The Bulletin of Symbolic Logic
Volume 4, Number 2, June 1998

ALONZO CHURCH'S CONTRIBUTIONS TO PHILOSOPHY AND INTENSIONAL LOGIC

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§0. Alonzo Church's contributions to philosophy and to that most philosophical part of logic, intensional logic, are impressive indeed. He wrote relatively few papers actually devoted to specifically philosophical issues, as distinguished from related technical work in logic. Many of his contributions appear in reviews for ***The Journal of Symbolic Logic***,[1] and it can hardly be maintained that one finds there a "philosophical system". But there occur a clearly articulated and powerful methodology, terse arguments, often of "crushing cogency",[2] and philosophical observations of the first importance.

Many of the less formal philosophical contributions center around questions concerning meaning, but there are important clarifications and insights into matters of the epistemology and ontology of the sciences, especially the formal sciences.

§1. Methodology, epistemology, and ontology.

1.1. The logistic method. Church's writings on philosophical matters exhibit an unwavering commitment to what he called the "logistic method".[3] The term did not catch on and now one would just speak of "formalization". The use of these ideas is now so common and familiar among logicians

Received December 10, 1997; revised January 29, 1998.

Managing Editor's note. The June 1997 issue of this Bulletin contained a note from the managing editor describing a series of four papers commissioned by the Bulletin about Alonzo Church and his work as well as the first two articles of the series by W. Sieg and H. Barendregt. This paper by C. A. Anderson and the following one by H. Enderton conclude the series. They too are dedicated to the memory of Alonzo Church.

[1] In referring to reviews and works reviewed by Church I will use the standard system of ***The Journal of Symbolic Logic*** namely the volume number in Roman numerals followed by the page number upon which the review begins. In the case of more than one such item on a given page, the notation is, e.g., VII 131(2) for the second review or article reviewed in Volume VII, p. 131. Typically the year of the volume will also be included.

[2] This is Church's characterization of Frege's argument that sentences denote truth values [X 101, 1945], but it is at least equally appropriate here.

[3] Church attributed the invention of the logistic method to Frege in the dictionary entry, "Frege, (Friedrich Ludwig) Gottlob" [14].

1079-8986/98/0402-0001/$5.30

and logically-oriented philosophers that they are simply taken for granted. But they deserve to be celebrated and re-emphasized, for there are (still) philosophers who seriously underestimate and even consciously reject these techniques.

Church's characterization of the logistic method is strikingly detailed. It is described at length in Chapter 0 of ***Introduction to Mathematical Logic*** (1956, hereafter "***IML***") and its merits are repeatedly stressed in critical reviews. One first sets up a *logistic system* by giving its *primitive basis*. This is done by first specifying the *primitive symbols* and the *formation rules*. The fact that primitive symbols are often specially invented and, perhaps, unfamiliar notations is not of major theoretical significance.[4] One could use the already available vocabulary of some natural language. But the formation rules are to be exactly stated, effectively specifying the *well-formed formulas* of the logistic system—certain sequences of the primitive symbols, typically those to which one intends to ultimately assign meanings. Here is the first step in what may seem to be a kind of bootstrap operation.[5] Using the natural language, a language which is itself vague, irregular, and confused,[6] one precisely specifies the syntax of the logistic system and thereby begins to articulate a theory or system of logical analysis.[7] Church predicts that such a language can be expected to deviate more and more from the natural language as progress in logical analysis is made [XVII 284, 1942].

Then there is to be an effective characterization of the *axioms* and *rules of inference*. That such things are explicitly stated is a striking difference from the natural language and it is controversial whether they are even present implicitly in that case.[8] A *proof* is a finite sequence of well-formed formulas, typically *sentences* (to be defined in giving the formation rules), each one of which is either an axiom or follows from previous well-formed formulas by a rule of inference. A *theorem* (of the logistic system) is a well-formed formula for which there exists a proof, in the mathematical sense of existence.

Church gives plausibility arguments for the various requirements of *effectiveness*. That there be an effective test or criterion for being well-formed is required if a logistic system is to be (theoretically) usable as a system of communication. "And to the extent that requirements of effectiveness fail, the purpose of communication fails."[9] If, for example, an expression of the language is uttered, or written, and if there is no effective criterion

[4]This point is made already in VIII 47, 1943, in XVII 284, 1952, and again in ***IML***, p. 3.

[5]See also XXI 77(4), 1956.

[6]See, for example, XXI 312(4), 1956.

[7]***IML***, pp. 2–3.

[8]It is nevertheless plausible to argue that our grasp of natural language requires the grasp of certain rules of inference—if we are to count as having actual understanding. And it is in many cases a short conceptual step from a rule of inference to an axiom.

[9]***IML***, p. 52.

of meaningfulness (alias, well-formedness), an auditor may fairly demand a proof that an assertion has been made. If such proof were required to establish the status of the utterance and were given, says Church, then the proof ought to be incorporated into the utterance itself and the rules of well-formedness modified to allow for this. In a similar vein, he argues that the very notion of a proof requires that there be an effective method for determining whether or not a given sequence of well-formed formulas is such: "Indeed it is essential to the idea of proof that, to any one who admits the presuppositions on which it is based, a proof carries final conviction".[10]

These conclusions about effectiveness do not actually seem to follow from the considerations offered. For example, a correct proof will carry final conviction if there is a positive test or criterion for its being so—*if* it is a proof, then this shall be verified in the fullness of time. Why must there also be a negative test? One might urge that the idea of a proof requires that one be able to settle the question, yes or no, but it is not quite clear that this is so. Even if it isn't so, it would clearly be at least an advantage over the informal notion of proof that *proof in such-and-such logistic system* be effectively decidable. And on these grounds alone the inclusion of such requirements in the definition of a logistic system is fully justified.

To obtain a *formalized language* from a logistic system, there must be added *semantical rules*. In ***IML*** there is some brief, but characteristically deep, discussion of the form which such rules should take. For the immediate purpose, the formulation of classical (extensional) logic, Church adopts semantical rules which give the *denotations* of various well-formed expressions,[11] the *values* of expressions with free variables for given systems of values of their free variables, and (as a special case of denotation) the *truth-values* of sentences. One does not find a definitive discussion in Church's other writings as to *intensional semantical rules*—rules which determine the intensional meanings (and intensional values for given systems of intensional values) of expressions in the language. In "The Need for Abstract Entities in Semantical Analysis" [20][12] (hereafter, "*NAESA*"), it is said that one should give *rules of sense* (and of *sense values*) but a few years later, Church allows also the desirability of specifying *rules of synonymy* and *rules of non-synonymy* in connection with a formalized language, at least for

[10]***IML***, p. 53. See also XV 221, 1950. Similar considerations about proof appear in John Myhill [52] and in the writings of other logicians.

[11]Typically, if one is just formulating a logic, rather than a full-fledged formalized language, there are few denoting expressions. In certain formulations of the propositional calculus there occurs a sentential constant, say 'f', which denotes the truth-value **f** (falsehood) and in Church's formulations of the Logic of Sense and Denotation (cf. §3.3.1), there are more denoters.

[12]Aptly renamed "Intensional Semantics" for publication in Martinich [50]. Church was never really happy with the original title.

certain restricted purposes.[13] Of course these latter could not take the place of the former—one has to be actually given the meanings, and synonymies need not do this. And in fact Church seems to see the rules of sense as rendering rules for synonymies and non-synonymies unnecessary, presumably on the grounds that one need only know what the senses are in order to be able to determine whether or not they are identical.[14]

This presupposes that the meta-language, initially a restricted portion of the natural language, is able to discuss and assign meanings—concepts, for example, and propositions. Ultimately, Church holds, the meta-language too should be formalized. He therefore sets out to clarify the intensional notions by, of course, the logistic method. Thus the *NAESA* project and the *logic of sense and denotation*, to be discussed below in §§2.4 and 3.3.1, respectively.

Church is emphatic that the procedure of translating into the natural language, even if the semantical rules for it are themselves made precise, is not necessary for interpreting a formalized language.[15] In fact, by itself the procedure of translating in this way could not have the desired effect—the vagueness and uncertainty of the former could hardly be prevented in this way from infecting the latter. And, he urged, there is no reason in principle why the formalized language could not be learned as a first language, precise syntax and semantics and all, so it is also not necessary to first specify English or some other natural language to partake of the fruits of the formalized language.

It should be mentioned that there is an important revision to Church's view on the learnability of formalized languages in his review of J. J. C. Smart's "Theory Construction" [XXXVIII 665, 1973]. There it is conceded that it may be that a first-learned language has to have some sort of "here-now" words, whose denotation and logic are not fixed, but are relativized to context. Rules of inference, for example, would not be stated absolutely, but as "valid-in-such-and-such-a-context." One supposes that such an argument as "He is here, now. It is now 2:30; hence he is here at 2:30" would be valid relative to a context.[16] Church conjectures that it would be desirable to set up such a semi-formalized language containing here-now words and to use it

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[13]There are some further brief remarks about semantical rules in "Intensional Isomorphism and Identity of Belief" [21].

[14]This assumption can be, and should be, disputed. More recent discussions make it quite plausible that one can "know the meanings" of two expressions, in the ordinary acceptation of that phrase, and not necessarily be able to determine that they are or are not synonymous. So as representing the competence of a complete (ideal) master of the language, we should conclude that this second type of rule is not dispensable.

[15]In XXI 77(4), 1956, in ***IML***, p. 48, and in *NAESA*.

[16]David Kaplan has worked out the logic of such context-relative arguments in his [47]. And there is a further extensive literature on context-sensitive languages, typically from a semantical point of view rather different from Church's.

as meta-language to formalize a system of the more usual "static" sort. Here one sees a more realistic and detailed scheme for "building a better ladder" (as opposed to pretending to climb it and then kicking it away). Church did not carry out this project in detail, at least in print, but it represents an important change of viewpoint and undoubtedly had time permitted, he would have toiled honestly as his methodology demanded.

Church seems to view the detailed construction and investigation of a formalized language as a sort of counterpart to experimentation in the natural sciences. One cannot really fully anticipate the sorts of logical problems and surprising consequences that may arise when the attempt to supply the details is actually made. God, then, or the Devil, is in the details.

There is here no extremist commitment to rigor. Church explicitly admits that formalization is not necessarily an aid to the practice of natural science itself. There is no demand that, for example, physical sciences must be axiomatized—except in special cases where it is plainly desirable, such as the proposed non-standard-logical approaches to quantum mechanics. But formalizability in principle is "important to formal logic and to theoretical methodology" [XXXVIII 665, 1973]. Whether actual formalization is likely to be valuable must examined in detail, case by case. He expresses reservations, for example, about the question of whether such formalization will be of use to psychology in its present stage [VII 100, 1942]. He does add [X 132, 1945] that eventually a "firm language" must be sought in such domains also.

To those (for example, Smart in XXXVIII 665) who would argue that the language of science actually demands vagueness, or is, or should be unformalizable (in principle), Church replies that the inexactness is in the fit between theory and world. He often cites Euclidean Geometry as an example of a precise theory applied to the real world. There is imprecision but this resides entirely in the applications. The very success of mathematics as applied depends, he thinks, on the fact that the rigor with which the theory is formulated exceeds the rigor of our control over the situation to which it is applied. It doesn't follow, of course, that the best method of doing science itself involves full formalization.[17]

Philosophy, on the other hand, can and must use the logistic method. Vague problems may be discussed, but "... discussion of them is profitable only so far as directed towards their more precise formulation ..." [XIII 226(1), 1948]. According to Church, the use of symbolic logic and the logistic method generally in philosophy is the valuable legacy of the Logical Positivists (especially, one supposes, Rudolf Carnap). No philosophical

[17]But it is, he says, "... a test of the perfection of a physical theory that it shall become a branch of pure mathematics by abstracting from the intended application ..." [Ibid., p. 667]. Perhaps here Church does not see, or ignores as unimportant to the point, any distinction between logic and mathematics.

system is worth more than preliminary discussion unless it at least aspires to such a formulation [XXIX 48, 1964]. But "... the value of logic to philosophy is not that it supports a particular system but that the process of logical organization of any system (empiricist or other) serves to test its internal consistency, to verify its logical adequacy to its declared purpose, and to isolate and clarify the assumptions on which it rests" [XXI 396, 1956].

Time and time again, Church criticizes articles for making vague suggestions about logical and philosophical matters without actually investigating their feasibility by constructing or beginning to construct a formalized language incorporating the idea.[18] If a philosophical position or question is declared incapable in principle of being formulated and investigated by such a method, then "may the suspicion of a pseudo-problem arise" [Ibid.]. This has a kind of Positivist flavor, but is of course a much more moderate view.

This then is the fundamental methodology of the formal sciences and of philosophy. Once formulated precisely, it remains to evaluate the resulting theory.

1.2. Hypothetico-deductive rationalism. We should see Church's approach to the epistemology of mathematics and logic[19] as a sort of "hypothetico-deductive[20] rationalism". After necessary preliminary philosophical criticism and evaluation, a proposed theory must be rigorously formulated via the logistic method. Acceptance of the axioms of such a theory can rest in part on intuition[21] or faith[22]

In discussing Paul Cohen's results on the Continuum Problem [27], Church says that the Axiom of Choice does not seem to him to have the sort of self-evidence of elementary principles of logic (or, one might add, of some of the other axioms of set theory). And he there seems to admit the initial legitimacy of "[t]he feeling that there is an absolute realm of sets, somehow determined in spite of the non-existence of a complete axiomatic characterization ... " (p. 18). So an intuition, or feeling, of "self-evidence" is *prima facie* evidence in favor of the truth of such principles, but "all presuppositions must receive repeated critical re-examination." [XIII 226(1), 1948].

Of course this feeling about "a problem which intuition seems to tell us must 'really' have only one true solution" [27, p. 18] is challenged, even if

[18] For example: VI 106, 1941; VI 108(1), 1941; XII 96(3), 1947; XXI 318, 1956.

[19] I concentrate on the formal sciences. However, there are places in which Church expresses doubts about the possibility of drawing a clear distinction between the purely logical (or mathematical) part of a theory and its empirical component, e.g., X 16, 1945. So there is evidence of a certain agreement with some of Quine's ideas about theories.

[20] I do not mean to imply by the use of the term "hypothetico-deductive" that Church was committed to any specific form of confirmation, only that *a priori* theories are to be evaluated partly in terms of their consequences.

[21] Church was not "... in sympathy with Brouwer's subjectivistic epistemology" [XIV 132(2), 1949] with its heavy reliance on individual intuition.

[22] See "Mathematics and Logic" [25].

not contradicted, by Cohen's result which, together with Gödel's prior work, entails that the best available candidate for the one true set theory cannot settle the continuum problem. Church goes on to say that the "informal" criterion of simplicity may be all one has to fall back on if the competing set theories (settling the Continuum Hypothesis in opposite ways, perhaps) are otherwise equally good.[23]

There is, in the case of intensional entities, such as propositions and concepts, even a kind of "observation" of the entities in question. He says in *NAESA*:

> But the preference of (say) *seeing* over *understanding* as a method of observation seems to me capricious. For just as an opaque body may be seen, so a concept may be understood or grasped. And the parallel between the two is indeed rather close. In both cases the observation is not direct but through intermediaries—light, lens of eye or optical instrument, and retina in the case of visible body, linguistic expressions in the case of the concept. And in both cases there are or may be tenable theories according to which the entity in question, opaque body or concept, is not assumed, but only those things which would otherwise be called its effects.

Notice that this type of observation is not directly relevant to the existence of other, non-intensional, logical or mathematical entities.[24]

In general theories are to be evaluated in terms of their consequences, including the ability to provide analyses for various notions and to solve pre-existing logical puzzles.[25] But he mentions, at various places, besides simplicity, also naturalness, convenience, workability, generality, and elegance as criteria (presumably "informal") for evaluating theories.[26]

Philosophers of science have not so far provided justification for the application of such standards in the natural sciences. Church's attitude seems to

[23]In his review of Stephen F. Barker, "Realism as a philosophy of mathematics" [XL 593, 1975], he suggests a different response to the situation: "There may under certain circumstances turn out to be alternative set theories whose sets alike have reality." Against the position of such a realist, objections based on the Cohen result have no force. But Church adds that, "Granted the realistic outlook, it still seems unlikely that anything presently known or fixed upon has singled out a unique reality to be called the domain of sets, or even has selected a small number of such."

[24]Possibly he thought that our *a priori* knowledge of extensional entities such as sets depends upon our grasp of and intuitions about propositions concerning those entities. Charles Parsons has observed that this last view is close to Gödel's.

[25]Church would have agreed with Russell's remarks in "On Denoting" [58] about keeping one's mind well-stocked with logical puzzles. See, for example, *NAESA*.

[26]It would be interesting to know if there was any influence, in either direction, between Church and Gödel on these matters. In his "Russell's Mathematical Logic" [39], Gödel says things about the "hypothesis" of abstract entities which have the same general tone as many of Church's ideas. According to a footnote, Church read that paper and suggested improvements in the English.

have been that the criteria are obviously reasonable in that domain (perhaps this fact is itself delivered by intuition) and make equally good sense when applied to the formal sciences. John Myhill, discussing the use of simplicity as a methodological principle in the *a priori* sciences, writes: "To accept such an extension one must be at once sufficiently a platonist to believe in a strong parallelism between the methods applicable to empirical and *a priori* matters, and sufficiently an empiricist to be hypothetical rather than dogmatic in the latter sphere" [52, p. 74]. Though not intended as such, this strikes one as a quite accurate characterization of Church's stance about the application of such principles.

Church is not generally sympathetic to empiricism, behaviorism, or nominalism.[27] He sometimes discusses the force of arguments when they are viewed from the empiricist perspective (e.g., in XXIII 344(2), 1958) but does not ever explicitly endorse that epistemology. And his remarks, quoted above, about understanding as observation are clearly opposed to an empiricism which confines the sources of knowledge to the senses, but not to a more general respect for the necessity of testing theory against observation.

He does however name the Verificationist Theory of Meaning as a philosophical contribution of the Positivists. One should not see this as an endorsement of its empiricist presuppositions, but rather as an acknowledgment of a laudable attempt to construct a precise theory of meaning. Indeed, in a famous review of the second edition of A. J. Ayer's ***Language, Truth and Logic*** [XIV 52, 1949], Church himself was to defeat an early attempt to give a clear statement of that theory.[28] The possibility of finding an acceptable formulation was not of course disproved by this, but it must have been a blow to the morale of the Positivists. Rudolf Carnap's much more sophisticated explication [7] suffered a similar fate at the hands of David Kaplan [46], and Carnap gave up. Thus Church played a role in bringing down the most powerful empiricist philosophy of modern times.

Nominalism, a philosophical viewpoint distinct from, but akin to empiricism, was also to suffer considerable damage from criticism by Alonzo Church.

1.3. Realism. Church is usually seen as a quite traditional Platonic Realist, but he himself eschewed that label, partly because he saw it as involving the thesis that only universals are real or that they are the ultimate

[27] Empiricism here means the view that all knowledge is ultimately to be based solely on sense experience. As applied to mathematics, Church held that one must defend "... naive empiricism against well-known objections, such as those of Frege ... against Mill" [XIV 126, 1949]. Presumably he doubted that such a defense would ultimately succeed.

Behaviorism is the view that the mental is somehow completely reducible to behavior. And nominalism is the doctrine that there are no abstract entities—semantics, mathematics, and the like are to be explained without appealing to such things.

[28] See XV 216(2), 1950, for a different criticism of the Verifiability Theory of Meaning—the view that the meaning of a sentence consists in its method of verification.

reality. Early on he espoused a kind of instrumentalism or fictionalism—mathematical entities are fictions, part of an abstract structure constructed by us to enable us to understand reality [Review of ***Principia Mathematica***, second edition, Volumes II and III, 1928]. This is the youthful Church and was pretty certainly not his mature view. Still, the view is stated again in [9] (pp. 348–349) and persisted until at least 1937 (see II 44(3), 1937).

We find him saying in 1939, in a review of W. V. Quine [IV 170], that the question of whether "abstract nouns" really denote abstract entities is illusory. Quine's view there is that a satisfactory criterion as to what entities there are, from the point of view of a particular language, depends crucially on the variables employed therein. Ontological commitments are determined in particular by the existentially quantified sentences of the theory. Church asserts that on this showing the answer is relative both to a choice of language and to a selection of a notation therein to stand for existential quantification. From this point of view, he says, the result of Quine's proposed criterion is to "emphasize the illusory character of the question of whether abstract nouns *really* have designata."

This may well signal a change of view. It's not quite clear what answer to the question should be given, according to the instrumentalist of fictionalist view. But one supposes that even after a particular language and symbolization of existential quantification is specified, all that is settled is what denotes-according-to-the-language. Fictionalism would seem to entail that either abstract nouns really don't denote (in any language) because the alleged entities are fictions, or that they can denote, but they will denote fictions—so that the question is not illusory.

Still, it is also not clear exactly what he is rejecting here, so the revised view is so far somewhat uncertain. He certainly thinks, early and late, that such questions are to be settled by constructing this or that precise theory-in-a-language and this idea persists. Maybe all that is being said is that *apart from a theory*, the question is illusory. One must construct a theory or language which postulates this or that, and the result is then to be tested against various objective standards. If an entity *exists according to such and such a theory* and the theory survives such evaluations, then the assumption is so far justified. He notes in the review presently being discussed that Quine's proposal of a nominalist language in which quantification over all alleged abstract entities is avoided would be extremely artificial.[29] Here and elsewhere (e.g., in V 81(4), 1940) he takes mathematical adequacy to be one of the criteria for evaluating a proposed language—meaning adequacy to at least reconstruct classical mathematics and mathematical reasoning.

If this was Church's view, then it is thoroughly respectable.[30] Don't try

[29]And in VI 108, 1941, he expresses doubts as to whether such a language is even possible.

[30]My hesitation here stems from various remarks he makes against "absolutism" in V 81(4), 1940, and in XIII 148, 1948, and his contrast in 1956, discussed immediately below,

to answer the question "Does such-and-such exist?" independently of all theory. Formulate the theory, postulate as you must and test the result. By contrast, his early view, denying the reality of mathematical and logical entities, suggests that some sort of negative answer be accepted prior to construction of the theory.

This general approach may lead one, as it does Church, to *postulate* such "odd" things as truth-values. Church explicitly contrasts his position with the view of Frege (as a "thoroughgoing Platonic realist") that the situation[31] "indicates that *there are* two such things as truth and falsehood" (***IML***, note 66, p. 25, Church's emphasis). The emphatic contrast seems a little odd. For Frege, the evidence "indicates" that there are two such things, and the same evidence induces Church to postulate two such things. Unless the difference is epistemological, Frege thinking that the thing has been proved and Church being more tentative and experimental, there is a slight suggestion here of something like the earlier instrumentalism.

It is clear that Church believed that an adequate logic capable of evaluating reasoning involving the propositional attitudes, such things as assertion and belief, would likely postulate propositions, properties, and "individual concepts." His own formulations, to be discussed below, postulate such things unabashedly.[32]

§2. Semantics: Meaning.

2.1. Nominalism: Inadequate. Various attempts to do away with propositions altogether, in favor of such relatively concrete things as sentences, were resisted by Church in several articles. In "On Carnap's Analysis of Statements of Assertion and Belief" [18], he refutes a series of nominalistic proposals of increasing sophistication. Church's refutations of the simpler of these views are as nearly conclusive as philosophical arguments ever get. Carnap's own analysis, though not obviously nominalistically motivated,[33] makes use of sentences instead of propositions as objects of belief. Church's critique of the more carefully formulated Carnapian proposal makes use of the now-famous "Translation Argument" and turns on assumptions which are somewhat more debatable.

In essence, Church's objection is that sentences reporting assertion and belief do not *mention* sentences of any language, neither of the reporting language nor of any other. Now, he holds, an analysis must preserve meaning—it must record a synonymy—and synonymous sentences must be

between "discovering" and "postulating" such things as truth-values.

[31] In the context, the "situation" is Frege's argument, to be discussed in §2.3, that sentences should be taken to denote truth-values.

[32] A quite similar attitude is expressed by Paul Grice, "My taste is for keeping open house for all sorts and conditions of entities as long as when they come in they help with the housework" [43, p. 31].

[33] But see note 39 below.

about the same things. Since C. H. Langford's simple device of translating a sentence into another language can be used to highlight the linguistic (or non-linguistic) subject matter of a sentence, it is not surprising that it is used by Church for just that purpose. Carnap uses the notion of *intensional isomorphism* in his analysis. Roughly, but precisely enough for the present purpose, two expressions are intensionally isomorphic if they have the same structure and differ at most in the replacement of necessarily equivalent simple parts. Using this idea, Carnap would analyze:

(1) Columbus believes that the world is round,

as

(1′) There is a sentence S in a semantical system $\mathbf{S}'$ such that Columbus is disposed to an affirmative response to S as sentence of $\mathbf{S}'$ and S as sentence of $\mathbf{S}'$ is intensionally isomorphic to 'The world is round' as sentence of English.

But, Church observes, the sentence (1) is not even a logical consequence of (1′)—one must depend on the further fact, not given by (1′), that 'The world is round' means in English that the world is round. That (1) and (1′) convey different information is emphasized by considering the translations of (1) and (1′) into German and testing them on a German speaker who knows no English. Church aims to show that the sentences have different meanings and if this is so, he argues, the one cannot be a correct analysis of the other.

A more intricate version of the argument is used by Church to deal with a possible reply to the objection that (1) is not a logical consequence of (1′). Here he makes use of two analyses—that just given and the analysis, as it would be given in German, of the translation of (1) into German. If we translate (1) directly into German, state the analysis (in that language) as it would be given, and translate back into English,[34] we get:

(2′) There is a sentence S in a semantical system $\mathbf{S}'$ such that Columbus is disposed to an affirmative response to S as sentence of $\mathbf{S}'$ and S as sentence of $\mathbf{S}'$ is intensionally isomorphic to 'Die Welt is rund' as sentence of German.

These transformations are supposed to preserve meaning, but obviously they do not—in fact, (1′) and (2′) are again not even necessarily equivalent. The semantical rules for German and English do not record necessary facts. To get from (1′) to (2′) one needs, in addition to what is given, some such thing as:

[34] In "A Formulation of the Logic of Sense and Denotation" [19] Church gives a slightly different version of this objection. There he translates analysans and analysandum into a different language and notes the non-synonymy. Here he has considered analyses in two different languages of a translated sentence. The present version, unlike the former, actually goes through on a weaker assumption about analysis than that it must be meaning preserving.

(4) 'The world is round' as sentence of English is intensionally isomorphic to 'Die Welt is rund' as sentence of German.

But this is a contingent fact about the two languages. And certainly sentences which are not even logically equivalent are not synonymous.

It should be noticed that even if we reject Church's requirement of synonymy for analysans and analysandum, and only take logical equivalence as a necessary condition, the proposed analysis is still refuted.[35]

Now there occurs an interesting twist in the dialectic. Church notes that Carnap will ultimately apply his analysis to "semantical systems", idealized interpreted formalized languages with regular syntactical and logical features. The objection so far, Church says, has assumed that the sense of the English words 'English' and 'German' makes reference to matters of pragmatics. For example, Church says, 'English' has been taken to mean something like 'the language which was current in Great Britain and the United States in A.D. 1949'. On that understanding, (4) is not a necessary truth and presumably must be supplied to establish the equivalence of (1′) and (2′). But, keeping Carnap's program in mind, if we take the sense of the word 'English' to be given by 'the language for which such and such semantical rules hold', and similarly for 'German', it might happen that (4) is a necessary truth.

One is tempted to interject at this point that the word 'English' does not mean either of these things, and similarly for 'German'. While this is correct as an immediate objection, it is short-sighted. Carnap has proposed an analysis. Church has considered a version of it according to which the names of natural languages occur in the analysans. Church's argument so far does not really depend on taking the sense of these names to involve matters of pragmatics. It is enough, for the argument to this point, if (4) is not a necessary truth. And this is so whatever the exact analysis of the meaning of names of languages therein, even if (heaven forfend!) the names have no sense at all, but instead denote the respective languages in the manner of Russellian proper names.

To avoid further dispute about the exact meaning of names of languages, Carnap needs only to modify his analysis so that these words do not occur at all, but rather definite descriptions of the sort Church offers as giving second possible meanings for the words 'English' and 'German'. And to the resulting analogue of (4) the objection that it is not logically necessary may just be false, depending on the details of the semantical rule descriptions. So the objection that (1′) and (2′) are not logically equivalent will not apply to their analogues, (1*) and (2*), on this alternative.[36]

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[35] Nathan Salmon makes this observation already in [59].

[36] I have not written out (1*) and (2*) to avoid further tedium. Imagine the words 'English' and 'German' replaced by descriptions of the form 'the semantical system whose semantical rules are . . . ', where these latter are written out in full.

It is here that Church points out that (1*) and (2*) are not acceptable translations of one another, that they are not intensionally isomorphic, and thus the analysis is unacceptable. Evidently Church's assumption that analysis requires synonymy is being used at this point—it may be worth noticing that all that need be assumed is that if the analysandum is translated into another language, then the resulting analysans in that language ought to be synonymous with the original analysans. This has some slight plausibility as a constraint on the concept of analysis, but is difficult to give any very convincing independent motivation.[37]

To drive the point home, Church embeds the two analysantia into a propositional attitude context and notes that this may produce sentences with different truth-values.

John believes that (1*)

may certainly differ in truth-value from

John believes that (2*).

Again it is being required that analysans and analysandum are synonymous—or at least closely enough so as to preserve truth-value when interchanged in belief contexts.

In reply, one may just reject the requirement that analysans and analysanda must be synonymous.[38] It can be quite convincingly argued that a useful analysis need not preserve meaning. Indeed if such is required, then there are virtually no known examples of analysis in mathematics or in philosophy. Certainly mathematics contains clear cases of successful analyses, but these are never identities of meaning in any strict sense. I believe that this is in fact a correct objection to Church's critique of Carnap's analysis. So, it would seem, the modified analysis has survived the Churchian onslaught. But the game is not over.

Church's criticisms are designed to tell against those who attempt to do without propositions and other useful abstracta.[39] A natural question is

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[37]John Kemeny, [48, p. 159], reports that Church suggested invariance under translation as a kind of test for a field of study, or the notions therein, as belonging to mathematics. The idea is evidently that language independence in this sense is a mark of objectivity and freedom from merely syntactical significance. Perhaps a case could be made that the notion of analysis should likewise obey such a condition, with a similar purpose. To really pursue this would require, I believe, a careful consideration of the reasons it is desirable to have an analysis of something. I suspect that the point would not hold up.

[38]This objection is made by Donald Davidson [36] and earlier by Hilary Putnam [55].

[39]Carnap, in ***Meaning and Necessity*** [6] thinks that certain kinds of propositions are perfectly acceptable—his intensions, which are identical when expressed by necessarily equivalent sentences. (Nathan Salmon makes this point in the paper mentioned in note 35.) But it may be that there is some nominalism hanging in the background. Carnap sees propositions in his sense as explicable via state descriptions, which are just sets of expressions. So in the end they are ontologically rather tame.

this: if a proposed "analysis" of sentences ostensibly about propositions is not required to preserve meaning, in what sense have propositions been "eliminated"? It was once common to reply at such junctures that "we can still say everything that we want to say", even without propositions, using instead the proposed (non-synonymous) analysis. But what does this really come to? Suppose that we want to say that Columbus believed that the world is round. Can we not do so?

Finally, insofar as Carnap's particular analysis is concerned, we should observe that Church will return to intensional isomorphism in "Intensional Isomorphism and Identity of Belief" [21] ("*IIB*") and, on independent grounds, argue against its use in analyzing belief.[40]

In "Propositions and Sentences" [23] and "On Scheffler's Approach to Indirect Quotation" [28], Church considers a quite overtly nominalistic proposal to do away with propositions in favor of inscriptions and predicates thereof. Here the proposal is by Israel Scheffler, but strongly influenced by the nominalistic analysis of language due to Nelson Goodman (early on in collaboration with W. V. Quine). Scheffler suggests that one can make do without names of propositions by using certain predicates of actually occurring and intentionally produced inscriptions—not expressions, for they too are abstract and hence, according to the nominalist, objectionable. In effect the predicates say that the inscription expresses such-and-such a proposition, but are themselves to be regarded as semantically indivisible. Church points out that this will result in difficulty in adequately analyzing such statements as "Church and Goodman have contradicted each other", "Goodman will speak about individuals", and "Some assertions of Velikovsky are improbable", things that seem to involve quantification over propositions. If new primitive predicates of inscriptions are introduced to deal with these examples, then ". . . it may be difficult or impossible to provide (axiomatically) for the logical connections between them, between them and the propositional predicates, and between them and the syntactical make-up of the inscriptions they apply to in a specified language." In effect, it is a challenge to provide the details of the theory so as to allow for the statement of quite ordinary things, the evident logical relations between such statements, and the ordinary needs of semantics. Apropos of this last, Church specifically points to the difficultly of providing a nominalistic definition of truth along the lines indicated.[41]

[40] I am indebted to the members of the Santa Barbarians Philosophical Discussion group for helpful observations on the Translation Argument. Nathan Salmon was especially helpful.

[41] A similar theme about the obstacles faced by nominalistic syntax appears in a letter to Nelson Goodman (quoted in the latter's [42, p. 153]). Goodman queries Church as to what tasks remain for nominalistic syntax, according to the platonist. Church responds by giving a list of the major results of classical syntax and semantics, including Gödel's Incompleteness Theorem. Amazingly, Goodman replies, in part, that "Incompleteness is not more to be cherished for the sake of Gödel's theorem than is crime for the sake of detection; banishment

His own description of the methodology to be used in constructing an analysis is given in "Logic and Analysis" [24]. An analysis of, for example, belief is not to be provided by a single statement of ordinary language but requires the full statement of a logic of the concept. To press the point he quickly and decisively dispatches a simplistic, single sentence, behavioristic analysis of belief due to A. J. Ayer.

The later Church does not anywhere express skepticism about propositions (and such related intensional things as propositional functions, in Russell's sense, and concepts). He thought that the best logical analysis of our ordinary discourse about belief, assertion, modality, and the like, will involve abstract intensional entities. He complains [e.g., in VI 108, 1941 and in XVII 284(2), 1952] that some philosophers want to discard propositions, but continue to speak and write in a way that seems to presuppose them or some near relative, such as states of affairs. Here also is the familiar complaint that these philosophers give only vague and sketchy indications of the language they would use instead.

2.2. Synonymy: Puzzles dispersed. In two reviews of versions of a paper by Nelson Goodman [XV 150, 1950; XXI 76(5), 1956] and in *IIB* Church takes on arguments to the effect that no two distinct expressions are ever synonymous. This conclusion would weaken, although it would not defeat, the motivation for postulating meanings. Goodman's argument [40] proceeds thus. Suppose that, for example, "triangle" and "trilateral" had the same meaning. Well, "triangle that is not trilateral" is a triangle-description but not a trilateral description. Hence the two words differ in "secondary extension". But synonymous expressions must have the same [primary and] secondary extension. So the expressions are not synonymous after all.

Church replies:

> The author states no decision as to whether 'triangle that is not a triangle' is a triangle-description or not. And he does not say under what circumstances he holds that the predicate $\ulcorner\zeta$-description, but not an η-description$\urcorner$ applies to an inscription of the form $\ulcorner\zeta$ that is not $\eta\urcorner$. What, e.g., if we take ζ to be '$\hat{x}\ (\exists z) \centerdot z^2 = x$' and η to be '$\hat{x}\ (\exists y) \centerdot y^2 = x$'? [XV 150].

Ever true to form, Church complains that Goodman's own proposal, to speak of likeness of meaning of greater or lesser degree, is not really worked out "by a detailed treatment within an adequate formalized meta-language."

In a reprint with some added notes Goodman supplies two principles, axioms in effect, to govern his use of $\ulcorner\zeta$-description$\urcorner$. Church then points out that one of the principles fairly straight-forwardly begs the question

of crime and incompleteness to the realm of fiction would hardly be a matter for regret." Thus the mentioned metatheorem is just "banished"! One wonders if Goodman might therefore in good conscience propose to a graduate student the task of producing a complete and consistent axiomatization of arithmetic.

against the synonymist [XXI 76(5)].

A more interesting contest is between Church and Benson Mates, in particular, as against the latter's famous argument from "Synonymity" [51]. Again the argument is essentially to the effect that no two distinct expressions are really synonymous.

Begin by assuming that two distinct expressions, say 'fortnight' and 'period of fourteen days', are synonymous. Then

> (14) Whoever believes that the seventh consulate of Marius lasted less than a fortnight believes that the seventh consulate of Marius lasted less than a fortnight.

and

> (15) Whoever believes that the seventh consulate of Marius lasted less than a fortnight believes that the seventh consulate of Marius lasted less than a period of fourteen days,

ought to be synonymous. But this cannot be since, according to Mates, it is true that:

> (16) Nobody doubts that whoever believes that the seventh consulate of Marius lasted less than a fortnight believes that the seventh consulate of Marius lasted less than a fortnight.

But it is not true that:

> (17) Nobody doubts that whoever believes that the seventh consulate of Marius lasted less than a fortnight believes that the seventh consulate of Marius lasted less than a period of fourteen days.

For example, philosophers who have considered the question of the criterion of identity of belief and have been led to hesitation or doubt may serve to falsify (17). Perhaps Mates himself is such a philosopher.

Church responds in *IIB* by once again applying the (Langford's) Translation Test. But this time, he concludes by way of translation that two things *are* synonymous, contrary perhaps to hasty first examination. Since German has no one word translation for 'fortnight', the translations into that language of (14) and (15) are both:

> (14′) (15′) Wer glaubt dass das siebente Konsulate des Marius weniger als einen Zeitraum von vierzehn Tagen gedauert habe, glaubt dass das siebente Konsulate des Marius weniger als einen Zeitraum von vierzehn Tagen gedauert habe.

Therefore, Church concludes, the translation into German of 'Mates doubts that (15) but does not doubt that (14)' is a direct self-contradiction and so is false. Hence, whatever Mates himself may say, he is not really a counterexample to (17). Now Church goes on to suggest that Mates and others in a similar state are confusing a doubt about meta-linguistic matters with the doubt that (15), this latter being identical with the doubt that (14)—namely, a question about who satisfies (in English) a certain sentential matrix involving the expessions 'fortnight' and 'a period of fourteen days'.

This is a fascinating and powerful response to Mates's challenge. But the reasoning requires a principle which is actually inconsistent(!) with things Church held about Fregean semantics as it is to be applied to the so-called Paradox of Analysis.[42] Why, philosophers have asked, is the sentence:

(18) Fortnights are periods of fourteen days,

an analysis and so informative (even if relatively trivial) as opposed to:

(19) Fortnights are fortnights.

In his review of a controversy between Max Black and Morton White about this puzzle [XI 132, 1946], Church invokes some ideas deriving from Frege. The sentence (18), if proposed as an analysis, is misleadingly stated. For that purpose, it is not equivalent to a statement about the class of fortnights. Rather, it must be construed thus:

(18′) (The concept) *fortnight* = (the concept) *period of fourteen days*,

where the concepts here are taken to be the ordinary (Fregean) *senses* of the expressions 'fortnight' and 'period of fourteen days' respectively. The corresponding analogue of (19) is:

(19′) (The concept) *fortnight* = (the concept) *fortnight*.

Now we may pose the question involved in the Paradox of Analysis: How can (18′) differ in meaning from the trivial (19′)? According to Frege an expression in an "indirect" or "oblique" context denotes, not its usual denotation (here a class), but its usual or ordinary sense, its meaning strictly and properly so-called. The verbs expressing belief, knowledge, and assertion produce, says Frege, such contexts.[43] Now in those contexts an expression will have a sense different from its ordinary sense, an indirect or oblique sense. So, Church argues, here, in (18′), the expressions 'fortnight' and 'period of fourteen days' are in oblique contexts, induced by the word 'concept'. Just so, urges Church, can (18′) be informative: the senses of '*fortnight*' and '*period of fourteen days*' in those, oblique, contexts are different. That is, the indirect senses of these expressions are distinct. On the other hand, both sides of the identity (19′) have the same sense, namely, the indirect sense of 'fortnight'.

Ingenious, compelling, and, one might think, undeniably correct.[44] But

[42]This has been noticed and discussed in print, in passing by the present author, and in explicit and careful detail by Nathan Salmon [60].

[43]It is more correct to say that it is the "that" following such verbs, explicit or implicit, which induces the oblique context.

[44]I personally doubt seriously whether (18′) qualifies as an analysis in any sense worth saving. But the puzzle about the apparent difference in cognitive content between (18′) and (19′) remains even so.

observe that Church has implicitly assumed, in his argument against Mates, that the indirect senses of 'fortnight' and 'period of fourteen days' are identical—because these two expressions are synonymous. Something must give. The most plausible line seems[45] to be to abandon the argument against Mates and use the already available mechanism of indirect senses to account for the possible difference in truth-value between (16) and (17). One will then need to maintain that two synonymous expressions may nevertheless differ in indirect sense. It follows that belief sentences in English involving embedded occurrences of 'fortnight' need not have precise, literal, and strictly meaning preserving translations into German—if that language lacks any means of reproducing the indirect sense of 'fortnight'.[46] Church explicitly urges that the absence of such a device in German can hardly be a real deficiency. And perhaps it is not, at least not in ordinary contexts.[47]

2.3. Semantics of sentences: Frege's scheme. As is well known, Church thought that Frege's ideas about sense and denotation provide the most promising foundation for the theory of meaning.

In his review [15] of Carnap's ***Introduction to Semantics***, Church says that he was converted from his former view, that sentences denote propositions, to the view that sentences denote truth-values under the influence of Frege. Church develops an argument which seems to be implicit in some of Frege's remarks, and directs it to the particular proposals of Carnap on the matter. Later, in ***IML*** [pp. 24–25] the argument is restated and generalized in certain respects. We describe only the latter in detail.

The argument is supposed to be particularly effective against those who hold that sentences denote propositions. Assume the plausible semantical principle that if two expressions A and B have the same denotation, then the denotation of a sentence[48] containing A will be the same as that of the corresponding sentence with the occurrence of A replaced by B. Now consider the following sequence of sentences:

(1) Sir Walter Scott is the author of *Waverley*.
(2) Sir Walter Scott is the man who wrote twenty-nine Waverley Novels

[45]I.e., the most plausible according to the present author. Salmon, in the paper cited in note 42 above, thinks it preferable to drop the idea of indirect senses and find some other solution to the Paradox of Analysis. As he notes, this involves abandoning some essentials of the Fregean semantical scheme. One suspects that Church would adopt the course advocated here.

[46]Nathan Salmon drew this somewhat counter-intuitive consequence to my attention. It doesn't follow that the belief sentences have no standard translation into German—only that such translation need not strictly preserve the Fregean sense of the sentence.

[47]The case for the proposed analysis of the situation can be strengthened by observing that the Fregean theory of indirect senses provides some machinery which may be able to deal with the puzzle in Saul Kripke's "A Puzzle About Belief" [49]. This application of the theory has been noticed in print by Nathan Salmon [61].

[48]If no indirect, or oblique, contexts are involved.

altogether.

(3) The number, such that Sir Walter Scott wrote that many Waverley Novels altogether, is twenty-nine.
(4) The number of counties in Utah is twenty-nine.

The sentences (1) and (2) must have the same denotation since 'the author of *Waverley*' and 'the man who wrote twenty-nine Waverley Novels altogether' have the same denotation. Similarly, (3) and (4) have the same denotation since 'The number, such that Sir Walter Scott wrote that many Waverley Novels altogether' and 'The number of counties in Utah' have the same denotation. The claim that (2) and (3) have the same denotation is justified by Church thus: "[I]f [(2)] is not synonymous with [(3)], [it] is at least so nearly so as to ensure its having the same denotation . . . " (p. 25). It would follow that (1) and (4) have the same denotation, but they ". . . seem to have very little in common. The most striking thing that they do have in common is that both are true." (Ibid.) Elaboration of such examples leads one to conclude, with at least plausibility, that all true sentences have the same denotation. Parallel examples strongly suggest the conclusion that all false sentences have the same denotation.

Certainly considerations of this sort point toward the Fregean conclusion. But an exponent of the view that sentences denote propositions might well complain that the argument has a gap. The transition from (2) to (3), on his view, is guaranteed to preserve the denotation of the sentence only if they are exactly synonymous, and not merely nearly synonymous.[49] An alternative premiss to justify the transition has sometimes been adopted in this connection, namely, that logically equivalent sentences must have the same denotation.[50] But it is difficult to see the proponent of the view that sentences denote propositions (or, for that matter, facts or states-of-affairs) as finding this principle acceptable—if propositions are to be viewed as the objects of assertion and belief.[51]

Even waiving the objection explained, the conclusion may be resisted by invoking Russell's Theory of Descriptions to fault the argument at another

[49] An anonymous referee made the observation that (2) and (3) differ in that (2) claims to uniquely characterize Scott, while (3) does not. That referee also observed that this defect in the argument is easily repaired.

Gödel [39] gives a rather different argument for the Fregean conclusion that any two true sentences "signify" the same thing. His argument also depends upon an assumption (which might be questioned) about the identity of meaning of distinct sentences.

[50] This assumption was made by Carnap in ***Introduction to Semantics*** and Church uses it in his version of the argument in his review thereof.

[51] Possibly the assumption of the Positivists that logically equivalent sentences "say the same thing" could be behind this point of view. This, in turn, depends on the idea that one can't learn anything new by deduction—a position which certainly will not survive critical scrutiny.

point. That theory avoids the conclusion by denying that there are any semantically complex names (other than sentences). In particular it is held that descriptions are not subject to the mentioned semantical principle. Church was well aware of this last maneuver and seems to have held throughout that Russell's theory is tenable but is inferior to Frege's. The matter is still not settled but some (including the present author) find the conclusion practically irresistible that the natural, parallel, and plausible analogue of the denotation of a complex name, as extended to sentences, is the truth-value.

2.4. General pure semantics: The NAESA project. The review of Carnap's ***Introduction to Semantics*** describes it as "an important pioneering work". Church himself "appreciate[s] the importance of semantical investigations in philosophy", and "agree[s] with Carnap in his expectation of the fruitfulness of this new line of approach". The new line of approach is to see semantics as consisting of a pure and a descriptive part. Pure semantics is an abstract discipline, a branch of mathematics which defines "semantical systems", formalized languages really, and investigates the consequences of such definitions. Descriptive semantics is concerned with describing the structure and meaning of historically given languages.

Church suggests and argues for two fundamental changes in Carnap's initial work. Sentences, for the Fregean reasons just discussed, should be seen as denoting or designating truth-values. And the notion of sense should be taken to play a role of equal importance with that of designation. Church had initiated his defense and elaboration of Frege's theory of sense and denotation in a talk to the Seventh Meeting of the Association for Symbolic Logic in the previous year, 1942 [VII 47]. From the abstract it appears that it was an informal description of Frege's views and a comparison with the views of Russell. Church had evidently been converted to Frege's views by then. In the review of Carnap he says, about the idea that sentences denote truth-values rather than propositions, "On this point the reviewer confesses to have changed his own former opinion, but not without compelling reason" [p. 299].

In XII 55, 1947, Church reviews a paper called "Metalypsis and Paradox in the Concept of Metalanguage" by Henry Winthrop. In spite of some criticisms, Church writes, "The rest of his discussion may, however, be read as urging the construction of a universal semantical meta-language (which would in particular serve as meta-language of itself)." But in spite of the excessive strength of some of Winthrop's claims, "... the reviewer would nevertheless agree in part with much that is said."

He goes on to say that what is needed is a definite proposal for such a universal semantics and "... it may be that no such language is consistently possible."

NAESA is Church's attempt to formulate something like a universal semantics. He was well aware that in detail it would be impossible for a language to serve as its own meta-language in a certain sense. But the ideal goal is that universal semantics is supposed to be a theory of meaning about all possible languages, "an abstract theory of the actual use of language for human communication—not a factual or historical report of what has been observed to take place, but a norm to which we may regard everyday linguistic behavior as an imprecise approximation ...". "We must demand of such a theory that it have a place for all observably informative kinds of communication—including such notoriously troublesome cases as belief statements, modal statements, conditions contrary to fact—or at least that it provide a (theoretically) workable substitute for them. And solutions must be available for puzzles about meaning which may arise, such as the so-called "paradox of analysis." But "[b]ecause of the extreme generality which is attempted in laying down these principles, it is clear that there may be some difficulty in rendering them precise (in their full attempted generality) by restatement in a formalized metalanguage." The principles are therefore stated informally (really semi-formally) with pretended or attempted full generality and the comment is added that "... it should be possible to state the semantical rules of a particular object language so as to conform, so that the principles are clarified to this extent by illustration." [note 11].

As one might expect, the principles are Fregean, but with some modification and generalization. The basic syntactical categories are *constant*, *variable*, and *form*—this latter being the general notion of an expression which has free occurrences of variables. *Sentences* are of course names of truth-values. There is no real attempt to minimize the primitives at this stage. Frege's problematic notion of a function as "unsaturated" is dropped and to sense and denotation is added the pair of semantical notions "(denotation) value"—of a form for a given assignment of values to variables—and "sense-value" of a form for a given assignment of sense-values to the variables. These last are the generalizations of Frege's notions which Church sees to be implicit in the theory: "For Frege's question, 'How can $a = b$ if true ever differ in meaning from $a = a$?' can be asked as well for forms a and b as for constants, and leads to the distinction of value and sense-value of a form just as it does to the distinction of denotation and sense of a constant." This leads Church to associate with variables, not only the usual range, but a range of sense-values.

Church then states sixteen principles about concepts (the senses of constants), variables, constants, forms, sense, denotation, sense-value, value, and substitution. Here is a sampling: (i) Every concept is a concept of at most one thing. (ii) Every constant has a unique concept as its sense. (iii) Every variable has a non-empty class of concepts as its sense-range. (iv) The denotation of a constant is that of which its sense is a concept. (v) The range of a variable is the class of those things of which the members of the sense-range are concepts. Using a terminology which Church would introduce later (in ***IML***), let us say that two forms are *concurrent* if they have the same value for every assignment of values to their free variables and *sense-concurrent* if they have the same sense value for every assignment of sense-values to their free variables. Sense-concurrence among forms is the analogue of sameness of sense, synonymy, among constants. Denotation is preserved if codenoting constants are interchanged—with fairly obvious analogues for concurrence and sense-concurrence of forms.

Principle (i) is the only one which is not obviously and directly about general semantics. The relation between a concept and that of which it is a concept is, as Church conceives it, a relation which obtains independently of this or that language. If a constant in a language expresses the concept, then that constant comes to denote that which falls under the concept. But even principle (i) might be semantical at bottom—if one were to define a concept as what is expressed by a constant in some (possible) language and define such a concept to be a concept of what is denoted thereby, then this too is (very) general semantics. This would not be Church's way. "In logical order, the notion of a concept must be postulated and that of a possible language defined by means of it."[52]

Church does not here follow Carnap's example in ***Introduction to Semantics***. Carnap thinks of the denotation relation of a semantical system (formalized language) as just being defined in the manner of Tarksi [63]. If so, then there is no need for a primitive notion of denoting, or of having values. A semantical system is just defined as having such-and-such syntax, and with denotation for that language defined as a certain correlation between the syntactical entities and some chosen entities.

One reason Church does not take this line is that he is apparently in the process of formulating a general theory of truth and meaning. Carnap, following Tarski [63], does not really envision such a theory. Rather, the "Tarski method" of defining denotation and truth on a case by case basis is to be employed. In a certain sense, there is no such thing as a general theory of (pure) semantics, only a general method. In addition, as already noted, Church would add the notion of sense as a semantical primitive and it is not immediately clear how to eliminate this along Tarskian lines.

But Church had another reason for resisting the Tarskian approach to semantics, even on the purely extensional side and as applied to particular languages. He sees Tarski's definition of truth (and the analogous definition of denotation) for a language as a reduction of a semantical notion to a syntactical(!) notion. Church had already reached this conclusion in [13] and it is discussed in ***Introduction to Mathematical Logic***, ***Part 1*** (1944) and in more detail in ***IML***. In the latter he would write,

As appears from the work of Tarski, there is a sense in which

[52] In lectures given in 1946 (notes by Leon Henkin), Church was more tentative. He there said that "[t]he question is open whether senses precede language or vice-versa."

> semantics can be reduced to syntax. Tarski has emphasized especially the possibility of finding, for a given formalized language, a purely syntactical property of the well-formed formulas which coincides in extension with the semantical property of being a true sentence. And in Tarski's *Wahrheitsbegriff* the problem of finding such a syntactical property is solved for various particular formalized languages. But like methods apply to the two semantical concepts of denoting and having values, so that syntactical concepts may be found which coincide with them in extension [***IML***, p. 65].

Why, one wonders, does Church call a Tarski-type definition of denoting for a particular formalized language *syntactical*? It would consist of a relation between the expressions of the language and the (typically) non-syntactical entities to be denoted. This is a relation between the language and the world and would seem to count as semantical. But evidently for Church this is not enough. No essentially semantical *concept* needs to be used in the definition. The meta-language contains, let us suppose, the object language as proper part. Given adequate, but purely logical, resources, we can define the idea that such-and-such an expression denotes so-and-so object without using any semantical notions in the meta-language. And this part of the meta-language Church therefore considers to be merely the syntax language for that object language:

> The concepts expressed by 'denote' and 'have values' as thus defined belong to theoretical syntax, nothing semantical having been used in their definition. But they coincide in extension with the semantical concepts of denoting and having values, as applied to the particular formalized language [***IML***, p. 65, note 143].

Perhaps calling such a definition "syntactical" strikes one as a bit odd, since non-syntactical notions will certainly be used in defining the abstract correspondence, but the idea is clear. There are, according to Church, characteristically semantical notions, denoting, being true, expressing, and such-like, and a definition which invokes none of them is non-semantical, even if it involves some abstractly defined relation between the language and the non-linguistic.[53] He goes on in a way which is harder to grasp:

> Semantics begins when we decide the meaning of the well-formed formulas by fixing a particular interpretation of the system. The distinction between semantics and syntax is found in the different significance given to one particular interpretation and to its assignment of denotations and values to the well-formed formulas, but within the domain of formal logic, including pure syntax and pure

[53]Tarski [63] himself emphasizes that no semantical concepts are used in applying his method to particular languages.

> semantics, nothing can be said about this different significance except to postulate it as different [***IML*** p. 65, note 143].[54]

But Tarski's method threatens to collapse the distinction.

> All this suggests that, in order to maintain the distinction of semantics from syntax, 'denote' and 'have values' should be introduced as undefined terms and treated by the axiomatic method [***IML***, p. 66, note 143].[55]

The suggestion seems to be that even if one is just formulating the semantics of a particular language, truth and denotation should be taken to be primitives of the meta-language with axioms about them simply postulated. And this is quite apart from the project of formulating a general theory of the semantical concepts. Of course this latter theory presumably would contain some of the desired axioms about truth and denotation, but would not entail the Tarski biconditionals for such-and-such a language.

Church's project here is interesting and important and has never been completed. He himself would concentrate his later attention on the logic of intensional entities, taking the concept relation as primitive and leaving aside any direct consideration of possible languages and their semantics. But in view of our hesitant grasp of the concepts and principles which should govern intensions—propositions, properties, and the like—perhaps there remains a quite considerable heuristic value in an approach like that of *NAESA* which treats simultaneously of intensions and the semantics of possible languages.[56]

There are a number of details of the *NAESA* theory which might be reasonably questioned. The idea of a "sense value" is not part of the present stock of tools used in logical semantics. This is partly because the current standard semantical approaches are often purely extensional and hence even the notion of sense is not employed.[57] But it is difficult to see that Church's

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[54] Church's full discussion in §09, "Semantics", of ***IML*** is still well worth reading. I see some of the ideas there as anticipating similar thoughts by Michael Dummett in "Truth" [38].

[55] In his ***Encyclopedia Britannica*** entry, "Denotation" [26], he says that denotation seems to be a primitive notion. Physicalists will not follow Church here since it is precisely the reduction of truth and denoting to non-semantical, physical terms that they see as the main philosophical value of Tarski's work. They evidently do not want to "maintain the distinction of semantics from syntax."

See further Donald Davidson's [37] for considerations which are relevant to this disagreement. The debate continues now between "deflationists" about truth, and their opponents.

[56] It should be noted that Church thought in 1947 that "... the investigation of the notion of meaning must proceed simultaneously with the enterprise of perfecting the grammar of both object language and of meta-language ..." [XII 96(3)]. That this is not quite the same thing as the present suggestion is seen from the fact that he is discussing a single meta-language for a particular object language.

[57] The now standard semantics for modal logic is a kind of intensional semantics, but is not sufficiently general for the purposes Church envisions.

idea of assigning a sense value to a variable will be useful in connection even with intensional semantics. There is no strikingly natural way to select a class of concepts to serve as the sense-range of a variable of a particular sort. The idea of using the set of senses of appropriate kind expressed already by expressions in the meta-language threatens to lead to circularity or an infinite regress.[58] And if the domain is non-denumerable, the mentioned set will not contain concepts of all the values of the variables which we might want to consider assigning.

Logicians and mathematicians typically specify the range of a variable just by introducing a kind—people, physical objects, or real numbers.[59] One might hope to specify the set of concepts as all those somehow involving the kind in question, not restricted to concepts directly expressible in the meta-language, and take this to be the sense-range of the relevant variables. However, here one might wonder if each thing of the kind in question falls under any concept at all. Of course, it may be that everything falls under some concept or other. Whether we say yea or nay depends in part on how much stress we put on "human" in the description of the theory as an abstract theory of human communication. The question is a delicate one of the degree of idealization to incorporate into the theory, but there is considerable plausibility in the negative decision.[60] In that case, principle (v) above will not serve to properly determine the desired (denotation) range.[61]

[58] I believe that this particular obstacle to using sense values can be overcome.

[59] Intensional semantics will very likely have to make reference to the kinds used to assign the ranges of the variables in specifying the senses of quantified sentences and designators involving bound variables.

[60] Church was ambivalent about the question. In the 1946 lectures mentioned in note 52, he first postulated that everything falls under some concept, only to drop the assumption when it led to a conflict with his attempt to formalize Alternative (0) (See below, §3.3.1.) He then poses the question whether or not there might be particular real numbers which are intrinsically unnameable. In "Outline of a Revised Formulation of the Logic of Sense and Denotation" [31] he says that his intention in "A Formulation of the Logic of Sense and Denotation" was to make it "an optional axiom schema" that there be no conceptless things. But it was already a consequence there and remained so in the reformulated version of Alternative (2). In his "A Revised Formulation of the Logic of Sense and Denotation, Alternative (1)" [34] he again states the principle as desirable—indeed, as an axiom. (Actually it follows, as before, from the other axioms.)

[61] Even waiving this objection, there is reason to fear that there will be no such set. Suppose that α is a concept of an object of the kind in question and imagine a name 'a' expressing that concept in some possible language. Now let π be any true proposition and 'S' be a sentence expressing π, again in some possible language. Then, one supposes, there will be an expression (in some possible language) also picking out the entity in question, but involving 'S'. For example, there is the possibility of a language containing both of these expressions, and some device for forming (Fregean) definite descriptions therein. Such a language would contain some such thing as 'The x such that $x = a \mathbin{\&} S$.' Now on a strict criterion of identity of concepts, if the sentences 'S' and 'S^*' are not synonymous (in their respective languages), then distinct concepts will correspond to the (possible) descriptions 'The x such that $x = a \mathbin{\&} S$' and 'The x such that $x = a \mathbin{\&} S^*$.' So there will be "as many" concepts

The decision to make it part of the essence of a language that it obey Frege's principles will be decried by those who hold that in actual languages there are names with denotation but no sense. Actually the theory as formulated so far would allow the possibility that sense and denotation might in some cases coincide, although Church would certainly have rejected the idea. But further, it might be desirable to loosen up the constraints on possible languages, introducing Fregean languages by definition, so as to allow the comparison within the theory of Fregean and non-Fregean "languages."

Church was fully aware of the main obstacle facing the theory. There is the danger that it will, when formalized, become inconsistent, containing as it does such ideas as "the value of x for assignment y is z (in possible language w)". If these variables are unrestricted, then paradox threatens. Church would presumably adopt some form of the theory of types, but such considerably limits the generality of the theory.

As already mentioned, Church later concentrated on the purely "ontological" aspects of the matter—the logic (or better: theory) of intensional entities.

§3. Intensional logic.

3.1. Logic of intensional entities: Preliminary discussion. Church's methodology demands that the theory of propositions and other intensions be rigorously formulated. Preliminary philosophical discussion and argumentation are fine as long as they are directed toward that goal. His initial and informal discussions of these matters appear primarily in reviews.

Church was aware of Frege's views about sense and denotation early on and mentions the ideas most conspicuously in reviews of articles by A. I. Melden [V 162, 1940] and W. V. Quine [V 163, 1940]. In reviewing Melden, he expresses full sympathy with the suggestion that intensional meanings are to be postulated to analyze such things as "I am hunting a unicorn" and "He deserves an everlasting monument."

The review of Quine's "Notes on Existence and Necessity" [VIII 45, 1943] is a classic contribution to the foundations of intensional logic.[62] Quine, apparently unfamiliar with Frege's ideas on indirect contexts, had argued that one must distinguish "purely designative" and "not-purely designative"

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of the thing in question as there are true propositions altogether. By the reasoning of the Russell-Myhill Paradox (to be discussed below in §3.3.1), we might be led to conclude that there is no such thing as the "totality" of all true propositions. It will then follow that there is no set of all the sense-values (concepts) corresponding to a single element of the domain.

One might hope to well-order the propositions and to select a set of concepts for the elements of the domain, but I doubt that this is possible.

[62]The review VII 100(2), 1942, already had begun to indicate the lines Church would prefer to use in constructing a logic of modality and other intensionalities. In particular, it contains the idea that modal operators should be prefixed to names of propositions, and not to sentences.

occurrences of expressions. The latter are the same as Frege's indirect occurrences of expressions. But Quine comes to the conclusion that variables in such contexts cannot refer back to a quantifier outside the context. Though Church is sympathetic to much of the argumentation in Quine's article, he questions this particular conclusion. Rather, he says, the conclusion should be that such a variable must have an intensional range.[63]

Church goes on to give an interesting example of a valid inference involving such a context using the possibility operator '$\Diamond$'. Adopt the following class names: 'b' for biped, 'f' for featherless, 'm' for men. From '$fb = m\,.\,\Diamond fb \neq m$' it follows, says Church: '$(\exists\phi)\,.\,(x)(\phi x \equiv x \in m)\,.\,\Diamond\sim(x)(\phi x \equiv x \in m)$', where '$\phi$' is a variable ranging over attributes. Such inferences, he says, should be retained if modal operators and quantifiers are both present. But Church is careful to explain that the notation here is "inaccurate". The premiss and conclusion of the inference are stated in the style of the usual formulation of modal logic, which he regards as misleading. And the latter paragraphs of the review implicitly carry the suggestion of a Frege formalization of statements of modality.

The remark is elaborated in "Postscript 1968" [29], where he explains that the preferred formalization of such a thing as that it is possible that featherless bipeds are exactly the men is not '$\Diamond fb = m$', but rather something like '$\Diamond \digamma i\beta \circ \mu$'. Here '$\digamma$' denotes the sense of '$f$', '$i$' denotes the concept of class intersection, 'β' denotes the sense of 'b', '$\circ$' denotes the sense of the identity sign '$=$', and 'μ' denotes the sense of 'm'. Then the expression '$\digamma i\beta \circ \mu$' is a name of a proposition, the proposition expressed by '$fb = m$', and the modal operator is to be seen as a predicate thereof.

If we adopt a different formalization and some definitions, to which Church need not object in principle, we can come quite close to the form of the argument as originally formulated. The result accords with the basic ideas of the logic of sense and denotation which was to be Church's rigorous formulation of intensional logic. The exercise of thus formalizing the argument may serve to highlight respects in which Church found the standard formalization misleading.

Let the (standard) name of the sense of an expression be formed by writing the expression in boldface. We may call this the 'first ascendant' of the original expression. Thus, under our convention the first ascendant of '$(x)(x \in fb \equiv x \in m)$' is '$\boldsymbol{(x)(x \in fb \equiv x \in m)}$', and this latter denotes the proposition which is expressed by the former. The inference then might

[63] Church nowhere in print expresses sympathy with the idea that there might be such a thing of a "de re" reading of a belief sentence or other sentence apparently involving obliquities. However, in lectures on open problems in intensional logic at U.C.L.A. in 1977 [recorded by Nathan Salmon] he says the idea is "tenable" but leads to some surprises. These might include the one discussed in Church's "A Remark Concerning Quine's Paradox about Modality" [32].

be analyzed like this. The initial premiss:

$$fb = m \,\centerdot\, \Diamond{\sim}(fb = m)$$

is rewritten as:

$$(1) \qquad fb = m \,\centerdot\, \Diamond{\sim}(\boldsymbol{fb = m}).$$

Here the possibility sign '$\Diamond$' is to be read as a predicate of the proposition, rather than as a sentential connective. From this it follows that

$$(2) \qquad (x)(x \in fb \equiv x \in m) \,\centerdot\, \Diamond{\sim}(\boldsymbol{x})(\boldsymbol{x \in fb \equiv x \in m}).$$

To see this, observe that the two propositions denoted by boldface expressions are necessarily equivalent and the equivalence of their possibilities will be (presumably) provable in the Frege-Church logic including modality. Let '$\triangle$' stand for the concept relation and adopt the abbreviative definition:

$$\boldsymbol{\phi}(x) \quad \text{for} \quad \exists c\, [\triangle(\boldsymbol{\phi}, c) \,\centerdot\, x \in c].$$

That is, "x has the property $\boldsymbol{\phi}$" is to mean the same as "There exists a class c, such that $\boldsymbol{\phi}$ is a concept of c and x belongs to c." Abbreviate also:

$$\phi(x) \quad \text{for} \quad x \in \phi.$$

We may assume that the property of being a featherless biped is a concept of the class of featherless bipeds.[64]

$$(3) \qquad \triangle(\boldsymbol{fb}, fb).$$

Now it will be a fundamental principle of the Fregean intensional logic that a concept is a concept of at most one entity (cf. Principle (i), p. 149 above). Hence, in particular, a concept of a class is a concept of that class only:

$$(4) \qquad \triangle(\boldsymbol{\phi}, c) \supset \centerdot \triangle(\boldsymbol{\phi}, c') \supset c = c'.$$

It follows from (2), (3), and (4) that:

$$(5) \qquad (x)(\boldsymbol{fb}(x) \equiv x \in m) \,\centerdot\, \Diamond{\sim}(\boldsymbol{x})(\boldsymbol{fb}(\boldsymbol{x}) \equiv \boldsymbol{x \in m}).$$

Therefore,

$$(6) \qquad (\exists\boldsymbol{\phi})\colon (x)(\boldsymbol{\phi}(x) \equiv x \in m) \,\centerdot\, \Diamond \sim (\boldsymbol{x})(\boldsymbol{\phi}(\boldsymbol{x}) \equiv \boldsymbol{x \in m}).$$

So the argument from (1) to this, using (3) and (4), is valid.

Here are the first beginnings of Church's intensional logic. He thought that certain additional questions bearing on that project deserve detailed philosophical discussion.

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[64]Similar things will be theorems in the Logic of Sense and Denotation.

3.2. Constraints on the logic of intensional entities: Criteria of identity (especially for propositions). One preliminary question which Church took quite seriously is "What is the criterion of identity for propositions?" That is, under what circumstances do two sentences express the same proposition (and analogously for other complex meaningful expressions). Of course one answer is "When they are synonymous", but because of the vagueness of the ordinary notion of synonymy, this is not very helpful. The whole question of what exactly a criterion of identity for a kind of thing is supposed to be, and why putative kinds without such criteria are suspect, is vexed. But to the extent that the problem is just that of making the theory of propositions more definite and the suggestion is that some heuristic principle relating sentences and propositional identity will help, then it is unobjectionable and probably even correct.

One odd, but possible, answer is: two sentences express the same proposition when they are both true or when they are both false. Church seems to have thought of the two truth-values as somehow resulting from this kind of principle, perhaps by abstraction. In "A Formulation of the Simple Theory of Types" [11], he first describes the theory as involving propositions and then contemplates adding an axiom of extensionality, namely that materially equivalent propositions are identical. Hence the usual, extensional, propositional calculus results. His considered view seems to have been that adopting this criterion of identity just amounts to postulating the two truth-values—and other criteria, better for some other purposes, will result in other kinds of propositions. But of course, then one shouldn't call all these various different kinds of things "propositions".[65]

Another possible criterion is that the propositions expressed by sentences A and B are identical when the material equivalence between A and B, $A \equiv B$, is not merely true, but true necessarily, or "on logical grounds alone."[66] This was adopted by Carnap[67] and was one of the criteria which Church developed in detail in his formal work.

The first criterion is of course adequate for extensional logic and the second is natural if one is dealing only with modality. Neither is adequate for propositions traditionally so-called and, presumably, as needed for the development of the logic of belief, assertion, and knowledge. To show the inadequacy of necessary equivalence for this purpose Church observed that it would make nonsense of the notion of mathematical proof, a process which is supposed to enforce belief. If B is a logical consequence of A, then AB is necessarily equivalent to A. Hence, on the given criterion, if A logically

[65]That Church held the various different criteria of identity correspond to different kinds of "propositions" is clear from the lectures mentioned in note 63.

[66]This was Church's preferred terminology as it was used in the 1977 lectures mentioned in note 63.

[67]For modality, in ***Meaning and Necessity*** [6].

implies B, then if you believe A, then you believe AB. And, if one can do conjunction elimination, it follows that you believe that B. And this is so even if B is regarded as by mathematicians as being an open problem.[68]

An important step forward in the project was Carnap's proposal of "intensional isomorphism" as criterion of identity, a notion which we have already briefly explained above. In "A Formulation of the Logic of Sense and Denotation" [19], hereafter "*FLSD*" Church even says that the criterion of identity which he calls "Alternative (0)" makes the notion of sense correspond to Carnap's idea—with the one difference that the intensional ideas are to be treated directly in the object language. But in *IIB* he criticizes Carnap's criterion as still too weak, proposes a modification ("synonymous isomorphism"), and explains that Alternative (0) should correspond to this instead. And to this criterion he explains that he "attaches the greatest importance … because it would seem that in this direction … that a satisfactory analysis is to be sought of statements regarding assertion and belief." [*FLSD*, p. 7, note 7].

The critique of Carnap is swift and decisive. It would be possible for a formalized language to have two primitive predicates 'P' and 'Q' which are necessarily equivalent because of the meanings assigned in the metalanguage. And it might be that knowledge of this equivalence would require the solution of a difficult problem. Church lets 'Pn' mean that the positive integer n is less than three and 'Qn' mean that there are (positive integers) x, y, and z such that $x^n + y^n = z^n$. Then it might even be that it can be proved that $(n)[Pn \equiv Qn]$ (as indeed it has been since) and so the equivalence is necessary. But surely someone could believe $(\exists n)[Qn \,.\, {\sim}Pn]$ and not believe that $(\exists n)[Pn \,.\, {\sim}Pn]$, at least before having knowledge of the proof of the equivalence. Yet the two have the same structure and corresponding parts are necessarily equivalent.

Church proposed to modify Carnap's criterion by requiring that the corresponding simple parts be *synonymous*. One might complain again that this idea is quite vague. But Church supposes that it will be given as part of the semantical basis of the language which primitive expressions are stipulated to be synonymous with each other and with complex expressions present in the language.[69] Here is a kind of recursive clarification of the notion of synonymy as it obtains between complex expressions. The complex expressions must have the same form, modulo replacement of (given) synonyms.[70]

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[68]Church makes this argument in detail in the lectures of note 63. His example for B was Fermat's Last Theorem, with A being the conjunction of Peano's Postulates (second-order), and assuming that B follows from A.

[69]Here we have returned to the notion of an *intensional semantical rule* which we discussed in §1.1.

[70]I have been ignoring a slight complication—alphabetic change of bound variables is regarded as preserving synonymy, Church apparently seeing this as implicit in the practice of logicians and mathematicians. The matter is discussed in the next section.

So there are initially two criteria of identity in need of development, logical equivalence and synonymous isomorphism. The former yields a notion of a proposition suitable for modal logic and the latter is hoped to do the same for the logics of knowledge and belief and of similar notions.

3.3. The logic of intensional entities: Details in logistic systems.

3.3.1. *Fregean logic of sense and denotation*: *The preferred theory*. The logic of sense and denotation was Church's attempt to incorporate these ideas into a rigorous formal logic. The basic logic is a typed[71] lambda calculus[72] and is constructed along the lines of "A Formulation of the Simple Theory of Types". The types themselves are domains consisting of the sorts of entities to be treated by the theory. There is to be selected a well-defined class ι of *individuals*. This may be any definite collection of entities whatever. The type o (omicron) is the class consisting of the two truth-values, that is, it is $\{\mathbf{t}, \mathbf{f}\}$. For any domains, types, α and β, there is a type, designated $(\alpha\beta)$, consisting of the set ${}^{\beta}\alpha$ of all functions defined everywhere in type β and taking values in type α. From the point of view of set theory, which we may pretend is included in our meta-language, this is just the collection of all sets of ordered pairs $\langle b, a\rangle$, such that $b \in \beta$ and $a \in \alpha$ and satisfying the conditions that it contain such a pair for every $b \in \beta$ and that no two such pairs have the same first term.[73]

To obtain the intensional types, the most direct clause to add to these two is perhaps: if α is a type, then there is a type α_1 consisting of the *concepts* of entities of type α. Of course, we really don't quite know what these are—it is part of the purpose of the theory itself to clarify the nature of these entities. Intuitively, the members of the type α_1 are the entities suitable to be the senses of names (in some possible languages) denoting entities of type α. The type o_1 is therefore the type of concepts of truth-values. Among these are the propositions, senses of sentences. The type ι_1 is the type of *individual concepts*, things suitable to be the senses of names of individuals. What these will be of course depends upon the domain of individuals selected and in that respect is unlike the type of concepts of truth-values, which is the same for every one of the intended interpretations.[74] The type ι_{11}, abbreviated as

[71]Church seems to have turned to type theory as the most reasonable approach to the logical paradoxes after the failure of his own ambitious attempt to formulate a general foundation for logic and mathematics [9, 10].

[72]See ***The Calculi of Lambda Conversion*** [12]. Some of the basic ideas of this have been important in Richard Montague's work on intensional logic and in natural language semantics.

[73]Functions of more than one variable are treated as singulary functions of functions in the manner of Schönfinkel. We suppress discussion of this feature in our exposition. It does not go without saying that this idea can be used if the Logic of Sense and Denotation is required to be able to reproduce intensional reasoning from ordinary language.

[74]This assumes that we have fixed one or the other of the alternatives as criterion of identity for propositions. The domain of individuals may still be varied without producing a

ι_2, is the type of concepts of concepts of individuals, and so on.

In the most straightforward development, $(\alpha\beta)_1$ would be the type of concepts of functions from objects of type β to objects of type α. But here Church saw the possibility of a reduction in the array of types. Given the types α_1 and β_1, there will be the type $(\alpha_1\beta_1)$, consisting of functions from type β_1 to type α_1. Now consider hypothetically two names '*F*' and '*B*' naming entities of types $(\alpha\beta)$ and β respectively, and having senses μ and ν respectively. In a certain (possible) language, the name '*FB*' will be well-formed and will denote the result of applying the function named by '*F*' to the argument named by '*B*'. This will be an entity of type α and the sense of the name '*FB*' will be a concept of that element of α. Thus corresponding to any concept μ of a function which would be in the type $(\alpha\beta)_1$, there will be a function from concepts to concepts, that is, there will be a function which is a member of the type $(\alpha_1\beta_1)$. And we may pretend that this latter function just *is* the function-concept μ in question. This effects not only a reduction (in some sense, though not in actual numbers) of the system of types but constitutes an analysis of some kinds of concepts in terms of others. A concept of a function is itself just a function of a certain kind—from certain intensional concepts of arguments to certain concepts of values. But Church went farther. Suppose a function ϕ in type $(\alpha_1\beta_1)$ has as value, for every concept of an argument of a certain function f, a concept of the value of f for that argument. Then we may say that the function ϕ *characterizes* the function f. Why not say that every such function ϕ is a concept of f?

As beautiful as it is, this may have been a misstep. One should notice that the proposal has the result of postulating an enormous number of concepts. Suppose, for example, that the domain is the natural numbers, a collection of entities each one of which is certainly concepted (fails under a concept) in virtue of the fact that the Arabic numerals all have definite senses. Then every function from natural numbers to natural numbers will be concepted and so there will be concepts of all (the representatives in this domain of) the real numbers. This *might* be correct but then again, are we really sure that there are concepts of all the real numbers?[75] The question may be partly one of decision. But given that we want our concepts to work like senses of names and to serve to underpin the intensional logic of such things as knowledge and belief, it is really a quite substantial assumption, and not at all evident, that every real number is concepted. It is even already a consequence of these ideas that everything whatever is concepted.

Be that as it may, the basic system of the logic of sense and denotation proceeds upon that assumption and it is incorporated into the basic axioms by formalizing the following two principles:

(C1) If ϕ is a concept of the function f, then for every concept α

non-principal interpretation.

[75]Compare notes 60 and 61.

and every entity x, if α is a concept of x, then $\phi\alpha$ is a concept of fx.

(C2) If for every concept α and every entity x, whenever α is a concept of x, then $\phi\alpha$ is a concept of fx, then ϕ is a concept of the function f.

Our statement of these principles assumes that everything is of the right type and (therefore) that x belongs to the range of f, that α belongs to the range of ϕ, and that ϕ and α are of the intensional types corresponding to f and x, respectively.

In certain respects principle (C1) is more fundamental than principle (C2). Even if one finds reason to reject Church's reduction of concepts of functions to functions on concepts, some analogue of (C1) will be wanted. Presumably there is some mode of combination whereby a function-concept ϕ together with an entity-concept α (of appropriate type) combine to form a complex intension, which we might write as $\phi\alpha$. But this latter would not necessarily be construed as an application of function to argument. And the corresponding version of (C1) will still be fundamental to the Frege-Church approach to intensional logic.

Principle (C2), on the other hand, might be seen as a kind of criterion of identity for function concepts—namely, that they are identical if they have the same value for every argument—and, as such, is not really neutral between alternative choices of such principles. It turns out that principle (C2) is really quite at odds with the heuristic ideas of Alternative (0).

There is really only one more basic assumption which is characteristic of the intensional part of the logic, independently of further details about the criteria of identity for complex concepts. Let A be some closed expression of the language. For every such expression there is required to be present in the language an expression **A** which denotes the sense of A. And thus we want, as axiom or theorem:

(C3) **A** is a concept of A.[76]

In Church's formalization, the expression **A** may be obtained from the expression A by increasing the subscript number of every subscript by one. The result, **A**, is said to be "the first ascendant"[77] of the expression A. This formal device is designed to eliminate the ambiguity which, according to Frege, results from the presence of oblique contexts in ordinary language. So, for example, instead of using the very same expression A within a belief context, where it will have a different sense and a different denotation, the formalized language replaces A by its first ascendant **A**. If such contexts are iterated in the natural language, then within such, **A** would have to be

[76] Natural generalizations of this involving variables are also wanted as theorems, but we confine attention to the simpler case.

[77] This is the official version of the terminology we have already used in §3.1.

replaced by its first ascendant, or the first ascendant of its first ascendant, and so on, depending on the number of iterations of the obliquity-producing contexts.

Notice that such things as (C3) will not be appropriate, as theorems of logic, for just any expression which denotes the sense of the expression A. Rather, **A** is here a kind of "standard name" of the sense of the expression A—an expression which denotes the sense of A as a matter of logical necessity. In English, if A is, say, the sentence "Snow is white", then the analogue of **A** would be "That snow is white", a standard name of the proposition expressed by the given sentence. But senses, like anything else, can be denoted by expressions which pick them out only contingently—definite descriptions of appropriate kind, for example.

Given principle (C1) and the basic logic, it suffices to postulate (C3) for expressions A which are primitive constants. In applications of the logic of sense and denotation involving additional primitive constants, the corresponding instances of (C3) will have to be added to the logic. This need not be seen as a defect, although it might generally be regarded as a deficiency of a logic that logical axioms have to be added for a particular application. Here one should just extend, once and for all, the basic language to contain infinitely many primitive constants of every type, together with the corresponding axioms (C3). Then an application is entirely a matter of supplying an interpretation of the logical system, adopting as many of these axioms as are well-formed in the applied logic.

It remains to supply particular criteria to be used in fixing the intensional entities more precisely. The three contenders are: (A_0) Alternative (0) (already mentioned), that the senses expressed by distinct expressions are identical if and only if the expressions are synonymously isomorphic, (A_1) Alternative (1) (to be explained in more detail below), that the senses expressed by distinct expressions are identical if and only if the expressions are "lambda-convertible" to one another, and (A_2) Alternative (2), that logically equivalent expressions have the same sense. In *FLSD*, Church mainly attempts to formalize Alternative (2). There are some false starts, and the matter is taken up again in the second paper, "Outline of a Revised Formulation of the Logic of Sense and Denotation (Part I)" [30], where models are supplied using the "possible worlds" semantics of standard modal logic as guiding idea. As far as is presently known, this formalization successfully captures the ideas of Alternative (2).

Alternative (1), as a criterion of identity, was really a kind of accident. In an abstract of the same name as *FLSD* [17], Church explains that his two heuristic principles for criteria of identity are: (A) the senses of distinct expressions are to be distinct in so far as this is compatible with the other assumptions of the logic, and (B) that logically equivalent expressions have the same sense. Carnap is not mentioned in connection with (A) and hence it is to be assumed that this is the prototype of the criterion of synonymous isomorphism, independently formulated. Church takes this to be the criterion intended by Frege.

Two expressions are lambda-convertible to one another if one can be obtained from the other by a sequence of applications of the rules of lambda-abstraction, lambda-contraction, and alphabetic change of bound variables.[78] Now because of the use of the (typed) lambda calculus in formulating the underlying logic, it follows automatically that closed expressions which are lambda-convertible to one another are synonymous. More precisely, it is always possible to prove the identity between the first ascendants of such expressions—hence the mentioned synonymy follows. In a note added to the abstract in 1948, Church writes that this result does not accord well with the tendency of principle (A). He therefore, in *FLSD*, distinguishes Alternative (1), which is the criterion of identity embodied in the false start, from Alternative (0), the heuristic principle intended in (A). Church seems to have thought that Alternative (1) may still be of independent interest, in spite of its unseemly origin. In particular, it is a kind of intermediate principle, stricter than Alternative (2), and as such the study of it may throw light on the more important (and difficult) Alternative (0).

Church's decision here shows that principle (A) does not really fully state the underlying guiding intuition by which he is proceeding. Observe that it is really not quite true that the described consequence of Alternative (1) is contrary to principle (A) as stated. The case in question is in fact one where it is already a consequence of the underlying logic that such lambda-convertible expressions have the same sense.[79] There must therefore be some deeper intuition behind Church's observation. In fact, in later reformulations, it is always accepted that alphabetic change of bound variables preserves sense and this is not thought contrary to the original intention. So some synonymies which are consequences of the other assumptions are accepted and some, those resulting from lambda-abstraction and contraction, are rejected.

Perhaps a satisfactory motivation for the ideas underlying Alternative (0) is the following. Sense is what is known when the language is understood. In accordance with this, the intensional semantical rules should state essential facts about the semantics, the mastery of which constitutes (ideal) competence with the language. These may include the rules of synonymy briefly described in *NAESA*. Now, for a formalized language, the clearest case of such a rule is one corresponding to a formal definition—this is a stipulated

[78]For terminology and details of the lambda calculus, see ***The Calculi of Lambda Conversion*** [12]. For the typed lambda calculus as it is used in the present context, consult "A Formulation of the Simple Theory of Types" [11].

[79]Of course this consequence is very closely tied to Church's treatment of concepts of functions as functions on concepts, something which is not itself really demanded by the underlying intuitive ideas.

synonymy and, as Quine has observed [56, pp. 25–26], is a very clear case of the phenomenon of synonymy. Of course there are various constraints on such stipulations, as Church notes in *NAESA*. One may not stipulate (or define) a primitive constant as being synonymous with two distinct expressions, or set two complex expressions synonymous, unless their synonymy is already a consequence of other definitions, and one may not invoke circular definitions or stipulations. And there are other conditions on the acceptability of definitions which will govern the permissibility of the corresponding stipulation.[80]

So given such primitive synonymies, or stipulations, what other expressions must be synonymous? Well, it is required that interchange of synonyms, in an unambiguous language, preserves synonymy. Alphabetic change of bound variables is not quite a case of this. While it is true that variables of the same type are by the practice of logicians treated as fully synonymous, the kind of meaning in question is not the same as that of a sense which determines a denotation. A variable will have a meaning, typically given implicitly by the semantical rules specifying the range of the variable, and this is indeed the same for all variables of that type. But the variables do not thereby denote that range in the manner of a name of a set. And distinct variables of a given type are not fully interchangeable since distinctness and identity of variables may have to be preserved. Still, one may see here something very close to the principle that interchange of synonyms preserves (in an unambiguous language) synonymy.

The idea of Alternative (0) is then really this. Stipulated synonyms, subject to various conditions, are given as synonymous by the very construction of the language. And interchanging synonyms, together with the closely analogous process of alphabetic change of bound variables, preserves synonymy. Now one adds a closure condition: no two expressions are synonymous unless their being so follows from these principles. Axioms governing the relations between complex concepts and their constituents (in a sense which itself is to be fixed by the theory) are to conform to, and have as consequences, these conditions on synonymy within this or that particular

[80]The details of the criteria of adequacy for definitions which have been proposed are well worth studying for the light they throw on the general concept of synonymy. There is some discussion of various formal constraints in Rudolf Carnap's ***Logical Syntax of Language*** [5]. Such studies have not been popular because the much favored course among logicians is to relegate definitions to the meta-language and to regard them as purely abbreviative. And, given their usual purposes, this is quite reasonable.

Church points out [***IML***, p. 76, note 168] that Lesniewski's observations about definitions in the object language show the need for precise rules of definition as part of the primitive basis of such a language. In the present connection such languages and rules of definition are of particular interest. Observe, for example, that Tarski's conception of definition [64] will not do for the present purpose, allowing as it does the "identification" by definition of complex expressions which may well have different meanings as a consequence of previous assignments of meanings.

(possible) language.[81]

Natural languages do not often contain actual formal definitions or stipulations, except in the case of their technical extensions, for example, mathematical English. But one may see such things as the synonymy of "procrastinates" and "puts things off" as the result of an "ideal" stipulation, implicit in the practice of using the language. Which of these things is to be incorporated into the very definition of the language, as required knowledge for mastery of the language, will of course be vague in some cases and subject to decision as the immediate purpose requires.

Alternative (0) has not been successfully formulated and developed in Church's work. The axioms for Alternative (1) in the *FLSD* were worked out in the hope that they would point the way to Alternative (0), but they already lead to contradiction. The contradiction, which is best perhaps called the "Russell-Myhill Antinomy", proceeds along the lines of the reasoning which Bertrand Russell discusses in Appendix B, section 500, of ***Principles of Mathematics***, and was independently rediscovered by John Myhill [52]. This is not directly related to the paradox which commonly bears Russell's name, but involves the alleged possibility of there being a totality of all propositions.

Briefly, and intuitively, the argument is this. Suppose that there is a set consisting of all propositions and let F and G be any two sets of propositions. If the sets F and G are concepted, say by ϕ and ψ, respectively, then we may consider the universally quantified propositions, about propositions, $(p)\phi(p)$ and $(p)\psi(p)$. Now if these are identical, then the principle of Alternative (0) demands that $\phi = \psi$. But if these concepts are identical, then so are the sets F and G which fall under them. Observe that it follows that there is, in effect, a one-one correspondence between sets of propositions and propositions—F corresponding to $(p)\phi(p)$, G to $(p)\psi(p)$, and so on. This contradicts Cantor's Theorem that there are more subsets of a given set than elements of the set.

The reasoning goes through in the system for Alternative (1) as formulated and a variant holds for Church's attempted formulation of Alternative (0) in 1974. In particular, the assumption, implicit in the argument above, that every set of propositions is concepted is a consequence of the treatment of concepts of functions as functions on concepts. Indeed, as already observed, it is a theorem that everything is concepted (in the appropriate type). And the totality of propositions is itself taken as one of the primitive types.

Church's strategy for dealing with this, and with other antinomies and paradoxes which threaten, is to adopt some version of Russell's Ramified

[81] I would argue that other synonymy producing stipulations are natural and desirable—for example, for expressions with appropriate extensions, it may be simply stipulated that the order of expressions is to have no significance. This leads to a criterion which is less strict than Alternative (0) but is still in harmony with the basic ideas thereof.

Theory of Types, adapted of course to the Fregean setting of the logic of sense and denotation. Instead of dividing the propositions into orders, he attempts to ramify only the concept relations $\triangle$. These are already of various types, but are further to be assigned orders, indicated by superscripts. This idea is developed for Alternative (0) in [31], but, as already noted, the formulation there provided does not evade the Russell-Myhill Antinomy.[82]

In a footnote in *FLSD*, Church says that for Alternative (0) the axioms which incorporate the treatment of concepts of functions as functions on concepts (namely axioms $16^{\alpha\beta}$, corresponding to our (C2) above) should simply be dropped. But in [31] and [34] (treating mainly Alternative (1)), he attempts to retain these ideas, with some modifications, even for Alternative (0). It seems plausible to see this as largely responsible for many of the difficulties with this alternative. It is strongly suggested that the reduction be abandoned and, along with it, the assumption that there is such a thing as the type, or set, of all propositions. This course entails abandoning type theory and proceeding by analogy along the path set by Zermelo-Fraenkel set theory.[83] The cost will be the need for a primitive indicating "composition" for complex concepts, but one benefit will be the disappearance of the complex array of types and levels of ramification.

We have concentrated on the attempted development of Alternative (0) because Church believed, rightly, we maintain, that here there is still promise for an adequate underlying logic for the propositional attitudes. Alternative (2), as explained, encountered some difficulties[84] of its own in the early stages of the project, but these seem to have been fully overcome. This alternative has received some attention in the literature.[85] Church returned to Alternative (1) in 1993 and offered a consistency proof. We reserve detailed evaluation of this last for a later date.[86]

[82] See Anderson [1] for details.

[83] Church mentions this as a possible course in the 1977 lectures (cf. note 63), but comments that the Fregean ideas do not fit well with set theory. In particular, he anticipates difficulty with the ideas that all constants have a sense (including, one supposes, '$\in$') and with the assumption that everything falls under a concept. But we suggest that one might abandon the idea that '$\in$' has a denotation, together with the principle that if a part of a complex expressions lacks a denotation, then so must the whole. And we have already indicated above possible grounds for being suspicious of the principle that everything is concepted.

[84] See Terence Parson's "The Logic of Sense and Denotation: Extensions and Applications" [54] for details of the difficulties and for a very promising idea for retaining Church's construction of concepts of functions. [*Added in proof*: But see the "Afterword" thereto.] The difficulties associated with Alternative (0) are not faced there and we believe that the course here advocated will still be found preferable for that alternative.

[85] David Kaplan [44, 45], has given a perspicuous formulation of a version of Church's Alternative (2) together with a "possible worlds" semantics. This work is extended and, in some respects, completed by Charles Parsons [53].

[86] Several people have claimed to find technical difficulties in this paper. In addition, at the present writing, we have some doubts about the proposal there of a way to deal with Alternative (0) by introducing a defined lambda operator. But the paper deserves a separate

3.3.2. *Russellian intensional logic: A viable alternative.* Church seems to have held throughout that Russell's approach to intensional logic is viable and made important contributions to the project of formalizing that logic. In ***Principles of Mathematics*** Russell is realistic about propositions and evidently intends a criterion of identity something like Alternative (0). By the time ***Principia Mathematica*** ["***PM***"] was being written, however, Russell had come to doubt the existence of propositions, apparently on the ground that false propositions are too strange to be admitted into a sensible ontology. There are places in ***PM*** where bound propositional variables are actually used and the most natural interpretation of the system has sentences denoting propositions, with propositional functions being functions from entities of various kinds to propositions. Church pointed out[87] that there is promised a contextual definition which is supposed to do away with the need for propositions, but this is never supplied. So in his careful reconstruction of the logic, Church simply assumes the originally intended interpretation. Now, Church alleged, complex expressions are never allowed in ***PM*** to stand in the argument place of a function. How, then, can one express, even if constants are added, such things as that Jones believes that all men are mortal? The solution, according to Church, is to add a connective expressing propositional identity. Identities between propositional variables can already be written: '$p = q$', for example, just abbreviates '$(F)(Fp \supset Fq)$', but identities between complex sentences (which denote propositions) are not well-formed. Add, then, a connective '$\equiv$', "four-line equality", with the syntax of a connective. Then, with a minimum of distortion to the conventions of the original system, we can formalize our belief sentence as $\exists p(p \equiv (x)(Hx \supset Mx) \mathbin{.} B(j, p))$—roughly, "There is a proposition which is identical with all men being mortal and Jones believes it)". Church also contemplated, in the 1977 lectures, altering the syntax so as to allow complex formulas of the type of sentences to occur in the argument place of a function, and conjectured that one could obtain in that way an equivalent formulation. But his view seems to have been that adding the four-line equality does less violence to the original structure and intent of ***PM***.[88]

Church then adds appropriate axioms for the four-line equality and describes a semantics for the system. One selects a non-empty set of "individuals", together with two sets, non-empty and mutually exclusive, $\mathcal{T}$ and $\mathcal{F}$ ("true propositions" and "false propositions"). The union of these last two

and detailed evaluation.

[87] In lectures on sense and denotation, recorded by the present author in 1973.

[88] In the lectures mentioned in note 87, Church expressed some concern that allowing complex formulas in the argument places of functions might re-introduce the sorts of problems which the theory of descriptions was designed to solve—what Church, following Carnap, called the Paradox of the Name Relation. This is the now well-known problem of the failure of substitutivity of identity in intensional contexts.

is taken to be the range of the propositional variables. The semantical rules corresponding to the connectives negation and disjunction require that the negation of a proposition in $\mathcal{T}$ shall be in $\mathcal{F}$, the negation of a proposition in $\mathcal{F}$ shall be in $\mathcal{T}$, that the disjunction of propositions both in $\mathcal{F}$ shall be in $\mathcal{F}$, and so on, in the natural way.[89] The rule for the quantifiers is a bit more complicated and for the four-line equality it is required that the very same proposition be denoted by the two sides (or, in the case of variables, that the values be the same).[90]

One can add axioms corresponding to the criterion of identity of Alternative (0) but there immediately results the Russell-Myhill Antinomy—as Russell in essence knew at the time of ***Principles of Mathematics***. Church gives a careful formulation of the argument [33], closely following Russell's reasoning. But the reasoning of the paradox fails for the ramified type-theoretical version of the logic.[91]

Thus the two plausible candidates[92] for an intensional logic adequate to formalize reasoning about assertion and belief are, according to Church, the Fregean logic of sense and denotation, with a ramified theory of types, and the Russellian intensional logic with the four-line equality, again with ramified types to fend off the Russell-Myhill antinomy about propositions. Propositions in both cases are to satisfy principles corresponding to Alternative (0).

§4. **Conclusion.** There are many other philosophical contributions to be found in Church's reviews and other writings.[93] But Church's elaboration of a methodology involving the logistic method, his philosophical criticisms of nominalism and his defense of realism, his argumentation leading to conclusions about the theory of meaning, and the detailed construction of the Fregean and Russellian intensional logics, are more than sufficient to place him high up among the most important philosophers of this century. Indeed, those works provide a persuasive case for the suggestion that Alonzo Church be taken to be the denotation of the definite description, "The most rational

[89] Here is the first formulation of the semantical ideas which have been adopted by some recent intensional logicians under the name of "algebraic semantics for intensional logic".

[90] We have simplified greatly in giving our crude description of the semantics. For example, one must consider the cases where free variables are present and the formula takes a value rather than just denotes.

[91] There are some other problems with the Russellian theory, as explained in my [2].

[92] In "Intensionality and the Paradox of the Name Relation" [35] Church develops an idea due to Kazimierz Ajdukiewicz, which involves constructing "proposition-surrogates" out of extensional materials. As an approach to intensional logic the theory is subject to drawbacks which Church there explains. The construction merits further study.

[93] For example, the anticipation of the basic features of Arthur Smullyan's Russellian reply to Quine on modality [VII 100(2), 1942] and powerful criticisms of Russell's Theory of Descriptions [*NAESA*, note 14] (insofar as it is supposed to deal generally with names and descriptions within intensional contexts), to mention only two.

man since St. Thomas Aquinas"—a designation which Church himself is said to have reserved for Kurt Gödel.[94]

REFERENCES

[1] C. Anthony Anderson, *Some new axioms for the logic of sense and denotation: Alternative* (0), ***Noûs***, vol. 14 (1980), pp. 217–234.

[2] ———, *Some difficulties concerning Russellian intensional logic*, ***Noûs***, vol. 20 (1986), pp. 35–46.

[3] C. Anthony Anderson and Mikhail Zeleny (editors), ***Logic, meaning and computation: Essays in memory of Alonzo Church***, Kluwer Academic Publishers, forthcoming.

[4] A. J. Ayer, ***Language, truth and logic***, Dover Publications, Inc., New York, 1946.

[5] Rudolf Carnap, ***The logical syntax of language***, Kegan Paul Trench, Trubner & Co., London, 1937.

[6] ———, ***Meaning and necessity***, The University of Chicago Press, Chicago, Illinois, 1947.

[7] ———, *The methodological character of theoretical concepts*, ***The foundations of science and the concepts of psychology and psychoanalysis*** (Herbert Feigl and Michael Scriven, editors), Minnesota Studies in the Philosophy of Science, vol. 1, University of Minnesota Press, Minneapolis, 1956.

[8] Alonzo Church, *Review of Alfred North Whitehead and Bertrand Russell,* ***Principia Mathematica,*** *second edition, volumes II and III*, ***Bulletin of the American Mathematical Society***, vol. 34 (1928), pp. 237–240.

[9] ———, *A set of postulates for the foundation of logic*, ***Annals of Mathematics***, vol. 33 (1932), pp. 346–366.

[10] ———, *A set of postulates for the foundation of logic* (*second paper*), ***Annals of Mathematics***, vol. 34 (1933), pp. 839–864.

[11] ———, *A formulation of the simple theory of types*, ***Journal of Symbolic Logic***, vol. 5 (1940), pp. 56–68.

[12] ———, ***The calculi of lambda-conversion***, Annals of Mathematics Studies, no. 6, Princeton University Press, Princeton, 1941.

[13] ———, *Truth, semantical*, ***Dictionary of philosophy*** (D. D. Runes, editor), Philosophical Library, New York, 1942.

[14] ———, *Frege,* (*Friedrich Ludwig*) *Gottlob*, ***Dictionary of philosophy*** (D. D. Runes, editor), Philosophical Library, New York, 1942.

[15] ———, *Carnap's Introduction to Semantics*, ***The Philosophical Review***, vol. 52 (1943), pp. 298–304.

[16] ———, ***Introduction to mathematical logic, Part I***, Princeton University Press, Princeton, 1944.

[17] ———, *A formulation of the logic of sense and denotation* (*abstract*), ***Journal of Symbolic Logic***, vol. 11 (1946), p. 31.

[18] ———, *On Carnap's analysis of statements of assertion and belief*, ***Analysis***, vol. 10 (1950), pp. 97–99.

[19] ———, *A formulation of the logic of sense and denotation*, ***Structure, method and meaning*** (Paul Henle et al., editors), The Liberal Arts Press, New York, 1951.

[20] ———, *The need for abstract entities in semantic analysis*, ***American Academy of Arts***

[94] I am grateful to editors Andreas R. Blass, Charles Parsons, and Richard A. Shore, to an anonymous referee, and to Nathan Salmon for helpful comments on a previous draft of this paper.

and Sciences Proceedings, vol. 80 (1951), pp. 110–133, reprinted as *Intensional Semantics* in Martinich [50].

[21] ———, *Intensional isomorphism and identity of belief*, ***Philosophical Studies***, vol. 5 (1954), pp. 65–73.

[22] ———, ***Introduction to mathematical logic***, vol. 1, Princeton University Press, Princeton, New Jersey, 1956.

[23] ———, *Propositions and sentences*, ***The problem of universals: A symposium*** (J. Bochenski, editor), University of Notre Dame Press, Notre Dame, 1956.

[24] ———, *Logic and analysis*, ***Atti del XII Congresso Internazionale de Filosofia***, vol. 4 (1958), pp. 71–81.

[25] ———, *Mathematics and logic*, ***Logic, methodology and philosophy of science, Proceedings of the 1960 international congress*** (Ernest Nagel, Patrick Suppes, and Alfred Tarski, editors), Stanford University Press, Stanford, 1962.

[26] ———, *Denotation*, in ***Encyclopedia Brittanica***, 1964.

[27] ———, *Paul J. Cohen and the continuum problem*, ***Proceedings of international congress of mathematicians*** (I. G. Pterovsky, editor), Mir Publishers, Moscow, 1968, pp. 15–20.

[28] ———, *On Scheffler's approach to indirect quotation*, ***Semantica filosofica: Problemas y discussiones*** (Thomas Moro Simpson, editor), Siglo XXI Argentina Editores S. A., Buenos Aires, 1973.

[29] ———, *Postscript 1968*, ***Semantica filosofica: Problemas y discusiones*** (Thomas Moro Simpson, editor), Siglo XXI Argentina Editores S. A., 1973, postscript, in Spanish, to the Spanish translation of [VIII 45, 1943].

[30] ———, *Outline of a revised formulation of the logic of sense and denotation* (*Part I*), ***Noûs***, vol. 7 (1973), pp. 24–33.

[31] ———, *Outline of a revised formulation of the logic of sense and denotation* (*Part II*), ***Noûs***, vol. 8 (1974), pp. 135–156.

[32] ———, *A remark concerning Quine's paradox about modality*, ***Analisis Filosófico***, vol. 2 (1982), in Spanish; English translation in Salmon and Soames [62].

[33] ———, *Russell's theory of identity of propositions*, ***Philosophia Naturalis***, vol. 21 (1984), pp. 513–522.

[34] ———, *A revised formulation of the logic of sense and denotation*, *Alternative* (1), ***Noûs***, vol. 27 (1993), pp. 141–157.

[35] ———, *Intensionality and the paradox of the name relation*, ***Themes from Kaplan*** (Joseph Almog, John Perry, and Howard Wettstein, editors), Oxford University Press, New York and Oxford, 1989.

[36] Donald Davidson, *The method of extension and intension*, ***The philosophy of Rudolf Carnap*** (P. A. Schilpp, editor), Open Court, La Salle, Illinois, 1963, pp. 311–349.

[37] ———, *The structure and content of truth*, ***The Journal of Philosophy***, vol. 87 (1990), pp. 279–328.

[38] Michael Dummett, *Truth*, ***Proceedings of the Aristotelian society***, new series, vol. 59, 1959, reprinted with a postscript in his ***Truth and other enigmas***, Cambridge, Massachusetts, Harvard University Press, 1978, pp. 141–162.

[39] Kurt Gödel, *Russell's mathematical logic*, ***The philosophy of Bertrand Russell*** (P. A. Schilpp, editor), Tudor Publishing Co., New York, 1944, pp. 125–153.

[40] Nelson Goodman, *On likeness of meaning*, ***Analysis***, vol. 10 (1949), pp. 1–7.

[41] ———, *On some differences about meaning*, ***Analysis***, vol. 13 (1953), pp. 90–96.

[42] ———, ***Problems and projects***, Bobbs-Merrill Company, Inc., Indianapolis and New York, 1972.

[43] Paul Grice, *Method in philosophical psychology* (*from the banal to the bizarre*), ***Proceedings and Addresses of the American Philosophical Association***, vol. 48 (1974–1975), pp. 23–53.

[44] DAVID KAPLAN, *The language of L-Church-O*, unpublished notes, 1974.

[45] ———, *How to Russell a Frege-Church*, ***Journal of Philosophy***, vol. 72 (1975), pp. 716–729.

[46] ———, *Significance and analyticity*: *A comment on some recent proposals of Carnap*, ***Rudolf Carnap, logical empiricist*** (Jaakko Hintikka, editor), D. Reidel Publishing Company, Dordrecht and Boston, 1975.

[47] ———, *On the logic of demonstratives*, ***Contemporary perspectives in the philosophy of language*** (Peter A. French, Theodore E. Uehling, Jr., and Howard K. Wettstein, editors), University of Minnesota Press, Minneapolis, 1979.

[48] JOHN KEMENY, *A new approach to semantics*: *Part II*, ***Journal of Symbolic Logic***, vol. 21 (1956), pp. 149–161.

[49] SAUL KRIPKE, *A puzzle about belief*, ***Meaning and use*** (A. Margalit, editor), Reidel Publishing Company, Dordrecht, 1979, pp. 239–283.

[50] A. P. Martinich (editor), ***The philosophy of language***, Oxford University Press, New York and London, 1985.

[51] BENSON MATES, *Synonymity*, ***University of California Publications in Philosophy***, vol. 25 (1950), pp. 201–226.

[52] JOHN MYHILL, *Problems arising in the formalization of intensional logic*, ***Logique et Analyse***, vol. 1 (1958), pp. 74–83.

[53] CHARLES PARSONS, *Intensional logic in extensional language*, ***Journal of Symbolic Logic***, vol. 47 (1982), pp. 289–328.

[54] TERENCE PARSONS, *The logic of sense and denotation*: *Extensions and applications*, in [3].

[55] HILARY PUTNAM, *Synonymity, and the analysis of belief sentences*, ***Analysis***, vol. 14 (1954), pp. 114–122.

[56] W. V. O. QUINE, *Two dogmas of empiricism*, ***From a logical point of view***, Harvard University Press, Cambridge, Massachusetts, 1953.

[57] BERTRAND RUSSELL, ***Principles of mathematics***, Cambridge University Press, Cambridge, 1903.

[58] ———, *On denoting*, ***Mind***, new series, vol. 14 (1905), pp. 479–493.

[59] NATHAN SALMON, *On the very possibility of language*: *A sermon on the consequences of missing Church*, in [3].

[60] ———, *A problem in the Frege-Church theory of sense and denotation*, ***Noûs***, vol. 27 (1993), pp. 158–166.

[61] ———, *Being of two minds*: *Belief with doubt*, ***Noûs***, vol. 29 (1995), pp. 1–20.

[62] Nathan Salmon and Scott Soames (editors), ***Propositions and attitudes***, Oxford University Press, Oxford and New York, 1988.

[63] ALFRED TARSKI, *The concept of truth in formalized languages*, ***Logic, semantics, metamathematics*** (J. H. Woodger, translator), Oxford at the Clarendon Press, Oxford, 1956.

[64] ———, *Some methodological investigations on the definability of concepts*, ***Logic, semantics, metamathematics*** (J. H. Woodger, editor), Oxford at the Clarendon Press, Oxford, 1956.

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