The Bulletin of Symbolic Logic
Volume 4, Number 4, Dec. 1998

CONSTRUCTION OF TRUTH PREDICATES: APPROXIMATION VERSUS REVISION

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§1. Introduction. The problem raised by the liar paradox has long been an intriguing challenge for all those interested in the concept of truth. Many "solutions" have been proposed to solve or avoid the paradox, either prescribing some linguistical restriction, or giving up the classical true-false bivalence or assuming some kind of contextual dependence of truth, among other possibilities. We shall not discuss these different approaches to the subject in this paper, but we shall concentrate on a kind of formal construction which was originated by Kripke's paper "Outline of a theory of truth" [11] and which, in different forms, reappears in later papers by various authors.

The main idea can be presented as follows: assume a first order language $\mathcal{L}$ containing, among other unspecified symbols, a predicate symbol T intended to represent the truth predicate for $\mathcal{L}$. Assume, also, a fixed model $\mathbf{M} = \langle D, I \rangle$ (the *base model*) where D contains all sentences of $\mathcal{L}$ and I interprets all non-logical symbols of $\mathcal{L}$ except T in the usual way. In general, D might contain many objects other than sentences of $\mathcal{L}$ but as that would raise the problem of the meaning of sentences in which T is applied to one of these objects, we shall assume that this is not the case. This will simplify matters and is not essential for our purpose here, but notice that this assumption is less innocent than it may seem: it may cause the lack of a compactness theorem for a logic of truth (see Kremer [10, Section 3, pp. 250–253]). As no restriction is imposed on I and D contains all sentences of $\mathcal{L}$, any kind of self-reference is allowed in $\mathcal{L}$ (with respect to $\mathbf{M}$). Now, can we assign any interpretation to T in such a way that T can be read as "true in $\mathcal{L}$ with respect to $\mathbf{M}$"? The answer, of course, depends on the properties we require from a truth predicate, but a good strategy in the search of such interpretation may be the construction of a sequence of successive approximations to the desired interpretation. This is one of the ideas contained in Kripke [11] and will be the main subject of this paper.

Received December 16, 1997; revised July 17, 1998.

1079-8986/98/0404-0002/$2.90

§2. Kripke's inductive construction of the least fixed point. As $\mathbf{M}$ provides an interpretation for every symbol in $\mathcal{L}$ except T, sentences not containing T can be evaluated with respect to $\mathbf{M}$ in the usual way, but if we wish to evaluate sentences in which T occurs, we must add to our base model $\mathbf{M}$ some interpretation for T. Given an interpretation S for T we can check whether a sentence A is true or not in $\mathbf{M} + S$ (the base model plus the proposed interpretation for T), or, in symbols, whether $\mathbf{M} + S \vDash A$ holds or not. It would be nice if $A \in S$ if and only if $\mathbf{M}+S \vDash A$, but this is not possible if S is a classical interpretation for T and there exist liar sentences in the language (i.e, if there is some term c such that $I(c) = \neg T(c)$). In order to solve this problem we could either restrict the language avoiding the presence of liar (and any other problematic) sentences or abandon classicality and bivalence. Kripke's paper chooses this second option and explores the possibility of assigning a partial interpretation to T, that is to say, a pair $S = \langle S^+, S^- \rangle$ where $S^+ \cap S^- = \emptyset$. Such an S will be called *partial predicate*. S^+, the *positive extension* of S, is the set of sentences which the partial predicate S declares true, and S^-, the *negative extension*, is the set of sentences declared false by S. Sentences not belonging to either of the two extensions are considered neither true nor false or, in other words, are undefined according to S. If a sentence A is declared true by S ($A \in S^+$) and $I(c) = A$, then Tc (the sentence saying "A is true") is true in $\mathbf{M} + S$. Consequently, Tc is true in $\mathbf{M} + S$ ($\mathbf{M} + S \vDash Tc$) if and only if the sentence $I(c)$ is declared true by S (if and only if $I(c) \in S^+$), and, similarly, $\mathbf{M} + S \vDash Tc$ if and only if $I(c) \in S^-$.

If T is interpreted by a partial predicate we can evaluate sentences in $\mathbf{M} + S$ by means of some partial valuation scheme. Kripke [11] suggests three posibilities: Kleene's strong and weak schemes and van Fraassen's supervaluations (see the Appendix for definitions). In any case, for any sentence A either $\mathbf{M}+S \vDash A$ or $\mathbf{M}+S \vDash \neg A$ or neither of the two possibilities holds. The goal is to find a partial predicate S such that for any sentence A, $A \in S^+$ if and only if $\mathbf{M} + S \vDash A$ (A is declared true by S if and only if A is true in $\mathbf{M} + S$), and $A \in S^-$ if and only if $\mathbf{M} + S \vDash \neg A$ (A is declared false by S if and only if A is false in $\mathbf{M} + S$). Kripke shows that there exist partial predicates with this property.

We need some preliminary concepts. First of all, given two partial predicates S and R we can define

$$S \cap R = \langle S^+ \cap R^+, S^- \cap R^- \rangle$$

and

$$S \cup R = \langle S^+ \cup R^+, S^- \cup R^- \rangle;$$

we shall write $S \subseteq R$ whenever $S^+ \subseteq R^+$ and $S^- \subseteq R^-$, and any other set-theoretical concepts may be interpreted similarly. Secondly, given S we can define a new partial predicate $\mathbf{J}(S) = \langle \mathbf{J}(S^+), \mathbf{J}(S^-) \rangle$, where

$$\mathbf{J}(S^+) = \{ A : A \text{ is a sentence and } \mathbf{M} + S \vDash A \},$$

and

$$\mathbf{J}(S^-) = \{ A : A \text{ is a sentence and } \mathbf{M} + S \vDash \neg A \}.$$

Thus, $\mathbf{J}(S)$ is the partial predicate which *declares* true (false) all those sentences which *are* true (false) in $\mathbf{M}+S$. The kind of predicate we wish to find, according to the previous paragraph, is a predicate S such that $S = \mathbf{J}(S)$, that is, a *fixed point* of the operation $\mathbf{J}$. Notice that $\mathbf{J}$ depends on both $\mathbf{M}$ and the chosen valuation scheme, but for any reasonable partial valuation scheme (and, in fact, for any of the three mentioned above), $\mathbf{J}$ is an operation on the set of partial predicates, called *jump* operation. Moreover, for any of the three schemes considered by Kripke, $\mathbf{J}$ is monotonic, in the sense that $S \subseteq R$ implies $\mathbf{J}(S) \subseteq \mathbf{J}(R)$. Any scheme whose corresponding jump operation is monotonic is called a monotonic scheme.

Let **ORD** be the class of all ordinals. We can now define an infinite partial predicate sequence $S_\alpha,\ \alpha \in \mathbf{ORD}$, as follows:

$$\begin{aligned} S_0 &= \langle \emptyset, \emptyset \rangle, \\ S_{\alpha+1} &= \mathbf{J}(S_\alpha), \\ S_\alpha &= \bigcup_{\beta<\alpha} S_\beta, \quad \text{for limit ordinals } \alpha. \end{aligned}$$

This is the construction presented in Kripke [11]. The intuitive idea is constructing a sequence of successive approximations to the desired partial predicate, starting with the empty predicate and including in the positive (negative) extension of each predicate those sentences which are true (false) according to the previous approximation. This gives us an increasing sequence: because S_0 is empty, $S_0 \subseteq \mathbf{J}(S_0)$; by monotonicity of $\mathbf{J}$, the sequence remains increasing along the successor ordinals and, finally, by the use of unions, the increasing character is preserved for limit ordinals as well. Notice that the union of partial predicates is not in general a partial predicate (the intersection of the positive and negative extensions might not be empty), but the union of a chain of partial predicates is always a partial predicate itself, so the increasing character of the sequence is crucial to ensure that all its elements (particularly those corresponding to limit ordinals) are partial predicates.

As the sequence is defined for every ordinal but all partial predicates form a set, we infer that for some ordinals α, β $S_\alpha = S_\beta$. As the sequence is increasing, $S_\alpha = S_\gamma$ whenever $\alpha \leq \gamma \leq \beta$ and, in particular,

$$S_\alpha = S_{\alpha+1} = \mathbf{J}(S_\alpha)$$

(in fact, the whole sequence is constant from α on). Thus, S_α declares true (false) just those sentences which are true (false) in $\mathbf{M} + S$, and so S_α is the kind of interpretation we were seeking for T. This property is shared by any *fixed point* of $\mathbf{J}$, that is, any partial predicate S such that $S = \mathbf{J}(S)$. Actually, Kripke observes that there are many fixed points over $\mathbf{M}$, and this fact is used to provide a classification of sentences: some sentences have one and the same truth value in every fixed point, and these are called *grounded sentences*. In fact, they are the sentences which have a truth value in the fixed point yielded by the construction given above, which happens to be the *least* fixed point over $\mathbf{M}$. There are also sentences (like the liar sentence) which lack a truth value in every fixed point, and these are called *paradoxical* sentences. Sentences like the truth-teller sentence (one saying of itself that it is true or, more precisely, sentences of the form Tc where $I(c) = Tc$) are true in some fixed points, false in some others and have no truth value in other ones. We can also find sentences which are true in some fixed points and undefined in some others, but cannot be false in any fixed point, etc. All these kinds of sentences can be distinguished if we consider the various fixed points over $\mathbf{M}$.

A particularly interesting kind of fixed points is the class of intrinsic fixed points. A fixed point (over $\mathbf{M}$) $S = \langle S^+, S^- \rangle$ is *intrinsic* if and only if for every fixed point (over $\mathbf{M}$) $R = \langle R^+, R^- \rangle$, $S^+ \cap R^- = \emptyset$ and $S^- \cap R^+ = \emptyset$, i.e., no sentence which is true (false) in S can be false (true) in any other fixed point over $\mathbf{M}$. A sentence has an intrinsic truth value if and only if it has a truth value in some intrinsic fixed point (see Kripke [11]). Obviously, the least fixed point is intrinsic, and all grounded sentences have an intrinsic truth value. Notice that being intrinsically true is not the same as being true in some fixed point and false in none. Consider the following two examples: let $I(c) = Tc$ (so that Tc is a truth-teller) and $I(d) = Td \vee \neg Td$. Clearly, $Tc \vee \neg Tc$ is true in some fixed points (those making Tc true and those making Tc false), undefined in some others (those in which Tc is undefined) and false in none, but it has not an intrinsic truth value, because $Tc \vee \neg Tc$ is true in fixed points assigning different truth values to Tc. Despite its similarity with $Tc \vee \neg Tc$, the sentence $Td \vee \neg Td$ is intrinsically true: it's true in all fixed points in which Td is true and undefined in all others. In this case, $\neg Td$ cannot be true in any fixed point. Interestingly enough, there exists a largest fixed point over $\mathbf{M}$, a point in which all intrinsically true (false) sentences are simultaneously true (false), and no other sentence has a truth value. It is a fixed point in which we assign as many truth values as possible without making any arbitrary choice. Thus, as Kripke points out, this largest fixed point is a significant theoretical object.

Let's recapitulate the facts which ensure that Kripke's construction presented above will eventually reach a fixed point:

(a) The fact that $S_0 = \langle \emptyset, \emptyset \rangle$ is a *sound* predicate, that is, that $S_0 \subseteq \mathbf{J}(S_0)$.
(b) The monotonicity of $\mathbf{J}$.

(c) The fact that $\mathbf{J}$ is an operation on the set of partial predicates, i.e., that $\mathbf{J}(S)$ is a partial predicate provided that S is.
(d) The particular definition of S_α when α is a limit ordinal.
(e) The fact that S_α is defined for all ordinals, which do not constitute a set while partial predicates do.

In the remaining part of this paper we shall examine how similar constructions can be carried out when facts (a)–(d) do not hold and the kind of results yielded by such constructions. Moreover, we shall see that some other constructions proposed in the literature can be studied from a uniform point of view.

§3. Other ways to Kripkean fixed points. A simple glance at the arguments presented above will show that the only property of the empty predicate $\langle \emptyset, \emptyset \rangle$ which is relevant in order to prove that Kripke's construction reaches a fixed point is soundness. So we can establish the following **Fact 1**: any sequence starting with a sound initial predicate S_0 will reach a fixed point for exactly the same reasons as the sequence originated by $\langle \emptyset, \emptyset \rangle$ and considered above. In this way, we can reach greater fixed points. So the emptiness of the initial predicate is not essential. But what about soundness? Is it essential?

An answer can be found in Fitting [3]. It can be easily seen that the same argument which shows that an increasing ordinal-length sequence must eventually reach a fixed point would prove that any decreasing sequence would culminate in a fixed point as well. Let S_0 be a *replete* predicate, i.e., a predicate such that $\mathbf{J}(S_0) \subseteq S_0$, and define $S_\alpha,\ \alpha \in \mathbf{ORD}$, as follows:

$$\begin{aligned} &S_0 \text{ as above (a replete predicate)},\\ S_{\alpha+1} &= \mathbf{J}(S_\alpha),\\ S_\alpha &= \bigcap_{\beta<\alpha} S_\beta, \quad \text{for limit ordinals } \alpha. \end{aligned}$$

Monotonicity, repleteness of S_0 and the intersection rule applied at limit ordinals ensure that the sequence so defined is decreasing and thus reaches a fixed point. Notice that the intersection of partial predicates is always a partial predicate itself. Thus, soundness is not essential, provided we suitably modify the rule for limit ordinals.

More interestingly, we can define partial predicate sequences starting with any initial predicate S_0 (with no requirement of either soundness or repleteness) and arriving at fixed points, by means of a new redefinition of the predicates placed at limit ordinal positions. In this general case we cannot define S_α (for limit α) as $\bigcup_{\beta<\alpha} S_\beta$ nor as $\bigcap_{\beta<\alpha} S_\beta$. The first would not work because, as the sequence defined before α need not be increasing, its union may not be a partial predicate (the intersection of positive and negative extensions might not be empty). And the second one is not better, because if S_0 happens to be sound, $S_0 = S_\alpha$ for every limit ordinal α and the whole sequence would indefinitely repeat its natural-numbered part. So we must look for a different definition for S_α. A good option is setting

$$S_\alpha = \bigcup_{\beta<\alpha} \bigcap_{\beta\leq\gamma<\alpha} S_\gamma$$

(the lower limit of the sequence $S_\beta,\ \beta < \alpha$). This amounts to including in the positive (negative) extension of S_α those sentences which belong to all positive (negative) extensions in the sequence (up to α) after some $\beta < \alpha$. Thus, S_α collects the sentences which seem to have stabilized at some ordinal β before α. In this way we obtain a kind of sequences which may be called *semi-inductive sequences* (the name is taken from Herzberger [9], who applies it to his revision sequences).

S_α, as defined above, is always a partial predicate. To see this, define $R_\beta = \bigcap_{\beta\leq\gamma<\alpha} S_\gamma$. Being the intersection of a set of partial predicates, each R_β is a partial predicate. The sequence $R_\beta,\ \beta < \alpha$, is a chain and, as noticed above, its union

$$\bigcup_{\beta<\alpha} R_\beta = \bigcup_{\beta<\alpha} \bigcap_{\beta\leq\gamma<\alpha} S_\gamma = S_\alpha$$

must be a partial predicate. So we have defined a sequence S_α of partial predicates, for every $\alpha \in \mathbf{ORD}$.

Observe that if the initial predicate S_0 is sound, the sequence generated is increasing and $R_\beta = S_\beta$, for R_β as above and every $\beta < \alpha$, and thus

$$S_\alpha = \bigcup_{\beta<\alpha} \bigcap_{\beta\leq\gamma<\alpha} S_\gamma = \bigcup_{\beta<\alpha} R_\beta = \bigcup_{\beta<\alpha} S_\beta.$$

On the other hand, if S_0 is replete, the sequence will be decreasing and so $R_\beta = R_\gamma$ for every $\beta \leq \gamma < \alpha$, and so

$$S_\alpha = \bigcup_{\beta<\alpha} \bigcap_{\beta\leq\gamma<\alpha} S_\gamma = \bigcup_{\beta<\alpha} R_\beta = R_0 = \bigcap_{\beta<\alpha} S_\beta.$$

This shows that the construction which employs lower limits at limit ordinals is in fact a generalization of the two earlier ones, the one employing unions (when the initial predicate was sound) and the one employing intersections (for replete S_0).

It is not obvious that such a sequence must reach a fixed point, being neither increasing nor decreasing, but the following argument proves that this is always the case (**Fact 2**). Given any sequence $S_\alpha,\ \alpha \in \mathbf{ORD}$, say that a partial predicate R *vanishes* if for some ordinal β it happens that $\gamma > \beta$ implies $S_\gamma \neq R$. Let the *vanishing ordinal* of R be the least β with this property. Let α be the supremum of the set of vanishing ordinals of all vanishing predicates. Such an α is called the *initial ordinal* of the sequence. This means that for any β above α, the partial predicate S_β reappears at arbitrarily large positions in the sequence. Predicates with this property are called *cofinal predicates* (more precisely, a predicate R is *cofinal* in a sequence if for any ordinal β there is some $\gamma > \beta$ such that $S_\gamma = R$). Define a (denumerable) sequence of ordinals $\alpha_n,\ n \in \omega$, as follows: let $\alpha_0 = \alpha$ (the initial ordinal), and let α_{n+1} be the first ordinal after α_n such that every cofinal predicate appears in some position between α_n and α_{n+1}. Let β be the supremum of the set $\{\, \alpha_n : n \in \omega \,\}$. Clearly, a sentence A will belong to the positive extension of S_β if there is some $\gamma < \beta$ such that it belongs to all positive extensions after γ and before β. But as every cofinal predicate must appear between γ and β, A must be true in every cofinal predicate. As the same is true for negative extensions, it follows that S_β is the intersection of all cofinal predicates. However, both S_β and $S_{\beta+1}$ being cofinal, S_β must be included in $S_{\beta+1}$, and so S_β is a sound predicate, and we can conclude that the sequence S_γ, for ordinals $\gamma > \beta$ is an increasing sequence. But as S_β is cofinal, it must appear again at some larger position in the sequence, i.e., for some $\gamma > \beta\ \ S_\gamma = S_\beta$, which implies that all predicates between β and γ equal S_β. Thus, S_β is a fixed point and, moreover, is the only cofinal predicate in the sequence.

§**4. Paraconsistent and approximate languages.** The core of Kripke's approximation to truth predicates is the use of a partial language or, more precisely, a partial interpretation for the symbol T. (Notice, however, that everything works equally well if we admit that any predicate or relation symbol in the language may have a partial interpretation instead of a classical one.) Liar sentences are then neither true nor false and the liar paradox is thus avoided. Some authors (see, for instance, Priest [16, 17]) have proposed a different strategy, according to which liar sentences should not be considered as undefined, but as contradictory sentences, sentences simultaneously true and false. The interpretation of T should then be a paraconsistent predicate, i.e., one which may be either true or false or both of any object (but not undefined). This kind of predicate may be represented as a pair $S = \langle S^+, S^- \rangle$ where $S^+ \cup S^- = D$ (or, at least, it should equal the set of all sentences, had we accepted the presence of non-sentences as elements of D). So, any sentence belongs to at least one of S^+, S^- and it may belong to both of them. We can evaluate sentences of $\mathcal{L}$ in $\mathbf{M} + S$ according to some monotonic scheme (the most usual option is the use of Kleene's strong scheme, but other possibilities are available).

There is an obvious duality between partial and paraconsistent predicates: take any partial predicate S and add to both S^+ and S^- all undefined sentences (sentences belonging neither to the positive nor the negative extension of S). The result is a paraconsistent predicate R which declares both true and false just the sentences which were undefined according to S. Similarly, for any paraconsistent predicate R we can define a partial predicate which declares undefined the sentences which R considers as contradictory ones. The correspondence is one-to-one. Notice that given two partial predicates S, S' and the corresponding paraconsistent ones R, R', $S \subseteq S'$ implies $R' \subseteq R$ and vice versa, so sound predicates correspond to replete ones and the empty partial predicate $\langle \emptyset, \emptyset \rangle$ corresponds to the paraconsistent predicate $\langle D, D \rangle$. This symmetry between partial and paraconsistent predicates suggest obvious ways in which the fixed point constructions considered above for partial predicates can be mimicked. Replace every partial predicate by the corresponding paraconsistent predicate and interchange unions and intersections. Then, Kripke's original increasing construction starting with the empty (partial) predicate corresponds to a decreasing sequence of paraconsistent pedicates starting with $S_0 = \langle D, D \rangle$, $S_{\alpha+1} = \mathbf{J}(S_\alpha)$ and $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$, for limit α. Similarly, we can construct decreasing sequences of paraconsistent pedicates starting with any replete S_0. We can construct increasing sequences if we start with a sound paraconsistent predicate and take unions at limits and finally we can construct sequences beginning with any paraconsistent predicate S_0 applying the rule

$$S_\alpha = \bigcap_{\beta<\alpha} \bigcup_{\beta \leq \gamma < \alpha} S_\gamma$$

(the upper limit of the sequence S_β, $\beta < \alpha$) at limit ordinals α. All these sequences eventually arrive at paraconsistent fixed points, as can be proven by suitable modifications of the proofs for the corresponding partial predicate sequences. As in the case treated above, paraconsistent predicates can be assigned not only to T, but also to any other predicate or relation symbol.

Up to now we have treated T as a three-valued predicate, whose non-classical value may be either a gap (neither true nor false) or a glut (simultaneously true and false). Both possibilities can be combined, thus accepting that T represents a four-valued predicate. In this case, the interpretation of T will be a pair $S = \langle S^+, S^- \rangle$ with no additional restriction concerning the intersection or union of S^+ and S^- (we shall call these pairs "approximate predicates", following Woodruff [21]). Thus, any sentence may be declared both true and false, i.e., contradictory, (if it belongs to both S^+ and S^-) or neither true nor false (if it belongs to neither S^+ nor S^-). This kind of strategy appears in Woodruff [21], Visser [19, 20], and Fitting [4]. In addition to the philosophical interest it may have, this approach has an important technical advantage: in the context of approximate semantics, fixed points constitute a lattice, instead of just a semilattice, as it was the case in the partial and paraconsistent three-valued semantics (for further information, see the papers cited above).

Now, three-valued monotonic valuation schemes can be easily extended to four-valued monotonic ones (see the Appendix) and so, assuming any fixed valuation scheme and given any approximate predicate S, we can define an ordinal-length sequence of approximate predicates. Approximate predicates corresponding to successor ordinals are defined as usual: just apply the jump operation. In order to define limit-ordinal predicates there are two obvious options: we can use either the lower or the upper limit rule. In both cases we would obtain a new approximate predicate. So we can produce two different sequences, S^{lo}_{α} and S^{up}_{α}, $\alpha \in \mathbf{ORD}$, depending on whether we are using the lower or the upper limit rule at limit stages. By the same argument applied to former constructions, each of these sequences must reach a fixed point. Let S^{lo} and S^{up} be the fixed points produced by means of the lower limit rule and the upper limit one, respectively. An easy transfinite induction shows that, for every ordinal α, $S^{\mathrm{lo}}_{\alpha} \subseteq S^{\mathrm{up}}_{\alpha}$ and so that $S^{\mathrm{lo}} \subseteq S^{\mathrm{up}}$.

So, for any initial S_0, we have two different constructions leading to fixed points, one employing the upper rule, the other the lower one. Let's see how each rule works. Both rules agree in the truth value assigned to those sentences which, up to α, seem to have acquired a stable truth value, but disagree with respect to those sentences whose truth value seems to change again and again before α. More precisely, given a limit ordinal α, we have that for every formula A:

- If for some $\beta < \alpha$, $\beta \leq \gamma < \alpha$ implies $A \in S^{+}_{\gamma}$, then both the upper and lower limit rules include A in the positive extension of S_{α}.
- If for some $\beta < \alpha$, $\beta \leq \gamma < \alpha$ implies $A \notin S^{+}_{\gamma}$, then neither the upper nor the lower limit rule includes A in the positive extension of S_{α}.
- If for every $\beta < \alpha$ there are γ, δ, $\beta \leq \gamma, \delta < \alpha$ such that $A \in S^{+}_{\gamma}$ and $A \notin S^{+}_{\delta}$, then the upper limit includes A in the positive extension of the new predicate, but the lower limit does not.

(Negative extensions behave in a similar way.)

Thus, a decision in favour of the lower or upper limit rule implies a corresponding decision about the treatment given to unstable sentences. But it might be that we find no reason to decide the question in favour of one strategy or the other. In such a case we could opt for defining sequences S_{α}, $\alpha \in \mathbf{ORD}$, which at limit ordinals arbitrarily use either the upper or the lower limit rule. There are, of course, many sequences of this kind, but a slight modification in the proof of Fact 2 will show that all of them reach a fixed point sooner or later (**Fact 3**). To see this, recall the proof given for our Fact 2, let α be the initial ordinal of the sequence, define the set $\{\alpha_n : n \in \omega\}$ and let β be its supremum, as in that proof. Now, the predicate S_{β} must be either the lower or the upper limit of the sequence S_{γ} $(\gamma < \beta)$. Suppose it is the lower one (the other case is similar, with obvious changes). Then, S_{β} is the intersection of all cofinal predicates (and is cofinal itself). As $S_{\beta+1}$ is cofinal as well, $S_{\beta} \subseteq S_{\beta+1}$, and S_{β} is sound. It can be easily proven (by transfinite induction) that for every $\gamma > \beta$, S_{γ} is sound as well. The proof is obvious for successor γ. For limit γ, notice that, S_{β} being the least cofinal predicate, both the upper and the lower limit rules yield exactly the same result, $\bigcup_{\alpha\leq\gamma<\beta} S_\gamma$. It is clear, then, that from S_β on, the sequence is increasing, and that the particular choice of the rule (lower or upper limit) applied at each limit ordinal after β is immaterial, since the result is the same. So the remaining part of the proof of Fact 2 applies here as well.

Moreover, it can be easily proven (by an obvious induction) that, for any one of these sequences S_α ($\alpha \in \mathbf{ORD}$) it happens that for every ordinal α, $S^{\mathrm{lo}}_\alpha \subseteq S_\alpha \subseteq S^{\mathrm{up}}_\alpha$. It follows that the fixed point of the sequence S_α ($\alpha \in \mathbf{ORD}$) is greater than S^{lo} and is included in S^{up}

We could be also interested in sequences S_α defined in such a way that for limit ordinals α, S_α is neither the upper nor the lower limit of the sequence S_β, ($\beta < \alpha$), but any approximate predicate between them. Sequences of this kind may not reach a fixed point, but no matter that, they must contain cofinal predicates (once more, because there exists the set of all approximate predicates) and, moreover, we can also prove in this case that for every ordinal α, $S^{\mathrm{lo}}_\alpha \subseteq S_\alpha \subseteq S^{\mathrm{up}}_\alpha$, so we can conclude that for any cofinal S_β, $S^{\mathrm{lo}} \subseteq S_\beta \subseteq S^{\mathrm{up}}$. Thus, even if we are not convinced about the philosophical significance of either S^{lo} or S^{up}, they seem to be the natural lower and upper bounds of any reasonable alternatives.

§5. Yablo's antimonotonic language. Up to now, monotonicity has been one of the key points in proving that a given ordinal-length sequence eventually reaches a fixed point. So, if we give monotonicity up, we should no longer expect to arrive at fixed points. However, the sequences obtained may still show some interesting regularities, depending on the kind of nonmonotonicity of the language, the operation defined on predicates employed at successor ordinals and the rule(s) used at limit ordinals.

Consider, as an example of this, the language and constructions presented in Yablo [22]. Yablo argues for a *strong* conception of truth. Let "A" be a name of the sentence A. The strong conception of truth means that T"A" should be true if and only if the sentence A is true, and that T"A" should be false otherwise. More precisely, $\mathbf{M} + S \vDash T$"A" if and only if $A \in S^+$, and $\mathbf{M} + S \vDash \neg T$"$A$" otherwise. Notice that, according to this conception, T"A" is always a bivalent sentence, despite the fact that the language, as a whole, is conceived as a four-valued one, all other predicates having approximate interpretations. This means that we need not at all the negative extension of S and that we can work just with S^+, so that unions, intersections and inclusions of truth predicates may now be referred just to their positive extensions. However, we shall mantain our representation of truth predicates by pairs of a positive and a negative extension.

The language so defined is clearly nonmonotonic. However, the bivalence of T being the only source of nonmonotonicity, Yablo is able to define a new operation $\mathbf{L}$ on predicates for this language, called the *leap operation*, and show that $\mathbf{L}$ is antimonotonic, in the sense that for any pair of truth predicates R, S $R \subseteq S$ implies $\mathbf{L}(S) \subseteq \mathbf{L}(R)$. (This operation is different from Kripke's jump, though similar in spirit; see Yablo's paper for details.) Define a new operator $\mathbf{L}^2$ as follows: for any (approximate predicate) S, $\mathbf{L}^2(S) = \mathbf{L}(\mathbf{L}(S))$ (that is, $\mathbf{L}^2$ is just a double application of $\mathbf{L}$). Clearly, antimonotonicity of $\mathbf{L}$ implies monotonicity of $\mathbf{L}^2$.

Let's say that a predicate S is *supersound* if and only if $S \subseteq \mathbf{L}(S)$ and $S \subseteq \mathbf{L}(\mathbf{L}(S))$. Given any supersound predicate S (Yablo argues that the best choice is the closure under certain rules of the empty predicate, which turns out to be supersound), define a sequence S_α, ($\alpha \in \mathbf{ORD}$) as follows:

$$\begin{aligned} S_0 &= S, \\ S_{\alpha+1} &= \mathbf{L}(S_\alpha), \\ S_\alpha &= \mathbf{L}\Big(\bigcap_{\beta<\alpha} \bigcup_{\beta\leq\gamma<\alpha} S_\gamma\Big), \quad \text{for limit } \alpha. \end{aligned}$$

The rule given for limit ordinals is the leap of the upper limit. However, as Yablo himself notices, everything would work equally well (with some minor changes) if S_α is defined as $\mathbf{L}(\bigcup_{\beta<\alpha} \bigcap_{\beta\leq\gamma<\alpha} S_\gamma)$, (the leap of the lower limit). The sequences so defined are called anti-inductive sequences in Yablo [22].

Call an ordinal α *odd* if and only if $\alpha = \beta + n$, for some limit (or zero) ordinal β and some odd natural number n (recall that every ordinal can be uniquely decomposed in this way). Otherwise, α is said to be an *even* ordinal. Assume now that S_0 is supersound, i.e., $S_0 \subseteq \mathbf{L}(S_0)$ and $S_0 \subseteq \mathbf{L}^2(S_0)$. Yablo proves that the following facts hold: (i) for any even α and odd β, $S_\alpha \subseteq S_\beta$; (ii) for every pair of even ordinals α, β, $\alpha < \beta$ implies $S_\alpha \subseteq S_\beta$, and (iii) for every pair of odd ordinals α, β, $\alpha < \beta$ implies $S_\beta \subseteq S_\alpha$. (i)–(iii) are easy consequences of antimonotonicity of $\mathbf{L}$. Moreover, (iv) there exists an ordinal α such that for every odd $\beta, \beta' > \alpha$ and any even $\gamma, \gamma' > \alpha$, $S_\beta = S_{\beta'}$ and $S_\gamma = S_{\gamma'}$. Thus, S_β and S_γ are fixed points of the double leap operation $\mathbf{L}^2$. This can be argued as follows: As S_0 is supersound, it is sound relative to $\mathbf{L}^2$, so the sequence S_α (α an even ordinal) is an increasing sequence and must culminate at a fixed point. So does the sequence of odd-numbered predicates, which is a decreasing one. These are the main technical results contained in Yablo [22].

Clearly, the existence of two fixed points of the double leap operation $\mathbf{L}^2$ in every anti-inductive sequence can be proven by essentially the same argument we used to prove Fact 1 and the symmetric result concerning decreasing sequences, taking into account that $\mathbf{L}^2$ is monotonic and that S_0 (S_1) is sound (replete), and noticing that an anti-inductive sequence can be decomposed into an increasing subsequence (its even-numbered part) and a decreasing one (the odd-numbered part). This suggests that similar results should obtain for sequences starting with arbitrary S_0, not necessarily supersound, and that is in fact the case, as we can see if we apply the arguments used in the proof of Fact 2 above. Let S_0 be an arbitrary predicate and S_α ($\alpha \in \mathbf{ORD}$) the sequence defined according to Yablo's rules and starting with S_0. Clearly, as $\mathbf{L}^2$ is monotonic, the even-numbered subsequence reaches a fixed point (this can be proven like our Fact 2), and the same is true for the odd-numbered subsequence. So the whole sequence S_α ($\alpha \in \mathbf{ORD}$) satisfies statement (iv) of the former paragraph, and there is some ordinal α such that all even-numbered elements in the sequence are equal to S_β (for even $\beta > \alpha$) and all odd-numbered elements are equal to S_γ (for odd $\gamma > \alpha$). S_β and S_γ are in fact the only cofinal elements in the sequence. Let δ be any limit ordinal greater than α. According to Yablo's rules, S_δ must be the leap of the upper limit of the sequence. This upper limit must be greater than all those elements which are cofinal in the sequence, which in this case means greater than S_β and S_γ. By antimonotonicity of $\mathbf{L}$, its leap, i.e., S_δ, must be smaller than all cofinal elements. But, at the same time, S_δ is cofinal and even-numbered, so we conclude that $S_\delta = S_\beta \subseteq S_\gamma$, and thus that all of (i)–(iv) hold for Yablo's sequences starting with arbitrary initial predicates and that these sequences yield the same result as those starting with supersound predicates.

§**6. Revision sequences.** Let's now forget the strong conception of truth and its leap operation and return to the weak conception and our well known jump operation. Suppose now that we wish to construct our sequences of (approximations to) truth predicates for a classical bivalent language, thus employing only classical predicates. A classical predicate can be represented by a single set (its positive extension), but we shall keep our representation by means of pairs $S = \langle S^+, S^- \rangle$, assuming that the negative extension is the complement of the positive one (relative to the set of all sentences). This will prove useful in order to compare this approach with previous ones. There is no problem in the definition of successor stages: the jump of a classical predicate is always a classical predicate. But, how can we define limit stages? Notice that neither unions, nor intersections, nor lower limits nor upper ones would, in general, yield classical predicates, so we must resort to some new idea. The main intuition here is that S_α (for α a limit ordinal) should be some classical predicate in between the lower and the upper limit of the sequence S_β ($\beta < \alpha$). Recall that both limits declare as true (false) those sentences whose behaviour becomes stably true (false) at some point before α (something which seems a desirable property for the classical predicate we are seeking), and that the difference between the two limits is the treatment given to those sentences not stabilized at any point before α: the lower limit considers them as undefined (so that they belong neither to the positive nor the negative extension), while the upper limit declares them inconsistent (thus belonging to both the positive and the negative extension). We need them to belong to just one of them, but which one? We can find at least three different proposals in the literature:

• Herzberger [9] proposes to consider these sentences as false, thus including them in the negative extension of S_α. Thus, the positive extension of S_α is the lower limit of the positive extensions of predicates $S_\beta,\ \beta < \alpha$, while the negative extension is its complement (i.e., the lower limit of the negative extensions of previous predicates plus all unstable sentences). In this way, we construct *Herzberger's revision sequences*. Notice that, if a predicate is represented only by its positive extension (which is enough whenever the predicate is classical) we can say that predicates at limit stages are just the lower limits of previous ones, as Herzberger does. But this hides the fact that, in a classical context, not declaring a sentence true amounts to declaring it false. This observation should be kept in mind while reading Herzberger's paper.

• Gupta [6] decides the question differently. He argues that sentences not stabilized before α (for limit α) should receive at that stage the same value they were assigned at the initial predicate. In other words, there being no particular reason to consider such sentences either true or false, let's keep faithful to our initial decision, i.e., to the value assigned at S_0. This procedure yields *Gupta's revision sequences*.

• Belnap [2] criticizes Gupta (and Herzberger) arguing that, there being no reason to consider the up-to-α undecided sentences either true or false, any decision about them is equally good. So any classical predicate in between the lower and the upper limits of S_β $(\beta < \alpha)$ should be admitted as S_α. Thus, for any given initial S_0 there are in general many *Belnap sequences* starting with S_0, instead of just one, as happened in all former procedures.

Revision sequences (of either type) do not reach, in general, fixed points: there can be no classical fixed point if the language contains, at least, one liar sentence. However, for reasons we already know, all of them must contain cofinal predicates. In Gupta's and Herzberger's sequences, cofinal predicates reappear again and again exactly in the same order (these sequences contain indefinitely repeating loops). This is not the case for Belnap's sequences, because we can make arbitrary choices at limit stages.

Let S_0 be any classical predicate and S_α $(\alpha \in \mathbf{ORD})$ a revision sequence (of any type) starting with S_0. Any classical predicate is also an approximate predicate (and, of course, a partial predicate and a paraconsistent predicate). So it can be used to begin the sequences S^{lo}_α and S^{up}_α $(\alpha \in \mathbf{ORD})$ of Section 4 (which, by the way, are just the sequences we would obtain considering S_0 as a partial or a paraconsistent predicate, respectively). Then S_α $(\alpha \in \mathbf{ORD})$ is the kind of sequence considered in the last paragraph of Section 4, so we can apply the result obtained there: every cofinal predicate in this sequence (and, consequently, in any revision sequence of any of the three considered types) includes S^{lo} and is included in S^{up}. Notice that this result is independent of the valuation scheme employed, provided that such a scheme collapses to the classical valuation scheme when dealing with classical predicates, as is the case of Kleene's strong and weak schemes and van Fraassen's supervaluations.

Revision sequences do not supply us with fixed points to be assigned as adequate interpretations of T. Instead of that, the revision theory of truth studies the stability of sentences in revision sequences: a sentence is *stably true* (*false*) in a sequence if and only if it is true (false) in every predicate cofinal in the sequence, and is *unstable* in the sequence otherwise. It follows from the result in the former paragraph that, for any classical initial predicate S_0, every sentence true (false) in the Kripkean fixed point constructed form S_0 is stably true (false) in every revision sequence starting with S_0 (independently of the partial valuation scheme adopted in the construction and the type of revision sequence considered). Moreover, it can be proven in a similar way that every sentence true (false) in the minimal Kripkean fixed point is stably true (false) in every revision sequence. It may be interesting to recall that, as shown in McGee [14], stability in Belnap's sequences implies stability in Gupta's sequences which implies stability in Herzberger's sequences. Obviously, every Herzberger's or Gupta's sequence is a Belnap's sequence (though not conversely), so clearly stability in Herzberger's or Gupta's sequences implies stability in Belnap's. To see that stability in Gupta's sequences implies stability in Herzberger's ones, notice that even when a Herzberger's sequence is not necessarily a Gupta's one, for every Herzberger's sequence we can find a Gupta's sequence with exactly the same cofinal predicates and, consequently, the same stable and unstable sentences: let β be the initial ordinal in the Herzberger's sequence (recall: the first ordinal such that every predicate $S_\gamma,\ \gamma \geq \beta$, is cofinal). Construct a Gupta's sequence $R_\alpha,\ \alpha \in \mathbf{ORD}$, such that $R_0 = S_\beta$. It's not difficult to see that $R_\alpha,\ \alpha \in \mathbf{ORD}$, is also a Herzberger's sequence, that every predicate in it is cofinal and that, for every $\gamma,\ R_\gamma = S_{\beta+\gamma}$. More details about the revision theory of truth can be found in works referred above and in the more recent Gupta and Belnap [8].

§7. Some final remarks. There are many different approaches to the semantic problems raised by liar sentences. Many of these approaches resort, in one way or another, to ordinal-length sequences. It has been our purpose in this paper to present several ways of constructing such sequences in a homogeneous setting, so that comparisons between them may became easier. However, we have been concerned just with technical aspects of these constructions, deliberately excluding philosophical motivations or interpretations of technical tools and results. The reader should be advised that the way in which some of these approaches have been introduced here may not reflect their authors' philosophical conceptions or may even conflict with them, and so is encouraged to consult the original works of the corresponding authors (this advice should particularly apply to the various versions of the revision theory of truth).

We have assumed from the beginning a fixed base model $\mathbf{M}$ which provides classical interpretations for all symbols of the language different from T. However, it should be clear that when we are seeking a partial (paraconsistent, approximate) interpretation for T, $\mathbf{M}$ may be assumed to provide partial (paraconsistent, approximate) interpretations for the remaining predicate and relation symbols of the language without affecting any of the results reported here.

We have said above that the work developed here is intended to facilitate comparisons among different constructions. There are many interesting questions which remain unanswered. Recall, for instance, that one of the main properties of supervaluations is that every classically valid sentence is true under any supervaluation. Thus, assuming a (partial) supervaluational valuation scheme, any Kripkean supervaluational fixed point declares true every classically valid sentence. On the other hand, classical validities are always stably true in any revision sequence, and, particularly, in any Belnap sequence. So consider any classical initial predicate S_0, let R be the supervaluational Kripkean fixed point generated from S_0 and Q the (partial) predicate whose positive (negative) extension is the set of all sentences stably true (false) in every Belnap's sequence starting with S_0. We can deduce from results above that $R \subseteq Q$, but does $R = Q$ hold? Is, in any case, Q a fixed point? Moreover, given a base model $\mathbf{M}$, let $S = \langle S^+, S^- \rangle$, where S^+ is the set of all sentences stably true in all Belnap's sequences based on $\mathbf{M}$ (for any initial predicate), and define S^- similarly. Is S a supervaluational Kripkean fixed point? Is it the least fixed point for $\mathbf{M}$?

There are other kinds of semantical approaches to self-referent languages and the liar paradox, the most original of which is the one presented in Barwise and Etchemendy's book *The liar* (Barwise and Etchemendy [1]). The book contains two different treatments of the liar paradox, the Russellian account and the Austinian account. The Russellian account has some similarities with Kripke's theory, while the Austinian one intends to capture the philosophical intuition that the truth value of a sentence depends on the context in which it is evaluated and, moreover, that the evaluation context changes when we reflect on the meaning of self-referential sentences like the liar or the truthteller. However, the main novelty of Barwise and Etchemendy's account is the use of Aczel's non-wellfounded set theory as the mathematical framework in which their (Russellian and Austinian) propositions are modeled (in fact, they prefer to deal with propositions instead of sentences). Aczel's set theory allows the existence of exactly one set a of the form $a = \{a\}$, and many other self-belonging sets. This feature is exploited in the construction of self-referential propositions (a curious consequence of the use of Aczel's theory by Barwise and Etchemendy is the existence, within the Russellian account, of exactly one liar sentence). We cannot enter in more details here. The reader can find more information in Martin [12] and Sheard [18] and, of course, in Barwise and Etchemendy [1], which is an enjoyable book. See also McLarty [15], where it is argued that the use of Aczel's set theory is not essential.

Finally, recall the five conditions (a)–(e) of Section 2. Conditions (a)–(d) vary in different ways along the various constructions considered above, but (e) remains true everywhere. It would fail only if all possible truth predicates did not constitute a set, but a proper class. This would be possible if the object language contained expressions referring to ordinals, predicates S_α in ordinal-length sequences and truths and falsities in S_α, or if such expressions could be introduced in the language along the construction of the sequence. In such a language, most of the considerations made above would fail, but it seems to me that we would come closer to the very roots of semantic paradoxes. For semantic paradoxes arise because of the presence within the object language of (some of) its own semantic concepts. Paradoxes are blocked when some crucial semantic concepts present in the metalanguage cannot be expressed in the object language, such as "undefined" in Kripke's theory or "stably true" in the Revision theory. If we add such concepts to the object language we fall into paradox again, for then we can formulate the kind of sentences known as *strengthened liar* sentences, that is, sentences like "this sentence is either undefined or false" or "this sentence is not stably true". So we should rule out the possibility of a semantically universal language (a language containing its own metalanguage). But we may consider a progressive but never-ending addition of semantical concepts to our language (Gupta [7] and Gupta and Belnap [8] apply a similar idea to their stability concept), just as in set theory we can think of a progressive construction of ordinals but we cannot think of a set of all ordinals (we cannot assume that such a process is finished) on pain of paradox. However, we shall not pursue this issue here.

Appendix: Three and four-valued valuation schemes.

Partial valuation schemes. By a *partial model* we mean a pair $\mathbf{N} = \langle D, I\rangle$ such that $D \neq \emptyset$, $I(c) \in D$ for every constant c and for every n-ary relation symbol R, $I(R)$ is a pair $\langle Q^+, Q^-\rangle$, where $Q^+, Q^- \subseteq D^n$ and $Q^+ \cap Q^- = \emptyset$. Given a partial model $\mathbf{N}$, sentences can be evaluated in it according to Kleene's strong and weak schemes or van Frassen supervaluational scheme as follows (we use $\mathbf{N} \vDash_{\mathrm{k}} A$ and $\mathbf{N} \vDash_{\mathrm{w}} A$ to represent that A is true in $\mathbf{N}$ according to the Kleene's strong and weak valuation schemes, respectively):

• Strong Kleene's valuation scheme

$\mathbf{N} \vDash_{\mathrm{k}} Rc_1 \dots c_n$ if and only if $\langle I(c_1) \dots I(c_n)\rangle \in Q^+$, for $I(R) = \langle Q^+, Q^-\rangle$,

$\mathbf{N} \models_{\mathrm{k}} \neg Rc_1 \ldots c_n$ if and only if $\langle I(c_1) \ldots I(c_n)\rangle \in Q^-$, for $I(R) = \langle Q^+, Q^-\rangle$,

$\mathbf{N} \models_{\mathrm{k}} A \wedge B$ if and only if $\mathbf{N} \models_{\mathrm{k}} A$ and $\mathbf{N} \models_{\mathrm{k}} B$,

$\mathbf{N} \models_{\mathrm{k}} \neg(A \wedge B)$ if and only if $\mathbf{N} \models_{\mathrm{k}} \neg A$ or $\mathbf{N} \models_{\mathrm{k}} \neg B$,

$\mathbf{N} \models_{\mathrm{k}} \forall x\, A$ if and only if for every constant c not occurring in A and every partial model $\mathbf{N}'$ which differs from $\mathbf{N}$ at most in the interpretation given to c, $\mathbf{N}' \models_{\mathrm{k}} A[c/x]$,

$\mathbf{N} \models_{\mathrm{k}} \neg\forall x\, A$ if and only if for some constant c not occurring in A and some partial model $\mathbf{N}'$ which differs from $\mathbf{N}$ at most in the interpretation given to c, $\mathbf{N}' \models_{\mathrm{k}} \neg A[c/x]$.

Notice that, as $Q^+ \cup Q^-$ need not equal the whole domain D^n, it may be the case that neither $\mathbf{N} \models_{\mathrm{k}} A$ nor $\mathbf{N} \models_{\mathrm{k}} \neg A$, for any sentence A. In such a case, A is *undefined* in $\mathbf{N}$ (it is neither true nor false in $\mathbf{N}$). Otherwise, A is *defined* in $\mathbf{N}$.

- Weak Kleene's valuation scheme

$\mathbf{N} \models_{\mathrm{w}} Rc_1 \ldots c_n$ if and only if $\langle I(c_1) \ldots I(c_n)\rangle \in Q^+$, for $I(R) = \langle Q^+, Q^-\rangle$,

$\mathbf{N} \models_{\mathrm{w}} \neg Rc_1 \ldots c_n$ if and only if $\langle I(c_1) \ldots I(c_n)\rangle \in Q^-$, for $I(R) = \langle Q^+, Q^-\rangle$,

$\mathbf{N} \models_{\mathrm{w}} A \wedge B$ if and only if $\mathbf{N} \models_{\mathrm{w}} A$ and $\mathbf{N} \models_{\mathrm{w}} B$,

$\mathbf{N} \models_{\mathrm{w}} \neg(A \wedge B)$ if and only if both A and B are defined in $\mathbf{N}$ and either $\mathbf{N} \models_{\mathrm{w}} \neg A$ or $\mathbf{N} \models_{\mathrm{w}} \neg B$,

$\mathbf{N} \models_{\mathrm{w}} \forall x\, A$ if and only if for every constant c not occurring in A and every partial model $\mathbf{N}'$ which differs from $\mathbf{N}$ at most in the interpretation given to c, $\mathbf{N}' \models_{\mathrm{w}} A[c/x]$,

$\mathbf{N} \models_{\mathrm{w}} \neg\forall x\, A$ if and only if for every constant c not occurring in A, $A[c/x]$ is defined in every partial model $\mathbf{N}'$ which differs from $\mathbf{N}$ at most in the interpretation given to c and in some of these models $\mathbf{N}' \models_{\mathrm{w}} \neg A[c/x]$.

Observe that, for every A, $\mathbf{N} \models_{\mathrm{w}} A$ implies $\mathbf{N} \models_{\mathrm{k}} A$, but not conversely.

- van Fraassen's supervaluations.

Consider a partial model $\mathbf{N} = \langle D, I\rangle$. $\mathbf{N}' = \langle D', I'\rangle$ is a classical extension of $\mathbf{N}$ if and only if $\mathbf{N}'$ is a classical model, $D' = D$, $I'(c) = I(c)$ for every constant c and for every n-ary predicate symbol R, if $I(R) = \langle Q^+, Q^-\rangle$ then $Q^+ \subseteq I'(R)$ and $Q^- \cap I'(R) = \emptyset$. Roughly speaking, all atomic sentences having a truth value in $\mathbf{N}$ keep that truth value in $\mathbf{N}'$, while atomic sentences lacking a truth value in $\mathbf{N}$ receive an arbitrary truth value in $\mathbf{N}'$. Then, we say that a formula A is supervaluationally true in $\mathbf{N}$, $\mathbf{N} \models_{\mathrm{sv}} A$ if and only if for every classical extension $\mathbf{N}'$ of $\mathbf{N}$, $\mathbf{N}' \models A$, i.e., A is classically true in $\mathbf{N}'$. It is not difficult to see that $\mathbf{N} \models_{\mathrm{k}} A$ implies $\mathbf{N} \models_{\mathrm{sv}} A$.

It should be noticed that it is not excluded that $Q^+ \cup Q^- = D$ whenever R is an n-ary relation symbol and $I(R) = \langle Q^+, Q^-\rangle$. If this is the case, $\mathbf{N}$ can be considered as a classical model, and in fact all of $\models_{\mathrm{k}}$, $\models_{\mathrm{w}}$ and $\models_{\mathrm{sv}}$ collapse to the classical truth relation $\models$. So classical models and the classical truth relation can be considered as particular cases of partial models and the partial truth relation.

Paraconsistent and four-valued valuation schemes. A *paraconsistent model* is a pair $\mathbf{N} = \langle D, I \rangle$ such that $D \neq \emptyset$, $I(c) \in D$ for every constant c and for every n-ary relation symbol R, $I(R)$ is a pair $\langle Q^+, Q^- \rangle$, where $Q^+, Q^- \subseteq D^n$ and $Q^+ \cup Q^- = D^n$. Sentences are evaluated in paraconsistent models by means of the Kleene's strong valuation scheme given above. Notice that, in this case, for every sentence A either $\mathbf{N} \models_{\mathrm{k}} A$ or $\mathbf{N} \models_{\mathrm{k}} \neg A$, and that both relations may hold simultaneously. Classical models can be seen as a particular kind of paraconsistent models, those in which for every n-ary relation symbol R, $I(R) = \langle Q^+, Q^- \rangle$ implies that $Q^+ \cap Q^- = \emptyset$.

An *approximate model* is a pair $\mathbf{N} = \langle D, I \rangle$ such that $D \neq \emptyset$, $I(c) \in D$ for every constant c and for every n-ary relation symbol R, $I(R)$ is a pair $\langle Q^+, Q^- \rangle$, where $Q^+, Q^- \subseteq D^n$, with no additional requirement on Q^+ and Q^-. Sentences are evaluated in approximate models according to the strong Kleene's valuation scheme. In this case, given a sentence A and a model $\mathbf{N}$, it may hold one of $\mathbf{N} \models_{\mathrm{k}} A$ or $\mathbf{N} \models_{\mathrm{k}} \neg A$, or both, or neither. Of course, classical, partial and paraconsistent models are all particular subclasses of approximate models.

Clearly, the weak Kleene's and the supervaluational schemes could be suitably adapted to be applied to paraconsistent and approximate models, but this possibility seems not to have been taken seriously by anyone up to now.

Acknowledgments. This research was partly supported by grant PB96-1301-C05-04 of the Spanish Ministerio de Educación y Cultura, Secretaría de Estado de Universidades, Investigación y Desarrollo, Dirección General de Enseñanza Superior, Subdirección General de formación y Promoción del Conocimeiento. Thanks must be given to an anonymous referee for his suggestions. I am especially indebted to the editor A. R. Blass for reading the manuscript and suggesting many changes to improve the readability of the paper.

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