(n)}$, is infinite, and hence the notion of the total length of an axiom set as a measure of its simplicity is in this case not applicable. (To save this situation one would have to introduce an *ad hoc* notion of the length of an axiom schema. In fact, in computing the length of the schema As. 11 we would treat the letters α and β as if they were variables of our geometrical theory, and not letters standing for formulas of this theory. In this case the measure of simplicity of $EG^{(n)}$ and $EH^{(n)}$ differs but little from that of $FG^{(n)}$ and $FH^{(n)}$.)

It seems, however, much more interesting to discuss the simplicity of the axiom sets using criteria which can equally well be applied to both finite and infinite axiom sets. One such criterion is the number of alternations of quantifiers. We assume that every sentence involved has been replaced by a logically equivalent sentence σ in prenex normal form,

$$\sigma = (Q^{(1)}x_{1,1}Q^{(1)}x_{1,2}\cdots)\cdots(Q^{(n)}x_{n,1}Q^{(n)}x_{n,2}\cdots)\varphi,$$

where n is positive integer, φ is any quantifier-free formula, and each $Q^{(i)}$, with $i = 1, \ldots, n$, is either the universal quantifier $\forall$ or the existential quantifier $\exists$, with the assumptions that for no $i < n$ are $Q^{(i)}$ and $Q^{(i+1)}$ both universal or both existential. The number $n-1$ is referred to as the number of quantifier alternations of the sentence σ. Thus a sentence with 0 quantifier alternations is either a universal sentence, $\forall x_0\ \forall x_1 \ldots \varphi$, or an existential sentence, $\exists x_0\ \exists x_1 \ldots \varphi$; a sentence with 1 quantifier alternation is either a universal-existential sentence, $\forall x_0\ \forall x_1 \ldots \exists y_0\ \exists y_1 \ldots \varphi$, or an existential-universal sentence, $\exists x_0\ \exists x_1 \ldots \forall y_0\ \forall y_1 \ldots \varphi$; etc. A set X of sentences in prenex normal form has (at most) n quantifier alternations if this is true of every sentence in X. In particular we can speak of universal, existential, universal-existential, etc., sets of sentences. When comparing two sentences, σ and τ, in prenex normal form, or two sets, S and T, of such sentences, it seems natural to consider σ, or S, as structurally simpler than τ, or T, if the number of quantifier alternations is smaller in the former than in the latter.

If from this point of view we consider the axiom sets $EG^{(n)}$ and $EH^{(n)}$, we notice at once that all the axioms of these sets which are not instances of As. 11 are either universal, existential, or universal-existential sentences, or else are trivially equivalent to such sentences. However this does not apply in general to axioms which are instances of As. 11. In fact, since the number of quantifier alternations in the formulas α and β appearing in As. 11 can be arbitrarily large, the same is true of the corresponding instances of As. 11 when they are written in prenex normal form. Nevertheless, a set Σ of sentences can be constructed which consists entirely of universal-existential sentences and which can equivalently replace the set of all instances of As. 11 in our axiom sets on the basis of the remaining axioms.[6] This observation is by no means obvious and depends upon some deep metamathematical results concerning elementary algebra and geometry (which are discussed in the second part of Schwabhäuser-Szmielew-Tarski [29]). At any rate, we conclude that, by modifying appropriately the axiom sets $EG^{(n)}$ and $EH^{(n)}$, we arrive at universal-existential axiom sets for elementary n-dimensional Euclidean geometry; this result was announced in Tarski [44], p. 24. The result cannot be improved: no axiom set can be constructed for elementary geometry which is structurally simpler in this respect, i.e., which has 0 quantifier alternations.

[6]The set Σ referred to above consists of sentences which, in geometrical language, provide a characterization of those fields which are real closed, thus of sentences to the effect that every real number or its negative has a square root, and every polynomial of odd degree with real coefficients has a real root.

The simplification of an axiom set in either of the two directions discussed in the preceding remarks may certainly prove helpful in the metamathematical study of the theory based upon this axiom set—in particular, in those situations when one attempts to show that certain mathematical structures are models of this theory. In addition, the simplification achieved in the second direction, i.e., in fact the construction of a universal-existential axiom set, has—as is well known—some interesting model-theoretic consequences of a general character.[7]

To conclude this section we mention still some features of Tarski's axiomatization (which, however, are certainly shared by various other axiomatizations known from the literature). First, the axiom sets for m-dimensional and n-dimensional geometry, $m, n \geq 2$ and $m \neq n$, are identical except for the upper and lower dimension axioms (Ax. $8^{(m)}$, Ax. $9^{(m)}$ in one case and Ax. $8^{(n)}$, Ax. $9^{(n)}$ in the other). The question naturally arises whether the set obtained from $EG^{(2)}$ or $EH^{(2)}$ by deleting Ax. $9^{(2)}$ (but not Ax. $8^{(2)}$) can serve as an adequate axiom set for the elementary dimension-free geometry, i.e., the set of first-order sentences valid in all n-dimensional Euclidean geometries for $n \geq 2$. The conjecture that this is indeed the case is essentially due to Scott [30], p. 66, and was affirmed definitively by Gupta [5], p. 407.[8]

Second, the sets $EG^{(n)}$ and $EH^{(n)}$ are complete, i.e., every first-order sentence, or its negation, is derivable from these sets. The axiom sets $FG^{(n)}$ and $FH^{(n)}$, in addition to being complete in the domain of first-order sentences, turn out to be categorical, i.e., any two models of them are isomorphic. As a consequence, they are semantically complete, i.e., every sentence formulated in the language of this system is either true in all models, or fails in all models, of the system. These results are discussed in detail in the second part of Schwabhäuser-Szmielew-Tarski [29].

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[7]*There is yet another criterion that can be employed to judge the simplicity of a finite set of axioms. One set of axioms (written in prenex normal form) can be regarded as being simpler than another if the number of distinct variables needed to formulate the axioms is smaller. Scott [30], p. 61, showed that any first-order sentence with $n+1$ distinct variables will be true in every Cartesian space of dimension at least n over the field of real numbers if and only if it is true in at least one such space. This theorem implies that axiomatizations of n-dimensional subsystems of n-dimensional Euclidean geometry, for example the theory of n-dimensional Cartesian spaces over Pythagorean or Euclidean ordered fields, must always involve at least $n+2$ distinct variables.

For $n \geq 3$, Pambuccian [20] gives a set of axioms equivalent to $CG^{(n)}$ that uses exactly $n+2$ distinct variables. The same paper gives sets of axioms equivalent to $PG^{(2)}$ and to $CG^{(2)}$ that use only six distinct variables and five distinct variables respectively; it is shown in *op. cit.* and in Pambuccian [21] that these results are optimal: there are no equivalent sets of axioms that use fewer distinct variables.

[8]Scott [30] writes: "Though all details have not been completely checked by the author, it would seem that an adequate axiomatization of the theory $\mathcal{E}$ [elementary dimension-free Euclidean geometry] would result by dropping axioms A11 and A12 [axioms Ax. $8^{(2)}$ and Ax. $9^{(2)}$ above] of the system given by Tarski in [3]". The cited work is Tarski [44].

§4. Observations concerning individual axioms. The remarks below, primarily of a historical nature, concern sentences occurring in $FG^{(n)}$ and $FH^{(n)}$. We disregard, however, the most elementary axioms Ax. 1–Ax. 4, Ax. 6, Ax. 15, and Ax. 18.

The Five-Segment Axiom, Ax. 5, plays the basic role in deriving the fundamental theorems on the congruence of angles and triangles from the assumptions involving only the congruence of segments (i.e., the equidistance relation). It is a slight (and inessential) modification of Axiom XI in Veblen [51]. A somewhat more complicated but essentially equivalent form of Ax. 5, the Six-Segment Axiom, was used as an axiom even earlier, in Mollerup [16].

The Inner Pasch Axiom, Ax. 7, and Outer Pasch Axiom, Ax. 7_1, are highly simplified and specialized variants of the well-known Pasch Axiom (Axiom IV in Pasch [22]; see also Axiom XXI in Pieri [23], mentioned above). In selecting Ax. 7_1 as an axiom in $EG^{(n)}$ and $FG^{(n)}$, and deriving Ax. 7 as a theorem, Tarski followed the procedure in Schur [28], p. 7, where Postulate 6 and Theorem 5 differ but very little from Ax. 7_1 and Ax. 7 respectively; on p. 9 of *op. cit.*, Moore [17] is credited with the idea of this development. A weaker form of Ax. 7_1 occurs as Axiom VIII in Veblen [50]. The proof that this procedure can be reversed, i.e., that Ax. 7 can be selected as an axiom and Ax. 7_1 derived as a theorem, is due to Gupta [5], Theorem 3.70.

We now turn to the lower and upper n-dimensional axioms. In earlier versions of his axiom set, Tarski used Ax. $8^{(2)}$ as the lower 2-dimensional axiom and Ax. $9_1^{(2)}$ as the upper 2-dimensional axiom. Ax. $8^{(2)}$ is a simple sentence which naturally suggests itself as a lower 2-dimensional axiom, and was used in all later versions of Tarski's system. As regards Ax. $9_1^{(2)}$, it is seen that this axiom, and even more so its simplified variant Ax. $9_2^{(2)}$, express the following idea: any straight lines in the Euclidean plane which intersect in three different points a, b, c, divide the plane into seven regions in such a way that every point x of the plane belongs to one of these seven regions, and this belonging is characterized by an appropriate betweenness relation holding among the point x, one of the vertices of the triangle $\triangle abc$, and a point y on the side of the triangle opposite this vertex; see, e.g., Enriques [3], pp. 85–86. Ax. $9_1^{(2)}$ and Ax. $9_2^{(2)}$ are formulated entirely in terms of the betweenness relation. Such sentences are useful if one desires to construct an axiom set for 2-dimensional affine Euclidean geometry, treating the betweenness relation as the only primitive notion; cf. Szczerba-Tarski [34] where Ax. $9_2^{(2)}$ is actually used for this purpose.[9] Ax. $9_2^{(2)}$ (and even Ax. $9_1^{(2)}$) is rather concise as compared with other known sentences which are formulated entirely in terms of the betweenness relation and can serve as upper 2-dimensional axioms. The situation changes essentially if we take

[9]The full version of this paper appeared soon after these notes were written; see Szczerba-Tarski [35].

up the task of constructing n-dimensional analogues, Ax. $8_2^{(n)}$ and Ax. $9_2^{(n)}$, of Ax. $8^{(2)}$ and Ax. $9_2^{(2)}$ (which we shall not formulate here explicitly): as n increases, the sentences become really involved. Independent of the specific axioms discussed here it should be pointed out that, by results in Kordos [6], no first-order universal-existential sentence formulated entirely in terms of betweenness can serve as a lower n-dimensional axiom for $n \geq 3$; hence, using dimensional axioms so formulated we can never construct a universal-existential axiom set for elementary geometry of dimension $n \geq 3$.

If only for these reasons it is more advantageous to use for elementary geometry dimension axioms involving the notion of equidistance. Such are indeed Ax. $9^{(n)}$ which, in case $n = 2$, occurs explicitly in later versions of Tarski's axiom set. We do not know of any place in the literature where these, or closely related, sentences were actually used in constructing geometrical axiom sets. The sentences Ax. $9^{(n)}$ are concise in form and have a clear mathematical content. Even in the case of $n = 2$, Ax. $9^{(n)}$ is considerably shorter than Ax. $9_2^{(2)}$, and the difference in length rapidly increases as n grows larger. It is also important to notice that, for all values of n, Ax. $9^{(n)}$ is a universal sentence, and hence Ax. $8^{(n)}$ can be formulated as an existential sentence.

It is worthwhile to recall that, by a result of Scott [30], pp. 63–66, every first-order sentence σ (formulated in terms of betweenness and equidistance) which, just as Ax. $9^{(n)}$, holds in all Euclidean spaces of dimension $\leq n$, but fails in all spaces of dimension $> n$, can equivalently replace Ax. $9^{(n)}$ in $EG^{(n)}$ or $FG^{(n)}$, and $\neg\sigma$ can so replace Ax. $8^{(n+1)}$ in $EG^{(n+1)}$ or $FG^{(n+1)}$. This provides us with a great variety of sentences which can be adopted as dimension axioms for Euclidean geometry. Interesting examples, different from those discussed above, can be found in the proof of Lemma 4.1 in Scott [30]. However, no such sentences known to us are simpler than Ax. $8^{(n)}$ and Ax. $9^{(n)}$.

The axiom Ax. 10 in $FH^{(n)}$, or its variant Ax. 10_1 in $FG^{(n)}$, is one of the least known and most concise forms of the famous Axiom of Euclid (the Parallel Axiom). It is a specialized variant of the statement found in Lorenz [10], vol. I, pp. 101–102, to the effect that every interior point of an angle lies always on a line intersecting the two sides of the angle.[10] As in the case of Ax. $9_1^{(2)}$, it is formulated entirely in terms of betweenness, and hence is useful in the construction of an axiom set for affine geometry.

There is a curious structural connection between Ax. 10_1 and the Outer Pasch Axiom, Ax. 7_1. In fact consider the Weak Pasch Axiom, Ax. 7_3. It is directly derivable from Ax. 7_1 with the help of Ax. 12. On the other hand,

[10] The reference to Lorenz is taken exactly from p. 11a, footnote 4, of Gupta [5]. In early Italian editions of Enriques [3] the origin of Ax. 10 is attributed to Legendre, in later editions to Lorenz, but no bibliographic details are given.

there is a striking similarity between Ax. 7_3 and Ax. 10_1. One could say that Ax. 7_3 is an inner form of Ax. 10_1 or, equivalently, that Ax. 10_1 is an outer form of Ax. 7_3. It is not known whether Ax. 7_3 can equivalently replace Ax. 7_1, nor whether Ax. 7_3 is derivable from Ax. 10_1. If both these problems, or even just the latter, had a positive solution, this would exhibit a rather unexpected connection between the Pasch and the Parallel Axioms.

Recall that, by a result in Szmielew [37], p. 51, Ax. 10_1 can be equivalently replaced in $EG^{(n)}$ (as well as Ax. 10 in $EH^{(n)}$) by any first-order sentence which is derivable from this axiom set, but ceases to be derivable if Ax. 10_1 is deleted. While Ax. 10_1 is formulated entirely in terms of the betweenness relation, some equivalent sentences are known which involve essentially the equidistance relation, but are as simple as, or even somewhat simpler than, Ax. 10_1 and have, perhaps, a somewhat clearer mathematical content. An example of such a sentence is Ax. 10_2, which implies Ax. 10_1 by results of Bolyai [1] and which expresses the fact that every triangle can be inscribed in a circle.[11] Another example of such a sentence is Ax. 10_3, which is somewhat longer than Ax. 10_1, but—in opposition to the later—is universal; it essentially expresses the fact that the sum of the angles of an arbitrary triangle is equal to two right angles.[12] The derivation of Ax. 10_1 from Ax. 10_3 is based on results of Saccheri [26]; see also Legendre [8].

It seems interesting that none of the above variants of the Parallel Axiom, Ax. 10–Ax. 10_3, is directly related in its intuitive mathematical content to the basic idea of Euclid's original axiom, i.e., to the non-existence of two intersecting straight lines parallel to a given third line; it seems that the expression of this idea in terms of our primitive notions is necessarily more complicated.[13]

We turn finally to Ax. 11. It is a modified form of the well-known Continuity Axiom which, in its application to the theory of real numbers, was first stated in Dedekind [2]. The modification consists in the removal of some additional conditions often imposed on X and Y such as the condition that $X \cup Y$ is a straight line. This both simplifies and generalizes the formulation

[11] As regards Ax. 10_2 [as a replacement for Ax. 10 in higher dimensions], I [Tarski] am convinced that it suffices for $n > 2$ if its conclusion is strengthened by the condition that x is coplanar with a, b, and c. I believe also that this stronger form can be derived from the present form of Ax. 10_2 for every $n > 2$, without the help of the Parallel Axiom (using the properties of perpendicular projections).

[12] Do you [Schwabhäuser] know whether there is an equivalent form of Euclid's Axiom which is formulated entirely in terms of betweenness and is a universal sentence? [Schwabhäuser subsequently provided a negative answer to this problem; see Theorem 6.42 in Part II of Schwabhäuser-Szmielew-Tarski [29].]

[13] In the above discussion of the Parallel Axiom, the historical references are taken from Enriques [3], where I [Tarski] do not find any further details, e.g., page numbers.

of the axiom, and in consequence facilitates its use. Ax. 11 appears to be simpler than all other statements securing the continuity of geometric spaces which can be found in the literature.

§5. Remarks concerning the independence of axioms and primitive notions. We recall that a sentence in a set Σ of sentences is said to be *independent* (in Σ) if it is not derivable from the remaining sentences in Σ; the set (whether finite or infinite) is called *independent* if every sentence in Σ is independent. From the discussion in Gupta [5], pp. 1, 40–41c, it follows that in $FG^{(2)}$ each of the axioms Ax. 2, Ax. 4, Ax. 5, Ax. $8^{(2)}$, Ax. $9^{(2)}$, Ax. 10_1, Ax. 11, and Ax. 15 is independent, i.e., cannot be derived from the remaining axioms of $FG^{(2)}$; however the independence of the three remaining axioms Ax. 1, Ax. 3, and Ax. 7_1 is still an open question. If, on the other hand, we modify $FG^{(2)}$ by replacing Ax. 10_1 with Ax. 10_2, then, according to Szczerba [33], axiom Ax. 7_1 proves to be independent as well. To our knowledge, the problem whether Ax. 7_1 is independent in $FG^{(2)}$ itself remains open. This problem seems to be connected with the problems concerning the weak Pasch axiom mentioned in the preceding section. The proof of independence for each of the axioms σ mentioned above is carried through in the usual way, i.e., by constructing an independence model in which all other axioms hold and σ fails. With the exception of Ax. 7_1, the constructions involved are either well known (as in the case of the Parallel Axiom) or else rather simple. As regards, however, Szczerba's construction of the independence model for Ax. 7_1 the situation is different; compare the remarks below concerning the definability of betweenness in terms of equidistance in 1-dimensional geometry.

On the other hand, the problem of whether the set $EG^{(2)}$ is independent has a trivial negative solution, since for every instance σ of the schema As. 11 we can obviously construct another instance σ' which is formally different from, but logically equivalent to, σ. As a consequence a new, rather imprecise problem arises: to describe in a "simple and natural" way a set Γ of first-order sentences which, together with all sentences of $EG^{(2)}$ that are not instances of As. 11, would form an independent set logically equivalent to $EG^{(2)}$. It is known that such a set Γ must be infinite; see Tarski [44]. We do not know whether this problem will ever find a positive solution.

We may of course apply to $EG^{(2)}$ a weaker notion of independence, treating the set of all instances in As. 11 as if it were a single sentence, i.e., we agree to consider As. 11 as being independent if at least one of its instances is not derivable from those axioms of $EG^{(2)}$ which are not instances of As. 11. In this case all the observations made so far concerning the independence of axioms in the original and modified versions of $FG^{(2)}$ easily extend to $EG^{(2)}$. It seems very likely that all the independence results stated above hold also for axiom sets of higher dimensional geometry. However we haven't checked this matter in detail.[14]

We now take up the problem of the independence of primitive notions. Since in our systems of geometry there are just two primitive notions, B and $\equiv$, we have only two independence problems. In a given geometrical system we say that one of our primitive notions, say $\equiv$, is *independent* if it is undefinable in terms of the other primitive notions, in this case B. Recall that $\equiv$ is *definable* in terms of B if there is a sentence which can be derived from the axioms of the system and which is a possible definition of $\equiv$ in terms of B, i.e., has the form

$$\forall x\ \forall y\ \forall z\ \forall u\ (xy \equiv zu \leftrightarrow \psi),$$

where ψ is a formula in which B occurs as the only non-logical constant. As is easily seen, we can also assume without loss of generality that just four variables, x, y, z, u, occur free in ψ. In case we consider a system of elementary n-dimensional geometry (for $n \geq 2$) based, say, on the axiom set $EG^{(n)}$, then ψ must be a first-order formula, i.e., contain only individual (that is, first-order) variables ranging over points of the space; $\equiv$ is then said to be elementarily, or first-order, definable in terms of B. If, on the other hand, we are interested in full geometry, based say on the axiom set $FG^{(n)}$, then ψ may also contain second-order variables ranging over sets of points, operations on, and relations among, points; ψ may even contain higher-order variables ranging over sets of sets of points, etc. (assuming that the fragment of set theory, or the higher-order logic, upon which our geometry is based admits such variables). The primitive notion $\equiv$ is then called second-order, third-order, etc., definable in terms of B.

It now turns out that, in all systems of geometry considered here, $\equiv$ is independent, i.e., not definable in terms of B. This can be proved by means of the well-known method of Padoa, indeed by exhibiting, for any given system, two relational structures $\langle S_1, B_1, \equiv_1\rangle$ and $\langle S_2, B_2, \equiv_2\rangle$ which are both models of the given system, and in which S_1 and B_1 respectively coincide with S_2 and B_2, while $\equiv_1$ and $\equiv_2$ do not coincide, i.e., for some x, y, z, u, one of the formulas $xy \equiv_1 zu$ and $xy \equiv_2 zu$ holds and the other fails.

[14]These remarks do not concern the axiom sets $FH^{(2)}$ and $EH^{(2)}$. I [Tarski] hope that you [Schwabhäuser] know what the situation [regarding the independence of the axioms] is for these sets. Certainly, various facts stated in Gupta [5] can be applied here, but there are also some important gaps. [It is remarked on p. 26 of Schwabhäuser-Szmielew-Tarski [29] that axioms Ax. 4, Ax. 5, Ax. 6, Ax. $8^{(2)}$, Ax. $9^{(2)}$, Ax. 10, and Ax. 11 are independent in $FH^{(2)}$.] Kordos [7] states without proof the following information: in $FH^{(2)}$, with Ax. 11 deleted and Ax. 10 replaced by Ax. 10_2, the only independence problem that remains open is the one concerning Ax. 1. I do not know whether, and to what extent, the information given by Kordos continues to hold if we do not delete Ax. 11.

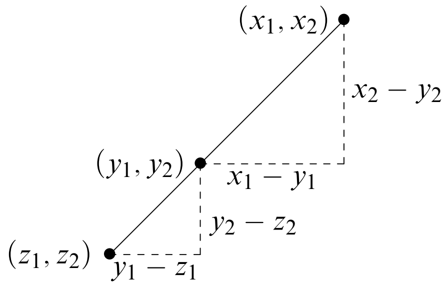

FIGURE 21. The definition of betweenness in the Cartesian plane.

To give two such models, consider the set S of all ordered pairs of real numbers; thus every $x \in S$ has the form $x = (x_1, x_2)$, where $x_1, x_2 \in R$. We define the relations B_1 and $\equiv_1$ among elements of S by stipulating that:

$$B_1(xyz) \leftrightarrow [(x_1 - y_1) \cdot (y_2 - z_2) = (x_2 - y_2) \cdot (y_1 - z_1)] \wedge \\ [0 \leq (x_1 - y_1) \cdot (y_1 - z_1)] \wedge [0 \leq (x_2 - y_2) \cdot (y_2 - z_2)]$$

(i.e., $B_1(xyz)$ if and only if the ratios and orientations of the corresponding sides of the triangle with vertices (x_1, x_2), (y_1, y_2), and (x_1, y_2), and the triangle with vertices (y_1, y_2), (z_1, z_2), and (y_1, z_2) are equal, so that the triangles are similar and similarly oriented—see Figure 21), and

$$xy \equiv_1 zu \leftrightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 = (z_1 - u_1)^2 + (z_2 - u_2)^2.$$

(this is just the definition of equidistance based on the Pythagorean Theorem: see Figure 22).

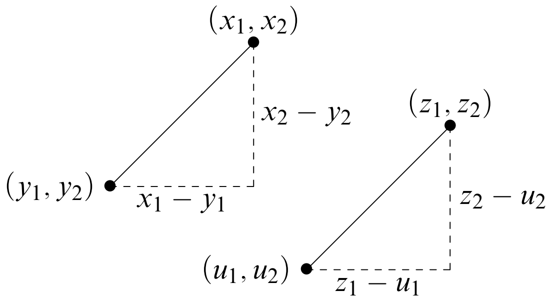

FIGURE 22. The definition of equidistance in the Cartesian plane.

It is well known that the structure $\langle S, B_1, \equiv_1 \rangle$ is a model of $FG^{(2)}$; in fact it is the ordinary Cartesian 2-dimensional space. Consider now the bijection f of S (i.e., the one-one mapping of S onto itself) defined by the formula

$$f(x_1, x_2) = (x_1, 2x_2).$$

If B_2 and $\equiv_2$ are the relations among elements of S which are the images under f of B_1 and $\equiv_1$ respectively, then obviously $\langle S, B_1, \equiv_1\rangle$ and $\langle S, B_2, \equiv_2\rangle$ are isomorphic (under f), and hence $\langle S, B_2, \equiv_2\rangle$ is also a model of $FG^{(2)}$. It is easy to check that actually B_1 and B_2 coincide. However, $\equiv_1$ and $\equiv_2$ do not coincide; we see, for instance, that

$$(0,0)(0,1) \equiv_i (0,0)(1,0)$$

holds for $i = 1$, but fails for $i = 2$.

It seems that the first proof of the undefinability of $\equiv$ in terms of B was implicitly given in Veblen [50]. However Veblen was not aware of this fact since, curiously enough, what he claims to have proved is that, to the contrary, $\equiv$ is definable in terms of B; see *op. cit.*, p. 344. Analyzing his proof we notice the following. He considers a system of affine geometry, say with the geometric space S and the betweenness relation B, and constructs in this system a quaternary relation $\equiv$ such that $\langle S, B, \equiv\rangle$ is a model of 3-dimensional Euclidean geometry. However his construction is not a definition in our sense, but what could be called a parametric definition, i.e., a definition in terms, not only of B, but also of a quintuple of arbitrary points $a_1, \ldots, a_5$ (subjected to certain conditions). Thus, strictly speaking, he defines a system of quaternary relations $\equiv_{a_1,\ldots,a_5}$ indexed by quintuples of parameters, and therefore he obtains, not a single structure, but an indexed system of such structures $\langle S, B, \equiv_{a_1,\ldots,a_5}\rangle$. It is easily seen that two relations $\equiv_{a_1,\ldots,a_5}$ and $\equiv_{b_1,\ldots,b_5}$ indexed by different quintuples in general do not coincide, and therefore the corresponding models provide a proof of the non-definability of $\equiv$ in terms of B. This observation was first made in Tarski [40] and again, with more details, in Tarski [42].

It should be mentioned that the proof of independence of a notion by the method of Padoa is model-theoretic in nature, and therefore its validity does not essentially depend upon the underlying logic or set theory. In particular the relation $\equiv$ is not first-order definable in terms of B, not nth-order definable for any $n > 1$, and in general not definable on the basis of any reasonable system of logic or set theory, strong as they may be. In connection with Padoa's method see Tarski [40].[15]

On the other hand, the relation B proves to be definable in terms of $\equiv$ on the basis of $EG^{(n)}$, and *a fortiori* of $FG^{(n)}$, for every $n \geq 2$. To show this we first define an auxiliary quaternary relation—the less-than-or-equal relation between segments, expressed by the formula $xy \leq zu$—by stipulating that

$$(1) \qquad xy \leq zu \leftrightarrow \forall v\,(zv \equiv uv \rightarrow \exists w\,(xw \equiv yw \wedge yw \equiv uv)).$$

(In 2-dimensional geometry the definiens asserts that for every point v on the perpendicular bisector of the segment zu, there is a point w on the perpendicular bisector of the segment xy whose distance from y is that same

[15]An English translation this paper appeared as Article X in Tarski [43].

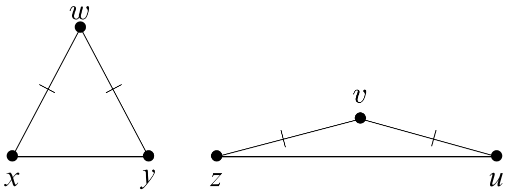

FIGURE 23. The definition of $\leq$ in terms of equidistance.

as the distance from v to u—see Figure 23). It suffices then to show that the following sentence, which is clearly a possible definition of B in terms of $\equiv$, is derivable from $EG^{(n)}$ with the help of (1):

$$B(xyz) \leftrightarrow \forall u\,(ux \leq xy \wedge uz \leq zy \rightarrow u = y). \tag{2}$$

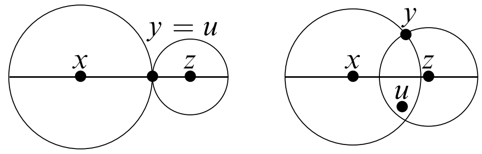

FIGURE 24. The definition of betweenness in terms of $\leq$.

(In 2-dimensional geometry the definiens asserts that the only way a point u can be in the circle with center x and radius xy, and also in the circle with center z and radius zy, is if $y = u$—see Figure 24.) This presents no essential difficulties. The result is essentially due to Pieri [23]; in the outline above we follow the argument in Robinson [25], pp. 70–71.

As a consequence of the definability of B in terms of $\equiv$ we can adopt $\equiv$ as the only primitive notion of Euclidean geometry, and construct an axiom set for this geometry involving exclusively $\equiv$. Such an axiom set can be obtained mechanically from any one of the axiom sets described in Section 2 by eliminating from it the relation B on the basis of (1) and (2). Clearly such an axiom set would be rather involved and unnatural from the point of view of the mathematical content of its axioms. In general it seems dubious whether, using $\equiv$ as the only primitive notion, an axiom set could be constructed which could compete with $EG^{(n)}$ (or $EH^{(n)}$, $FG^{(n)}$, $FH^{(n)}$) in simplicity of form and clarity of mathematical content.[16] Probably this applies even more strongly if, instead of $\equiv$, the ternary relation expressed by the formula $xy \equiv yz$ is used as the only primitive notion of geometry;

[16]*Since these words were written, concise axiom systems that use the equidistance relation as the only primitive notion have been given for various important, finitely axiomatized subsystems of 2- and 3-dimensional elementary Euclidean geometry. See, for example, Schnabel [27], Grochowska [4], and Richter-Schnabel [24].

the fact that this ternary relation can be so used follows from results of Pieri [23].

There are various other quaternary and ternary relations among points, each of which can be used as the only primitive notion of geometry. Such are, in particular, the quaternary and the ternary relations expressed by the formulas $xy \leq zu$ and $xy \leq yz$. It seems plausible that these relations, especially the former, are better suited for constructing desirable axiom sets.

§6. Subsumption of 1-dimensional geometry under geometries of higher dimension. We shall make here some remarks concerning the selection of axioms and primitive notions for 1-dimensional Euclidean geometry. The most natural way of setting up a foundations for this geometry would seem to consist in taking as a point of departure any of our four systems of n-dimensional geometry with $n \geq 2$, preserving its primitive notions and all axioms, with the exception of the dimension axioms, and using respectively Ax. $8^{(1)}$ and Ax. $9^{(1)}$ as lower and upper dimension axioms. It turns out, however, that the sentence Ax. 22, which certainly is true in 1-dimensional Euclidean geometry, is not derivable from any of the axiom sets thus obtained, and that Ax. 6, which occurs in $FH^{(n)}$, is not derivable from the set obtained from $FG^{(n)}$ by modifying it in the described way. We therefore add Ax. 22 to the axiom sets proposed for 1-dimensional geometry, and moreover, we replace Ax. 15 by Ax. 6 in the axiom sets obtained from $EG^{(n)}$ and $FG^{(n)}$. We denote the resulting set by $EG^{(1)}$, $EH^{(1)}$, $FG^{(1)}$ or $FH^{(1)}$, as the case may be. The sets $EG^{(1)}$, $EH^{(1)}$ are complete and the sets $FG^{(1)}$, $FH^{(1)}$ are categorical.

As could be expected, the axiom sets just mentioned can be considerably simplified. In fact, the axioms of a less elementary character, i.e., Ax. 5, Ax. 7 (or Ax. 7_1), Ax. 10 (or Ax. 10_1), and As. 11 can be either replaced by some elementary consequences which express natural properties specific to the straight line, or else entirely eliminated. It seems, for instance, very plausible that Ax. 10 can be entirely eliminated, while Ax. 5 can be replaced by Ax. 23 and Ax. 24. Also it seems plausible that As. 11 can be replaced by an infinite collection of sentences which, loosely speaking, express the fact that every segment can be divided into p congruent parts, for each prime number p. (Since each of these sentences clearly implies Ax. 22, the latter may then be omitted.) In this way we could arrive at natural and elegant axiom sets for elementary and full 1-dimensional geometries. However we have not considered this matter in details.

A substantial difference between the axiomatization of 1-dimensional and higher dimensional geometries comes to light when we turn to the problems concerning the independence of primitive notions. As in the case of higher dimensional geometries, $\equiv$ is not definable in terms of B in 1-dimensional geometry. To show this we proceed as in the case of $FG^{(2)}$ discussed in

Section 5 above. Indeed we consider the usual Cartesian model $\langle R, B_1, \equiv_1\rangle$ of $FG^{(1)}$, where R is the set of real numbers while B_1 and $\equiv_1$ are defined by the stipulations

$$B_1(xyz) \leftrightarrow (x \leq y \leq z) \vee (x \geq y \geq z),$$
$$xy \equiv_1 zu \leftrightarrow (x + u = z + y) \vee (x + z = u + y),$$

and we choose an appropriate bijection f of R which preserves the relation B_1, but not $\equiv_1$. As such a bijection we can use, for instance, the function f defined by the stipulations: $f(x) = x$ for $x \geq 0$, and $f(x) = x + x$ for $x < 0$.

On the other hand, in opposition to higher dimensional geometries, the problem of the definability of B in terms of $\equiv$ is rather involved. According to the observations of Lindenbaum announced (without proof) in Lindenbaum-Tarski [9], the situation in the full system $FG^{(1)}$ can be described as follows. Using the Cartesian model we easily see that the problem discussed is equivalent to an analogous problem concerning the full system of the theory of reals based upon $+$, 1, and $\leq$ as the only primitive notions; in fact to the problem of the definability of $\leq$ in terms of $+$ and 1. Rather unexpectedly, the latter problem proves in turn to be equivalent to that of the non-existence of a function g from and to reals which is not Lebesgue-measurable, and satisfies some additional conditions. If the fragment of set theory underlying the theory of reals contains the axiom of choice, then we can show that such a function g does exist, and consequently that $\leq$ is not definable in terms of $+$ and 1. If, however, in our fragment of set theory we can show that every function from reals to reals is measurable, then we may well be able to construct an explicit definition of $\leq$ in terms of $+$ and 1.

Elaborating on these observations, we should like to point out that for g we can take any bijection of the set R of reals for which the Cauchy equation

$$g(u + v) = g(u) + g(v)$$

is identically satisfied and which is neither increasing nor decreasing. As is well-known, with the help of the axiom of choice we can establish the existence of such a function g and show, without using the axiom of choice (see Sierpiński [31], p. 442), that no such function is measurable. From properties of g it follows easily that $g(1) \neq 0$. Hence we can define a new function f by stipulating that $f(x) = g(x)/g(1)$ for every $x \in R$. Of course the function f has all the properties of g mentioned above, and in addition we have $f(1) = 1$. Using f in the same way as we did in earlier proofs of independence, we construct the structure $\langle R, +', 1', \leq'\rangle$ in such a way that f maps $\langle R, +, 1, \leq\rangle$ isomorphically onto $\langle R, +', 1', \leq'\rangle$. Clearly, because of the Cauchy equation and the formula $f(1) = 1$, we see that $+$ coincides with $+'$ and 1 with $1'$. Since, however, f is not increasing, $\leq$ does not coincide with $\leq'$. Therefore, $\leq$ is not definable in terms of $+$ and 1. Thus we apply here essentially Padoa's method, with this difference however, that we do not construct the function f and the model $\langle R, +', 1', \leq' \rangle$ explicitly, but only prove their existence with the help of the axiom of choice. On the other hand, if in the underlying system of set theory we can prove that all functions from reals to reals are measurable, then, as is well-known, every solution g of Cauchy's equation is monotonic and indeed satisfies the condition $g(x) = x \cdot g(1)$ for every $x \in R$. Hence we can write explicitly a possible definition of $\leq$ in terms of $+$ and 1:

$$x \leq y \leftrightarrow \exists g\ \forall u\ \forall v\ ([g(u+v) = g(u) + g(v)] \wedge [x + g(g(1)) = y]),$$

where the second-order variable g ranges over functions from R to R. (Since $g(g(1)) = g(1) \cdot g(1)$, the definiens essentially asserts the existence of a non-negative number which, when added to x, yields y.) Regarding systems of set theory in which the fact that all real functions are Lebesgue-measurable can be proved, see Mycielski-Świerczkowski [19] (where bibliographic references to earlier papers of Mycielski and Steinhaus can be found) as well as Solovay [32].[17]

It should be pointed out that the observations regarding the connections between the definability of B in terms of $\equiv$ and some properties of the underlying set theory apply also to the problem of the independence of the Pasch Axiom Ax. 7_1 from the remaining axioms of geometry, and to Szczerba's solution of this problem (see Section 5 of these notes).

So far in this section we have been dealing primarily with full geometry. The situation changes rather substantially if we now turn to elementary geometry. As before, $\equiv$ is not definable in terms of B. Because of its character, the proof of this fact which we outlined before for full geometry actually applies to all reasonable logical formalisms. On the other hand, we are now able to give a straightforward proof that B is not first-order definable in terms of $\equiv$, or in an equivalent algebraic formulation, that $\leq$ is not first-order definable in terms of $+$ and 1. In fact, we can give a precise description of all sets of reals which are definable in terms of $+$ and 1 (such a description can be derived as a very particular consequence of the results in Szmielew [36], but it can also be obtained in a much more direct way—see, e.g., the proof of Theorem 1 in Tarski [39], p. 232), and from this description we easily see that, e.g., the set of non-negative reals is not so definable, although it is clearly definable in terms of $+$ and $\leq$.

As regards relations which can serve as the only primitive notion of 1-dimensional geometry, we notice that, just as in the case of higher dimensions, the ternary relation of Pieri expressed by the formula $xy \equiv yz$ is elementarily interdefinable with $\equiv$. In fact, the definition of the quaternary

[17] It seems to me [Tarski] very likely that the contents of the "elaborating remarks" made above were known to Lindenbaum. The explicit definition of $\leq$ in terms of $+$ and 1 appears here for the first time.

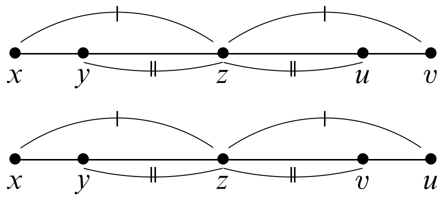

FIGURE 25. The definition of equidistance in terms of Pieri's relation.

relation of equidistance in terms of Pieri's relation is quite simple in the present case:

$$
\begin{aligned}
&xy \equiv uv \leftrightarrow \\
&\quad \exists z \, ([xz \equiv zv \wedge (x = v \rightarrow x = z) \wedge yz \equiv zu \wedge (y = u \rightarrow y = z)] \\
&\qquad \vee [xz \equiv zu \wedge (x = u \rightarrow x = z) \wedge yz \equiv zv \wedge (y = v \rightarrow y = z)])
\end{aligned}
$$

(see Figure 25). Thus Pieri's relation cannot be used as the only primitive notion for elementary 1-dimensional geometry, nor for full 1-dimensional geometry if the underlying set theory contains the axiom of choice. On the other hand, again just as in the case of higher dimensions, the quaternary relation $xy \leq uv$ and the ternary relation $xy \leq yz$ can be used this way. We may notice that, in the case of 1-dimension, B and $\equiv$ can be defined in terms of $xy \leq uv$ in an extremely simple way:

$$B(xyz) \leftrightarrow [yx \leq xz \wedge yz \leq zx]$$

(see Figure 26) and

$$xy \equiv uv \leftrightarrow [xy \leq uv \wedge uv \leq xy].$$

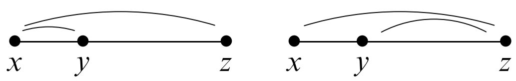

FIGURE 26. The definition of betweeness in terms of $\leq$ in 1-dimensional geometry.

There is still another problem concerning the primitive notions of elementary 1-dimensional geometry, of a more serious nature, which has recently been raised and investigated by Makowiecka in a series of papers. In opposition to higher dimensional geometries, the 1-dimensional geometry based upon B and $\equiv$ has a rather meager power of expression. There are many geometrical notions which are elementarily definable in the case of higher dimensions, but, when restricted to collinear points, are not so definable in the case of dimension one (although they are second-order definable in full

1-dimensional geometry). These are notions which, loosely speaking, are essentially involved in constructing within the geometrical space (i.e., within the standard model of our system of geometry) the field of real numbers, and in proving that the geometrical space is isomorphic to the Cartesian model over this field. (When developing 1-dimensional elementary geometry with B and $\equiv$ as the only primitive notions, we can still construct within geometrical spaces algebraic structures in which the spaces can be isomorphically represented; however, in opposition to higher dimensions, we take for these structures not fields, but much simpler structures, in fact ordered Abelian groups.)

To avoid such situations and to be able to subsume the 1-dimensional case under the development of geometries with arbitrary dimensions, we have to change our conception of 1-dimensional geometry and provide it with stronger primitive notions. As an example we may consider the quaternary relation K discussed in Makowiecka [11], [12]. In the field of real numbers it is defined by the elementary formula

$$K(xyzu) \leftrightarrow [(x-u)^2 = (y-u)\cdot(z-u)].$$

The relation K is a geometric notion in the sense that it is preserved under every similarity transformation of R onto itself. It is easily seen that the basic field operations $+$ and $\cdot$ are elementarily definable in terms of K and two arbitrary distinct real numbers, say 0 and 1. As a consequence, every relation among numbers which is geometric in the above sense and definable, or elementarily definable, in terms of $+$ and $\cdot$ is also definable, or elementarily definable, in terms of K. This applies, in particular, to the relations B and $\equiv$. Indeed we have:

$$B(yuz) \leftrightarrow \forall x\ (K(xyzu) \rightarrow x = u)$$

(the definiens asserts that the product $(y-u)\cdot(z-u)$ is non-negative only in the trivial case when it is 0—in other words, one of y and z coincides with u or else $y-u$ and $z-u$ have opposite signs, so that u is greater than one of y and z and less than the other),

$$xu \equiv uy \leftrightarrow K(xyyu)$$

(this is a definition of Pieri's relation in terms of K—the definiens asserts that $(x-u)^2 = (y-u)^2$), and

$$\begin{aligned} & xx' \equiv yy' \\ & \leftrightarrow \exists u\ ([xu \equiv uy \wedge (x = y \rightarrow x = u) \wedge x'u \equiv uy' \wedge (x' = y' \rightarrow x' = u)] \\ & \quad \vee [xu \equiv uy' \wedge (x = y' \rightarrow x = u) \wedge x'u \equiv uy \wedge (x' = y \rightarrow x' = u)]) \end{aligned}$$

(this is just the definition of the equidistance relation in terms of Pieri's relation that we encountered previously). For this reason K can be used as the only primitive notion for the new conception of elementary 1-dimensional geometry.

On the other hand K can be shown not to be elementarily definable in terms of B and $\equiv$ (although it is second-order definable in these terms). The proof of this result reduces to showing that in the field of real numbers $\cdot$ is not elementarily definable in terms of $+$, $\leq$, and 1. The latter fact can be established as follows: an immediate consequence of Theorem 2 in Tarski [39], p. 233,[18] is that the only particular real numbers elementarily definable in terms of $+$, $\leq$ and 1 are rational numbers; however the irrational number $\sqrt{2}$ is obviously elementarily definable in terms of $+$ and $\cdot$ (see also Makowiecka [13], pp. 677–678).

Makowiecka has given several other examples of geometric relations which are elementarily interdefinable with K in 1-dimensional space, so that each of them can serve as the only primitive notion of elementary 1-dimensional geometry. However she has also shown that no ternary relation can be used for this purpose, and that actually no set of ternary relations can serve as the set of primitive notions for elementary 1-dimensional geometry. In this connection see Makowiecka [12], [14], [15].

For any given $n \geq 1$ we can consider a quaternary relation K_n among points of n-dimensional Cartesian space defined by the formula

$$K_n(xyzu) \leftrightarrow \left[\sum_{i=1}^{n}(x_i - u_i)^2 = \sum_{i=1}^{n}(y_i - u_i)\cdot(z_i - u_i)\right],$$

where

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad z = (z_1, \ldots, z_n), \quad u = (u_1, \ldots, u_n)$$

are arbitrary n-tuples of real numbers. As in the 1-dimensional case, K_n can serve as the only primitive notion of elementary n-dimensional geometry; hence, in opposition to the 1-dimensional case, it is interdefinable with $\equiv$ or with Pieri's relation whenever $n \geq 2$. Obviously the set of all n-tuples $(x_1, \ldots, x_n)$ with $x_i = 0$ for $i = 2, \ldots, n$ forms a 1-dimensional subspace of our Cartesian space which may be identified with the usual 1-dimesional Cartesian space. It becomes then clear that K_n, when restricted to this subspace, coincides with $K_1 = K$. Thus the connection between K_n and K_1 is exactly the same as the one between the n-dimensional and 1-dimensional equidistance relations. It seems therefore proper to speak of all relations K_n as the same relation K and treat it as the common primitive notion for Euclidean geometries of any number of dimensions. Compare Makowiecka [13].[19]

[18] For an English translation see [43], p. 134.

[19] Makowiecka does not seem to be concerned with non-elementary definability. From a certain point of view a defect of her approach is that it compels us to base geometry on primitive notions with less intuitive, geometrical content.

§7. Historical remarks concerning the development of geometry on the basis of Tarski's system. To our knowledge the publication of Schwabhäuser-Szmielew-Tarski [29] marks the first time that a detailed and systematic development of geometry based on some variant of Tarski's system of axioms and primitive notions will appear in print and become generally accessible. However, unpublished versions of such a detailed development have existed for over fifty years.[20] In fact, a systematic presentation was given for the first time by Tarski himself in his course at the University of Warsaw during the academic year 1926–27 (see Tarski [45], footnote 34, page 45). Various versions of such a presentation were later given in courses on the foundations of geometry, held at the University of California, Berkeley, by Tarski, e.g., in 1956–57 and 1961–62, and by Szmielew during her visiting professorship in 1959–60.

In the early 1960s Szmielew and Tarski decided to join their efforts to prepare a comprehensive treatise on the foundations of geometry. In opposition to earlier works of this kind, the intended topic of the treatise was the study of various systems of geometry—in particular parabolic (i.e., Euclidean), hyperbolic, and elliptic geometries—developed within the framework of contemporary mathematical logic and investigated by means of contemporary metamathematical (both proof-theoretical and model-theoretical) methods. A systematic development of Euclidean geometry (with any given dimension) was to constitute the first part of the treatise. During her stay in Berkeley as a visiting research worker Szmielew, in collaboration with Tarski, prepared a manuscript of this part of the treatise (the first draft completed in 1965, the second in 1967).[21]

The manuscript contains the development of 2-dimensional geometry, including the introduction of coordinates and the representation of every model as a Cartesian space; a final chapter with indications how the development can be extended to higher dimensions is lacking. As was pointed

[20]Recall that these words were written around 1978.

[21]I [Tarski] made some search, both in my memory and in the NSF project reports, Tarski *et al.* [48], [49], and [47]. I reached the following conclusions concerning the chronology of Wanda's work on the manuscripts involved; they are not certain, but seem to me very plausible. Some time during Wanda's stay in Berkeley in 1960, or perhaps during my visit to Warsaw in 1961, we decided to start working on a treatise on the foundations of geometry, and to prepare the development of Euclidean geometry, based upon some variant of my axiom system, as the first part of the treatise. Not much progress was made between 1961 and 1964, when I was again for a short time in Warsaw, but probably Wanda's ideas regarding the development ripened in this period. She worked intensively on our project during her stay in Berkeley in the spring and summer of 1965, and prepared a draft of the work; this is, I believe, the draft which she made available to you [Schwabhäuser]. She continued the work during her next visit, in the spring and summer of 1967, and completed a manuscript which was ready to be put in final form for publication, except for an addition concerning the extension of the results to higher dimensions. I do not remember whether we ever returned to the development of this manuscript in later years.

out in Section 2, the axiomatic foundations adopted in the manuscript are essentially a variant of Tarski's system: the logical framework and the primitive notions are the same; the axiom set varies from $EG^{(2)}$ and $FG^{(2)}$ in that Ax. 15, Ax. 7_1, and Ax. 10_1 have been respectively replaced with Ax. 6, Ax. 7, and Ax. 10_2. Thus, this axiom set is very closely related to the one adopted in Schwabhäuser-Szmielew-Tarski [29]. However, the actual development of geometry on this basis differs rather substantially from those presented by Tarski in his courses (and, as it seems to us, from other earlier developments that can be found in the literature). While the presentation in Tarski's courses followed essentially the classical lines of Hilbert's work,[22] the manuscript embodies a number of Szmielew's own original ideas which, in our opinion, result in a new, elegant, and concise presentation.

The manuscript served as a base for courses on the foundations of geometry given subsequently by Szmielew at the University of Warsaw. It was used also for this same purpose by Schwabhäuser both during his visit to Berkeley in 1965–66, and in later courses which he taught in Germany.

For various reasons work on the treatise was not pursued intensely after the manuscript had been completed, and the untimely death of Szmielew in 1976 brought the whole project to a final close. Quite independently from Szmielew and Tarski, Schwabhäuser started work on a treatise of a similar character several years ago. He made substantial use of their manuscript in preparing for publication the first part of his treatise, i.e., the first part of Schwabhäuser-Szmielew-Tarski [29]. Indeed, Sections 1–8, 10, and the initial part of Section 9 of that work follow strictly the lines of their manuscript. In view of the whole situation he proposed in 1977, and Tarski concurred, that the first part of the treatise appear in print as the joint work of the three authors involved.

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