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LOGICAL CONSEQUENCE REVISITED

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Tarski's 1936 paper, "On the concept of logical consequence", is a rather philosophical, non-technical paper that leaves room for conflicting interpretations. My purpose is to review some important issues that explicitly or implicitly constitute its themes. My discussion contains four sections: (1) terminological and conceptual preliminaries, (2) Tarski's definition of the concept of logical consequence, (3) Tarski's discussion of omega-incomplete theories, and (4) concluding remarks concerning the kind of conception that Tarski's definition was intended to explicate. The third section involves subsidiary issues, such as Tarski's discussion concerning the distinction between material and formal consequence and the important question of the criterion for distinguishing between logical and non-logical terms.

§1. Preliminaries. In this paper an *argument* is a two-part system composed of a set of propositions **P** (the premise-set) and a single proposition **c** (the conclusion). The expression '**c** is a [logical] consequence of **P**' is used with the same meaning as the expression '**c** is [logically] implied by **P**'. The expressions 'is a logical consequence of' and the converse 'implies' are relational. Often, I shall be talking in the same sense of validity of an argument. Validity is a property of arguments; an argument with premise-set **P** and conclusion **c** is valid if and only if **P** implies **c**; i.e., **c** is a logical consequence of **P**. Notice that this notion of argument is strictly ontic; it does not involve any agent that thinks, determines or establishes that a given proposition is or is not a consequence of a given set of propositions[1].

Tarski uses the word 'sentence' to refer to the objects of which truth and falsehood are [coherently] predicable; i.e, to the bearers of truth and falsehood. In a strict sense this use does not seem quite appropriate. In the logical tradition it has been customary to distinguish sentences, which are strings of symbols formed according to rules, from propositions, which are

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[1]This concept of a purely ontic nature contrasts with others of an epistemic nature, such as "argumentation", "deduction", "proof", etc. See Corcoran [8].

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expressed by sentences. It is worth pointing out that this characterizes sentences as purely syntactical objects. Tarski is referring to expressions which are distinguished by their purely structural characteristics. Finally the word 'statement' has been used to mean something that amounts to an ordered pair composed of a sentence and a proposition, where the proposition provides the intended meaning of the sentence. In other words, statements, in this sense, are interpreted sentences. Incidentally, the word 'proposition' has frequently been used for this notion. The word 'statement' has also been used as a *default* word in the presence of the philosophical objections to the others.

Given this situation, it is sensible to interpret Tarski's remarks to the effect that a given sentence is true (or false) as elliptical for one to the effect that the proposition expressed by that given sentence is true (or false). Tarski does not mention propositions by name, but his 1933 truth paper clearly indicates that in order to be true or false it is necessary for a sentence to *say* something, to be fully interpreted. This is the use of the word 'sentence' that we find in his 1936 consequence article.

If this is correct, it seems advisable to make a parallel distinction between arguments and argument-texts that express them. In a strict sense it is arguments, and not the syntactic argument-texts that express them, that are properly said to be valid or invalid. Accordingly a Tarski remark to the effect that a given argument-text is valid will be understood as elliptical for a remark to the effect that an argument represented by the text is valid in the strict sense.

Once more, whether we consider sentences or propositions expressed by sentences, the important point is that these entities are taken to be objective, ontic or absolute in the sense that they do not depend for their existence or for their truth-value on an agent, whether an individual or a community of thinkers. A sentence is true if and only if what the sentence says is the case. There is nothing in Tarski's view that would allow for an epistemic or pragmatic or psychological conception of truth. Nothing in Tarski's writings suggests the idea that a sentence is true if and only if it is known or believed or held or accepted or felt to be true. Occasionally, Tarski uses the expression 'asserted sentence' to refer to theorems, or provable sentences (see [26, p. 166]). By the same token, it is clear that when Tarski defines validity, he intends an objective, pure ontic property; for any given argument-text, either it is valid or it is invalid, regardless of whether the argument-text is known, believed, held, accepted or felt to be valid.

1.1. Three conceptions of logical consequence. In a strict sense the problem that Tarski addresses in the 1936 consequence paper is to determine a necessary and sufficient condition for an argument expressed by a given argument-text to be valid. More precisely, Tarski's problem was to explain what is meant by the expression 'is a logical consequence of' when that expression is being used correctly by members of a certain community of mathematicians and logicians. In a broader sense the issue is to give an analysis of this fundamental notion. This important issue involves matters of a philosophical and mathematical character. Let us begin with some philosophical considerations.

The expressions 'the ordinary usage', 'the common usage' and 'the common concept' of logical consequence show up very often in Tarski's discussion and they can be misleading. Therefore the task of clarifying Tarski's motivations in this article is fundamental. Tarski's work is not an investigation in the philosophy of language, nor a study of natural languages per se. However, it is important to realize that the expression 'logical consequence', like most other expressions in the English language, is ambiguous or admits of variation within a range of correct usage. Moreover, it goes without saying that this expression 'logical consequence' has also been misused; and finally, that there are different but co-extensive senses that have been attached to this expression. Tarski is concerned in his elucidations with some usages of certain portions of natural languages that logicians and mathematicians use in their ordinary professional practices. If this is correct, I believe it is quite appropriate to talk about *conceptions of logical consequence* and to restrain ourselves from using expressions that are misleading, such as '*the* intuitive notion of logical consequence'. For, we all may have different intuitions and, of course, an intuitive notion as such can be uncritical and confused. On the other hand, the word 'conception' can suggest a certain development of an idea, perhaps even that an initial intuition has been sharpened in the light of a tradition of thought and endowed with a certain capability to withstand further criticism.

Briefly, there are three basic conceptions of logical consequence, each of which captures an important insight about this notion as it has been normally understood in the classical tradition of logic and mathematics:

1. The information containment conception: **P** implies **c** if and only if the information of **c** is contained in the information of **P**. In this sense, if **P** implies **c**, then it would be redundant to assert **c** in a context where the propositions in **P** have already been asserted; i.e., no information would be added by asserting **c**.

2. The necessity conception: **P** implies **c** if and only if it is logically necessary for **c** to be true were all the premises in **P** true. In this sense, if a person knew that **P** implies **c**, then, if the person knew that the propositions in **P** were all true, then he could use those propositions as conclusive evidence for inferring **c**.

3. The impossibility conception: **P** implies **c** if and only if it is logically impossible for the propositions in **P** to be all true with **c** false. In this sense,

a person who knew that **P** implies **c** and who knew that **c** is false could use the negation of **c** as conclusive evidence that at least one of the propositions in **P** is also false.

It is a reasonable hypothesis that these three conceptions of logical consequence or validity, though conceptually distinct, are nevertheless coextensive. In order to determine the validity of a given argument-text, sometimes it may be easier to use one of the three than either of the other two. For example, what is called a direct deduction corresponds to the necessity conception. In a direct deduction the conclusion is shown to follow of necessity from the premises. On the other hand, in an indirect deduction, it is shown that it is [logically] impossible for all the premises to be true and the conclusion false; here determination of validity corresponds to the impossibility conception[2].

The information containment conception is connected with the logical positivist emphasis on tautologies as propositions devoid of information and on tautological processes of reasoning as ones that do not add information to the premise-set. What is important about this conception is that it is heuristically and pedagogically useful and that it has been used by distinguished logicians such as Mill, De Morgan, Carnap, and more recently, Corcoran [9, 10]. Nevertheless it involves a certain obscurity. This conception presupposes the existence of information, something whose ontological status is to be distinguished from that of consequence, from propositions and from propositions related by consequence.

Logical necessity and logical impossibility are both on the same ontological level, so to speak, and these two conceptions of logical consequence connect logical consequence with something else of the same sort. These two conceptions add little to the information containment conception, and they do not help our intuitions much beyond that mentioned above. However, if we consider the idea of "possible worlds" to furnish an account of these two conceptions, we would have first, that in order for **P** to imply **c** it is necessary and sufficient for the conclusion **c** to be true in every possible world in which the premises **P** are true and we would have, second, that in order for **P** to imply **c** it is necessary and sufficient for there to be no possible world in which the premises are true and the conclusion false. In these cases we explain logical consequence by reference to entities of a completely different sort, just as the information containment conception involves entities of a different sort. Likewise, just as "information" is philosophically problematic, "possible worlds" are also philosophically problematic. There are fundamental questions concerning their ontological nature, the existence and the identity of the objects in their domains, and so forth.

[2] The question comes up whether every valid argument that can be determined to be valid directly can also be determined to be valid indirectly and conversely.

These three conceptions illuminate the issue and give some background about the notion that Tarski wanted to define, but of course, this is not enough. Tarski's 1936 paper does not clarify which of the previous three conceptions (if any) Tarski had in mind. I will get back to this issue in my concluding remarks.

In any case, his result should be localized in scientific semantics, a fundamental field in the general program of the methodology of the deductive sciences. Tarski's definition of *one* concept of logical consequence belongs to the *purely ontic* side of the methodology of the deductive sciences. Specifically, Tarski's definition is devoid of reference to the problem of how to determine validity or with how people have established validity in the past. A fortiori, his definition has no direct bearing on the *heuristics, epistemics, or pragmatics* of consequence, or with the context of discovery of the validity of a given argument.

§2. Logical consequence defined: the fixed-domain, variable content view. In the 1936 paper, Tarski deals with argument-texts formulated in a given interpreted language. On the basis of an assumed distinction between what he calls "logical and extra-logical terms", he implies that for every given argument-text **A** there exists an *argument-text function* $\mathbf{A}^*$ which is obtained by a one to one substitution of variables of the appropriate types for the extra-logical terms of the given argument-text. Sequences provide values of the variables. In the terminology of his 1933 truth paper, a sequence is a function whose domain is the set of positive integers and whose values are in the hierarchy of types based on *the* universe of discourse. The variables of the language under consideration have been enumerated in an order, and for any given sequence, the k-th object in the sequence is appropriately assigned as a value to the k-th variable. A sequence that *satisfies* a given set $\mathbf{L}^*$ of sentential functions obtained as above from a set **L** of *sentences* is called a *model* for that set **L** of sentences. Using these terms Tarski [29, p. 417] gives his famous "no-countermodels" definition of logical consequence:

> *The sentence X follows logically from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*

From the standpoint of modern mathematical logic this definition may look unsatisfactory. As noted by Corcoran in 1972 [4, p. 43] the expressed definition may only countenance sequences generated from just *one* universe, the universe of the given interpretation of the language. Logical properties, particularly validity of an argument-text, should be independent of the particular universe of discourse that the language happens to be interpreted in. Tarski seems to be giving a condition that is necessary but not sufficient for validity and hence making a mistake similar to his predecessor Bolzano.

Does this mean that Tarski in the thirties did not think of our present non-fixed domain, variable-content viewpoint? In particular, does this mean that Tarski did not conceive of the possibility of restricting the range of the individual variables to a subclass (perhaps even finite) of his universe of discourse? Does this mean that Tarski did not realize that by dint of having an assigned range to the variables the variables must also be regarded as being extra-logical symbols in the same way that an individual constant, predicate constant or relational symbol would be? As Church [3, Section 2] indicates, in an interpreted language just as an individual constant refers to a particular object and a predicate symbol refers to a particular class of individuals, likewise an individual variable denotes in a sense its range. It is remarkable that during this period, distinguished logicians like Carnap [2, p. 155] and Quine [18, p. 43] took variables to be logical signs[3]. If variables were logical signs, that would tend to mask the question of whether they should be reinterpreted when reinterpreting the remaining content-terms.

It is fair to think either that Tarski did not express what he intended, which presupposes that he already knew the non-fixed domain variable content viewpoint, or that he expressed what he actually intended, which presupposes that he did not accept the modern view of the import of the universe of discourse, or finally, a possibility compatible with both hypotheses, that for some reason he was just cautiously evasive about the whole point.

The point is that in this paper Tarski never says that he is not allowing the universe of discourse to change even though it is clear in other papers that he changes universe of discourse. On the other hand, what he does (perhaps instead of changing the universe of discourse) is to put restrictions into the sentences. We shall see this below when considering his argument-texts in the omega-concepts paper [28], where Tarski employs universal sentences with restrictive relative clauses[4].

In his paper on truth [26] there is plenty of evidence that Tarski was acquainted with the relativized notion of truth. Referring to the school of Hilbert, he says: "[. . .] another concept of a relative character plays a much

[3]Quine [18, p. 43] says the following: "All true statements which (like '$(x)(x = x)$') contain only logical signs are naturally to be classified as logically true". This means that all true cardinality sentences are tautological, in agreement with Wittgenstein, Ramsey and others. Furthermore in Quine [19, p. 244] a related view can be found: "The logical signs are the truth-function signs, the quantifiers, the variables of quantification and the identity sign". In a similar vein, Carnap [2, Section 7, p. 155] says: "Every sentence which contains only logical signs is L-determinate". Previously in Section 5 Carnap indicates: "The logical signs are divided into logical constants ('ist', 'nicht', 'wenn', 'so', 'fuer', 'jedes') and variables ('x', 'y' etc.)" So, it is clear that both authors took variables to be logical signs.

[4]As a rule, in the case of universal propositions, the addition of a restrictive relative clause attached to the subject of the given universal proposition restricts the range of objects being subjected to predication and, thus, drops information. For example "every number is even" implies "every oblong number is even".

greater part than the absolute concept of truth and includes it as a special case". Furthermore, in the same section he defines truth in a given domain with k-elements and truth in every individual domain for the calculus of classes (see Tarski [26, pp. 199–201]). On the other hand, Wilfrid Hodges [14] has advanced the hypothesis that the 1936 paper was written as an address to philosophers and Tarski did not think that philosophers would be interested in relativized notions of truth. Hodges quotes from Footnote 2 on p. 199 of Tarski [26]:

> *The discussion of these relativized notions is not essential for the understanding of the main theme of this work and can be omitted by those readers who are not interested in the special studies in the domain of the methodology of the deductive sciences* [...]

Yet there is also some room for the possibility that Tarski had reasons to be cautious. Hodges points out that the proper mathematical concept that is needed to explicate the relativized notion of truth is the concept of "truth in a structure" and, according to Hodges, in the thirties there was still no general agreement among logicians about what kind of object a structure or system is.

There was also a philosophical debate going on during the thirties concerning the status of the axiom of infinity in type theory. It is extremely difficult to find clear philosophical commitments in Tarski's writings beyond his usual technical considerations of the simplicity and clarity of the mathematical or logical tool employed. However, since Tarski's articles in the thirties were written in the framework of the *Principia Mathematica* I think it is safe to advance the hypothesis that he was aware of the criticism of the axiom of infinity and, hence, that a certain wariness of this previously accepted principle was required. Following Hodges's first suggestion, we may say that perhaps Tarski was just reluctant to deal with this issue with philosophers if he did not have confidence in the clarity and precision of an eventual answer. I will get back to this issue.

Be that as it may, it is worth exploring Tarski's discussion of the omega-concepts in his 1936 article, since this issue, besides being puzzling in itself, may give us more insight into Tarski's viewpoint on consequence at that time.

§**3. Tarski's discussion of omega-incompleteness and validity.** Tarski points out that it is a mistake to identify logical consequence with any syntactic or proof-theoretic concepts by acknowledging the existence of omega-incomplete theories. Tarski's point was that not only it is an ontological mistake to try to reduce logical consequence to a syntactic concept, but that it is also a mistake as far as extensions are concerned. Moreover, Tarski thought that there was a notion of logical consequence that was entrenched in the community of mathematicians and logicians at that time and that some authors—particularly Carnap—were trying to identify this notion with a proof-theoretic notion. For Tarski however, logical consequence is fundamentally a semantic concept, and even if it were coextensive with an appropriate syntactic concept it would still be a mistake to identify them[5]. The example of the omega-incomplete theories points out that there are valid arguments with infinitely many premises that cannot be deduced, since no deduction can use infinitely many premises; every deduction is inherently finite. In order to develop my reasoning it is convenient to say what I mean by an *omega* set of sentences, by an *omega-incomplete* set of sentences and by an *omega* argument. In order for a set of sentences about the natural numbers to be omega it is necessary and sufficient for it to be the set of all the numerical instances of some universal sentence about the natural numbers. In order for a set of sentences about the natural numbers to be omega-incomplete it is necessary and sufficient for it to imply the set of all of the numerical instances of some universal sentence about the natural numbers that it does not imply[6]. In order for an argument-text to be omega it is necessary and sufficient for it to have as its conclusion a universal sentence whose numerical instances form the premise-set. Here is Tarski's own example:

> Some years ago I gave a quite elementary example of a theory which shows the following peculiarity: among its theorems there occur such sentences as:
>
> A_0. 0 *possesses the given property P*,
>
> A_1. 1 *possesses the given property P*,
>
> and, in general, all particular sentences of the form
>
> A_n. *n possesses the given property P*,
>
> where '*n*' stands for any symbol which denotes a natural number in a given (e.g., decimal) number system. On the other hand the universal sentence:
>
> A. *Every natural number possesses the given property P*,

[5]More precisely, Tarski was clear that logical consequence and deducibility were two distinct concepts. It is unfortunate, that expressions, such as 'the syntactic concept of logical consequence' or 'the proof-theoretic concept of logical consequence' are a commonplace in contemporary philosophical discussion. This is an example of misguided rhetoric that has been taken up in the field.

[6]In other words, in order for a set of sentences about the natural numbers to be omega-incomplete it is necessary and sufficient for it to imply the set of all of the numerical instances of a universal sentence about the natural numbers *while not implying this universal sentence itself*. The omega-incompleteness concept has been generalized by B. Mates and others to a set of sentences interpreted in an arbitrary universe of discourse.

> cannot be proved on the basis of the theory in question by means of the normal rules of inference. This fact seems to me to speak for itself. It shows that the formalized concept of consequence, as it is generally used by mathematical logicians, by no means coincides with the common concept. Yet intuitively it seems certain that the universal sentence A follows in the usual sense from the totality of particular sentences $A_0, A_1, \ldots, A_n, \ldots$. Provided all these sentences are true, the sentence A must also be true.
>
> (Tarski [29, pp. 410–11])

This rather vague description is particularly puzzling since it shows—the way it is written—that Tarski believed that this argument was valid in the usual sense despite the fact that it was not syntactically deducible "in the sense" that its conclusion cannot be derived from the premise-set by means of a finite number of applications of the normal rules for deduction.

By appealing to the information containment conception, it is easy to see why arguments of this sort are not necessarily valid. No sentence in the premise-set says that there is no instance of the universal sentence outside of that set. In other words, there is no piece of information in the premise-set that tells you that every number has been accounted for[7]. Tarski says all are there but there is no premise to that effect. Moreover, he does not give any reason that even combining the sentences in the premise-set together gives you this information.

The fact that from our present model-theoretic viewpoint some omega arguments are invalid contributes to rendering the previous quotation paradoxical. Incidentally, at this point in our discussion we could take the existence of a countermodel in a different domain as evidence that perhaps Tarski was not aware of the import of the concept of a universe of discourse. In order to see it, let us formalize the semi-informal expression of Tarski's example in our current framework, assuming that the universe of discourse has been established as the set of natural numbers. Let us use the language of first order arithmetic with 'zero' and 'successor' as primitives. Take 'P' to indicate the property of being identical with zero or with the successor of some number. Here displayed is the argument-text in question:

A_0. $(0 = 0 \vee \exists y(0 = sy))$
A_1. $(s0 = 0 \vee \exists y(s0 = sy))$
and in general all particular sentences of the form

[7] It is clear, however, that an omega-set of sentences implies the corresponding universal if the premise-set is contradictory or the conclusion is tautological. For example, "$\neg(0 = 0)$", "$\neg(1 = 1)$", ... , etc., implies "$\forall x \neg(x = x)$". Likewise, "$\forall x(x = x)$" is implied by "$0 = 0$", "$1 = 1$", etc.

$$\frac{A_n.\ (sn = 0 \lor \exists y(sn = sy))}{A.\ \forall x(x = 0 \lor \exists y(x = sy)).}$$

Notice that each of the premises is tautological. A countermodel for this argument-text is constructed by taking the universe of discourse to be the class of non-negative rationals and providing the standard denotation for '0' and 's'. $1/2$ is a counterexample for the universal in the conclusion. In fact all of the added numbers that are smaller than 1 are counterexamples.

It is even more interesting to realize that this argument-text also has a countermodel in the same universe of discourse. Just assign the number 2 to the numeral '0'. Take the function **d** to be the function that is the identity on zero and the successor everywhere else. Let the function **d** be assigned to the functor 's'. So reinterpreted, all the premises are true and the conclusion false. One is the only counterexample for the conclusion since it is not denoted by '0' nor is it in the range of the function assigned to 's'.

It might be useful to emphasize that the formalized argument-text in its intended interpretation is materially valid. If the premises are true then the conclusion is true. However, material implication is not the issue, logical implication is. The point is that by means of re-interpretations arguments in the same form as the formalized argument are obtained in which all the premises are true and the conclusion false. In order to construct the countermodels, the standard model theoretic procedure has been used: a (re)interpretation for an argument-text formalized in a given language **L** is provided by specifying a domain **D** and a function **f** that assigns to the non-logical constants of **L** appropriate denotations in **D**.

The two previous countermodels show a common feature already indicated in terms of the information containment conception. Both exhibit the logical possibility that the premise-set in the given argument-text does not account for every instance of the corresponding universal. In every countermodel there will be a "number" which is not assigned to any numeral in the argument-text.

The question arises, what can we make of these examples? Why does Tarski believe the previous argument-text to be valid, though the semantic account renders the logical possibility of a countermodel? Why does Tarski think that argument-texts of this sort are all valid, when it is obvious to us that they are not all valid? Many other examples of invalid argument-texts of this sort can be provided. Add to the previous first order language the one place predicate 'P' and consider the argument-text whose conclusion is a universal sentence '$\forall xPx$' and whose premise-set consists in all of the instances 'Pn', where 'n' is a numeral, '0', '$s0$', '$ss0$', etc. formed by substituting numerals for 'x' and dropping the quantifier. This argument-text is obviously invalid.

Take the universe to be the natural numbers with '0' denoting 1, 's' denoting the successor function and 'P' denoting the property of being positive.

Of course our intuitions can go wrong in many ways. However it is striking to notice that Tarski is completely certain of the validity of all omega arguments not only in his 1936 paper but also in Tarski [28] where he claims that "from the intuitive standpoint" the truth of the universal sentence "is without doubt" a consequence of the truth of all the instances. (See Tarski [28, p. 294].)[8]

I propose to explore three explicative hypotheses that could explain Tarski's claim that omega arguments are all valid. Some of them are related to the issue of the fixed and non-fixed domain conceptions of logical consequence.

Hypothesis 1. Considering the universe of natural numbers; i.e., the intended interpretation of Tarski's example, suggests that he may have had confused the material validity of arguments with the logical validity of argument-texts. Material implication is completely determined by (actual) truth-values. Logical implication is not determined by truth-values except in the case of all true premises and false conclusion. It is clear that every omega-argument is materially valid since if the premises are all true, then every instance of the conclusion is true; and if every instance of the conclusion is true, then the conclusion is true. The conclusion is true if and only if all of its instances are. In fact, the conclusion is materially equivalent to the premise-set. This hypothesis may look rather implausible since there is an extensive footnote on page 419 in which Tarski distinguishes the concepts of "formal consequence" and "material consequence" without hesitation. However it is worth noting that perhaps things are not so obvious concerning this point if we consider it in connection with his discussion of the division between logical and extra-logical terms in Tarski [29, p. 419]:

> *It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage. In the extreme case we could regard all terms of the language as logical. The concept of formal consequence would then coincide with that of material consequence.*

The last quoted sentence does not seem right. There are argument-texts which contain no non-logical constants which are materially valid but not logically valid. Let us consider the language of quantificational pure identity

[8]Tarski immediately adds that it is clear that from the point of view of deducibility, the universal sentence, "need not at all be a consequence of" its numerical instances. From the point of view of deducibility the class of "valid" arguments has been defined with respect to a set of rules of inference. Namely, the consequences of a given set of premises are obtained by means of the four operations of substitution, detachment, and the insertion and deletion of the universal quantifier. See Tarski [28, p. 285, Def. 9].

interpreted on the universe of natural numbers. Take as the premise the sentence "there are at most two numbers" and for the conclusion the sentence "there is at most one number".

$$\exists x \exists y \forall z (x = z \vee y = z)$$
$$?\, \forall x \forall y (x = y).$$

Since the premise is false, this argument-text is materially valid. However, it is easy to see that it is logically invalid. Just consider a universe with two individuals[9].

It might be of interest to note that there is a little difference between taking the variables to range over a restricted subdomain of *the universe* and taking the variables to be unrestricted with a special primitive predicate to denote the subdomain that serves as the intended universe of discourse. The choice here is mainly one of convenience[10].

It is clear that this example contains no content-terms if regarded from the traditional division of logical and extra-logical terms. Again we seem to obtain more evidence to the effect that Tarski did not consider the non-fixed domain viewpoint in this article. Now the question comes, is there any justification for this surprising omission? Besides, if there is a justification, is there any way to satisfactorily explain his apparently unsatisfactory comment about material and logical implication? I will get back to this point.

Hypothesis 2. The second hypothesis amounts to admitting that Tarski was not certain about the division of logical and extra-logical terms. For example, Etchemendy [11, p. 85] has suggested that if numerals and the expression 'natural number' are included in the set of logical terms, then the omega argument-text above comes out valid. It is true that under this peculiar selection of logical terms, countermodels are ruled out. However, I believe this possibility is rather unlikely. Despite the fact that some unusual choice of logical terms may involve consequence relations that might be in agreement with certain ordinary usages, Tarski is aware that the selection is not "quite

[9]This problem is pointed out in J. Etchemendy [11] and in G. Sher [24] although no explication is made in neither of these books about the underlying reasons that justify why this is so. I shall explain this matter in my third hypothesis.

[10]The universe of discourse of a mathematical theory has been treated in different ways according to the logical framework employed, as well as the predilection and sense of simplicity of each author. For example, in Peano's axiomatization of arithmetic the universe of discourse is unrestricted. In this system, the specific domain of objects of the science, namely, the natural numbers, is denoted by a primitive predicate 'N'. In contrast, in Gödel's 1931 axiomatization of arithmetic, the universe of discourse is established as the set of natural numbers, and hence, it is restricted before hand as a semantic assumption. In this formalization, the presence of a special primitive predicate denoting the intended domain is not required. This second procedure is also simpler for notation since quantifiers are not syntactically relativized by means of a primitive predicate.

arbitrary". This point requires some further comment since it could give the wrong impression that the choice of logical terms at that time was a purely conventional matter. Assuming the fact that Tarski (then) lacked a criterion for the distinction should not lead us astray on this issue. Although the logical terms could be listed, the following hypothesis will make clear the fact that the inclusion of the numerals and the expression 'natural number' as logical terms involves deeper considerations than a simple conventional division of the language. Very often conventions are the result of laborious processes. Some unusual choice could have led to the definition of a different concept of consequence in agreement with one ordinary use rather than another. However, we should keep in mind here that Tarski is after the concept of consequence as it is "generally used by mathematical logicians" (See Tarski [29, p. 411]). Likewise, it might be useful at this point to indicate that neither numerals nor the predicate 'natural number' are logical constants in the standard first order formalization under Tarski's criterion of invariance under permutations of the universe. This criterion is developed in Tarski [38] but it is already hinted at in Tarski and Lindenbaum [40, p. 385].

Before discussing hypothesis three I need to elaborate again on the distinction between the fixed and non-fixed domain viewpoint. The fact that I have adopted a certain conventional decision, permitting myself a license, in the way I have dealt with the omega argument-text should not be disregarded. I have dealt with an omega argument-text formulated in the language of first order logic with identity whose intended interpretation has as its universe of discourse the class of natural numbers and it has a symbol '0' for zero and a symbol 's' for successor. We define a numeral in the usual way with the symbols '0', '$s0$', '$ss0$', and so forth. Furthemore, we define the omega-set for a universal sentence '$\forall x P(x)$' to be the class of instances, namely, the set consisting of '$P(0)$', '$P(s0)$', '$P(ss0)$', etc.; we also define an omega-argument to be an argument whose conclusion is the universal sentence '$\forall x P(x)$', and whose premises are the instances '$P(0)$', '$P(s0)$', '$P(ss0)$', etc. Finally, we define a set of sentences to be omega-incomplete if it implies the set of all of the instances of the universal that it does not imply. So an omega-incomplete set of sentences implies the set of numerical instances that is the premise-set of an invalid omega-argument whose conclusion it does not imply. However, it is sensible to ask whether Tarski would have formulated it within the limits of expression of first order logic. In fact, the system of natural numbers does not have a categorical characterization if the set of its axioms are formulated in first order[11]. Notice that in our example I have referred to the property

[11]This fact follows, of course, from the Löwenheim-Skolem theorem, which was well-known by Tarski as is indicated in Footnote 2 of Tarski [26, p. 241]. Once more, as a collateral point, we have evidence that Tarski knew at that time the limitation results that involved considerations about domains of different cardinalities.

of being identical with zero or identical with the successor of some number, which is coextensional with the property of being a natural number.

There is evidence in the omega paper [28] that Tarski in fact had in mind higher order logic with respect to the 1936 omega argument-scheme. The system developed in the 1933 omega article is, in fact, a reformulation of the language of *Principia Mathematica* (see Tarski [28, p. 279]. So, Tarski's language is a language of types. Specifically, a type-theoretic language has infinitely many variables of different orders which range over individuals (objects of first order), classes of individuals (objects of second order), classes of classes of individuals (objects of third order), and so on.

It is important to recall that in this framework, a natural number is not a member of the universe of discourse but it is a class of classes of individuals. More precisely, among the classes of classes, some of them are distinguished by the property that any member of that class is in a one-to-one correspondence with any other member and any class in one-to-one correspondence with a member is also a member. These classes of classes are known as cardinal numbers. So, for example, the cardinal number zero is the class whose only member is the null class, the cardinal number one is the class whose only members are all of the singletons, the cardinal number two is the class whose only members are all of the pairs, and so forth.

In this framework, the symbol '0' is now, not an individual name but it is a second order class name and in its intended interpretation it denotes the class whose only member is the null class. The symbol 's' no longer denotes a function from the universe into itself but it indicates a function that is defined on the class whose members are classes of classes. So, in particular, the domain of this function includes all of the natural numbers and many other things, (e.g., any subclass of any natural number, or the union of two natural numbers, etc.). This function is defined as follows: when applied it to a given class, it gives the class obtained from the given class by adding one member to every class in the given class.

In this type-theoretic situation, in addition to the symbols for zero and successor, which are logical constants, we also need a symbol, let us say, '$\mathbb{N}$' for the class of natural numbers and we have to consider universally quantified sentences with restrictive relative clauses of the form 'For every k, if k belongs to $\mathbb{N}$, then $P(k)$' , where 'k' is a variable of the same type as the natural numbers. Tarski [28] considers the following three argument-texts.

A)

$$
\begin{array}{c}
\bigcap_1^2(\beta^0 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\bigcap_1^2(\beta^1 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\cdots\cdots\cdots \\
\bigcap_1^2(\beta^n \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\hline
\bigcap_1^2(\beta^{n+1} \to \overline{\bigcup_1^1 \in_{1,1}^1})
\end{array}
$$

B)

$$
\begin{array}{c}
\bigcap_1^2(\beta^0 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\bigcap_1^2(\beta^1 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\cdots\cdots\cdots \\
\bigcap_1^2(\beta^n \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\hline
\bigcap_1^2(\delta_1 \to \overline{\bigcup_1^1 \in_{1,1}^1})
\end{array}
$$

C)

$$
\begin{array}{c}
\bigcap_1^2(\beta^0 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\bigcap_1^2(\beta^1 \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\cdots\cdots\cdots \\
\bigcap_1^2(\beta^n \to \overline{\bigcup_1^1 \in_{1,1}^1}) \\
\cdots\cdots\cdots \\
\hline
\bigcap_1^2(\delta_1 \to \overline{\bigcup_1^1 \in_{1,1}^1}).
\end{array}
$$

The following is a translation of argument-text **A** into English:

A_0. For any given set x, if that set has at most 0 individuals then there exists an individual that does not belong to x.

A_1. For any given set x, if that set has at most 1 individual then there exists an individual that does not belong to x.

. . . .

. . . .

A_n. For any given set x, if that set has at most n individuals then there exists an individual that does not belong to x.

A. For any given set x, if that set has at most $n+1$ individuals then there exists an individual that does not belong to x.

Argument-text **B** says:

A_0. For any given set x, if that set has at most 0 individuals then there exists an individual that does not belong to x.

A_1. For any given set x, if that set has at most 1 individual then there exists an individual that does not belong to x.

. . . .

. . . .

A_n. For any given set x, if that set has at most n individuals then there exists an individual that does not belong to x.

A. For any given set x, if that set is finite[12] then there exists an individual that does not belong to x.

Argument-text **C** says:

A_0. For any given set x, if that set has at most 0 individuals then there exists an individual that does not belong to x.

A_1. For any given set x, if that set has at most 1 individual then there exists an individual that does not belong to x.

. . . .

A_n. For any given set x, if that set has at most n individuals then there exists an individual that does not belong to x.

. . . .

A. For any given set x, if that set is finite then there exists an individual that does not belong to x[13].

Tarski, on this occasion, establishes the non-deducibility of the argument-texts **A** and **B** by means of the so-called matrix method extensively used by Lukasiewicz. He produces a syntactic property that belongs to the axioms of his type-theoretic system and which belongs to the premises of the argument and which is preserved under the rules of inference, but which does not belong to the conclusion of the argument . Now here is Hypothesis 3.

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[12]In other words, if the set is "inductive" in the sense of *Principia Mathematica*, namely if the set is obtained by adding individuals one at a time to the null set.

[13]In other words, the premises of these argument-texts can be read as "No set that contains at most zero elements exhausts the universe", "No set that contains at most one element exhausts the universe", etc. The conclusion of argument-text **A** reads "No set of at most $n+1$ elements exhausts the universe" and the conclusion of argument-texts **B** and **C** reads "No finite set exhausts the universe of discourse".

Hypothesis 3. Tarski held, at the time, the logicist fixed-domain viewpoint. It should be noted that for a logicist any cardinality sentence, or quantitative sentence as Tarski called them, if true, it is logically true. Logicists—particularly Ramsey [21] and Carnap [1][14]—held that the number of objects in the world is a logical matter. In other words, in this view every materially true cardinality sentence is also logically true and every materially false cardinality sentence is also logically false. Had Tarski held this logicist belief, the alleged charges of confusing material and logical implication discussed in the first hypothesis would be unjustified (and this is irrespective of the question whether this logicist principle is true or false). However, I have not been able to find any single remark in his writings to sustain any philosophical commitment in this direction. Nevertheless, he could have adopted this logicist principle as merely a technical convention without assuming any corresponding philosophical debts.

This idea, however, seems to be in conflict with Tarski's intuitive remarks in the omega paper explaining his proof of the non-deducibility of the arguments **A** and **B**. He clearly shows his awareness of the semantic effect of relativizing truth to a certain universe of discourse when he points out that restricting a universe of discourse to $n+1$ objects makes all the premises true of argument-texts **A** and **B** above. (See Tarski [28, Footnote on p. 291]). To be precise, in a strict sense, Tarski does not in this article establish the model-theoretic invalidity of any argument, but he clearly hints at semantic methods involving domains of specific cardinalities.

Given this language of types, the important point is that Tarski is looking at argument-texts with no individual constants. In fact the sentences in the previous cases contain only variables and logical constants of the sort allowed under the framework of *Principia Mathematica*. The presence of the membership relation in the previous argument-texts with the omega premise-set requires some further comment. It might be worthwhile to emphasize that the membership relation in question is not the first order membership relation. In Tarski [38] the logicality of the type-theoretic membership relation is justified under the criterion of invariance under all transformations of the universe in a hierarchy of types. A transformation in

[14] Ramsey [21, p. 223] says: "That is, 'There are at least *n* individuals' is always either a tautology or a contradiction, never a genuine proposition. We cannot, therefore, say anything about the number of individuals, since, when we attempt to do so, we never succeed in constructing a genuine proposition, but only a formula which is either tautological or self-contradictory. The number of individuals can, in Wittgenstein's phrase, only be shown, and it will be shown by whether the above formulae (for any *n*) are tautological or contradictory." Carnap [1, p. 141] also holds this existence assumption in his coordinate language II where the axiom of infinity is demonstrable. Carnap adds the following remark: "But the doubts previously mentioned are not relevant here. For here, those sentences only mean, respectively that for every position there is an immediately succeeding one, and that at least one position exists. But whether or not there are objects to be found at these positions is not stated."

this context is a one-to-one mapping of the universe onto itself. Since any transformation of the universe of individuals induces a transformation in the entire hierarchy, it is easy to see that the membership relation is invariant between adjacent types and hence, it is a logical notion[15].

In his undecidability paper, [31, p. 18], Tarski considers a theory T_1 whose only non-logical axiom is the sentence '$\forall x \forall y(x = y)$'. He writes: "This sentence clearly expresses the fact that the universe consists of only one element; although non-logical constants do not occur in the sentence, it is not a logical axiom since it is not satisfied in every realization". The terminology and the idea expressed in this paragraph are perfectly comprehensible in light of our current understanding of model theory. However, one is left to wonder how he could possibly have missed this essential point in 1936. It is clear from what has been said that Tarski was aware of the notion of truth relativized to a domain in 1936. However, this idea—and this is the point to be stressed—is alien to the framework of the *Principia Mathematica*. I conclude, then, that if hypothesis three is correct then all of the Tarski's omega-arguments are valid.

§4. **Concluding remarks.** I have not yet said anything about the *formality* requirement. In fact, this is the only feature of logical consequence which Tarski explicitly mentions. I believe it is this condition that provides some hint as to where to locate his conception. Moreover, the formality requirement is at the heart of both the syntactic and the semantic characterizations of logical consequence. Formality is involved in syntactic characterizations in that logical axioms and applications of rules of inference satisfy a "principle of form". Formality is involved in semantic characterizations in that every model (in the sense of the 1936 Tarski paper) of a given set of sentences is also a model of any other set of sentences having the same logical form.

Tarski indicates that if a sentence X is a consequence of a set K of sentences, then the same holds in *every* formally similar case; therefore, the non-logical (non-formal, content) symbols can be replaced by suitable variables in a one-to-one fashion. This means that two argument-texts in the same form are both valid or both invalid. Therefore, any argument-text in the same form as an invalid argument-text is invalid and any argument-text in the same form as a valid argument-text is valid. These are, again, purely ontic principles.

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[15]Nevertheless it is interesting to point out Tarski's conventional attitude towards the membership relation as is seen in a letter to Morton White in which Tarski asks, showing a certain candor, whether the epsilon should be taken to be a logical or non-logical sign: " ... sometimes it seems convenient to include mathematical terms, like the $\in$-relation, in the class of logical ones, and sometimes I prefer to restrict myself to terms of 'elementary logic'. Is any problem involved here?" (Tarski [39, p. 29].)

In order to understand Tarski's elucidations it is useful to take into account the informal mathematical practice current during and preceding the time of the article and to introduce some considerations of the related pragmatic issue of how mathematicians established the validity of a given interpreted argument-text. Although these are absent from Tarski's explicit discussion, they are part of the presupposed common background of Tarski's audience and they may cast some light upon his insights on formality.

Settling a hypothesis; i.e., determining the truth value of a proposition not yet proved or disproved, involves *deduction*. If a given hypothesis is deduced from premises already known to be true, the hypothesis is thereby *proved*; i.e., its truth is determined. On the other hand, if a proposition known to be false is deduced from the hypothesis alone, or from the hypothesis augmented by propositions accepted as true, then the hypothesis is proved to be false. Now, very often fallacies can vitiate applications of the deductive and the hypothetico-deductive methods. "Begging the question" arises when a proposition not accepted as true is used as a premise as if it had been accepted. On occasion, a proposition that is not actually in the premise-set may be used as a premise; this of course is "premise-smuggling".

This predicament is where the *formality* requirement in Tarski's discussion gains its full meaning. The definition of a key-concept for the foundations of the methodology of the deductive sciences must rest on solid ground. Historically, *disinterpretation* has been the traditional procedure to provide this ground. Disinterpretation amounts to "bracketing" content or subject-matter and, hence, to treating the non-logical symbols as if they were variables. It is precisely by attending to the formality requirement that premise-smuggling is avoided or detected. Tarski emphasizes the fact that consequence and hence deduction is formal, or subject-matter neutral, not sensitive to content. In [29, pp. 414–15], he says: "[. . .] *this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences in the class K refer*".

Tarski's point is that our intuitions concerning consequence should be independent of our intuitions about content terms. This is one feature that Tarski wants to capture and this is a feature that is carried over to his definition.

Although Tarski makes no explicit reference to this, it seems clear that the so-called epistemic principle of *economy of thought* is to be kept in the foreground. Its methodological value lies in the fact that once a conclusion has been *deduced* from a set of premises about a given universe of discourse, the deduction is "exportable" so to speak to a different domain of investigation. By the same token, once it is known that a given conclusion is not a consequence of a given premise-set, the same is known to hold in every formally similar case. Tarski builds this into his definition by making a countermodel for one argument automatically serve as a countermodel for every argument in the same form.

The natural question arising is what sort of conception of logical consequence underlies Tarski's formality condition. In other words, what does it mean to say that "the same holds in *every* formally similar case"? The presence of the universal quantifier seems to suggest that Tarski's definition reflects on a previous conception based on *generalization*. Generalization of a certain kind however, since the class of interpretations is determined by the topic neutrality of logical terms and not by the extrinsic influence of knowledge of the extra-logical terms. In fact, in his 1941 textbook, [30, Section 38, p. 117], Tarski terms an argument-text function, a *general argument*, which is of such a nature that "*it belongs no longer to our theory but to logic; and as a result of this generalization we obtain a general logical statement* [...] *which asserts that the theorem in question, when suitably interpreted, is satisfied by every model of our axiom system*."

Briefly stated, a generalization account of logical consequence holds that if a given argument-text is valid, then every argument-text in the class of the argument-texts having the same form as the given argument-text, is materially valid, and conversely. This is not what Tarski is doing and this shows that Tarski had a subtle insight of validity. It is clear that if a given argument-text is logically valid then every argument-text in the same form as the given argument-text is materially valid. However, the converse of the previous conditional is not true since it amounts to the *no expressible counterarguments* viewpoint of validity. Tarski was clear that to have no expressible counterarguments was a necessary but not a sufficient condition for an argument-text to be valid. One of Tarski's main achievements in the 1936 consequence article is to employ devices, such as *sequences*, *satisfaction* and *model* that he developed in earlier papers to extend the classical Russellian treatment of consequence. The comparison of Tarski's treatment of validity in the 1936 paper and Russell's treatment of validity in *Introduction to Mathematical Philosophy*, pp. 196–8 is truly astounding. At first, it may seem that Tarski's is just a more mathematically precise explication of consequence. Closer examination however, shows that much more is involved, namely, the extension of the ideas from the case of the Russellian generalized object-language material implication to the Tarskian generalized meta-language implication conception. When discussing formal implication, Russell transforms a given argument-text into a generalized object-language material conditional. In other words, he considers the universal closure of a material conditional obtained by appropriately substituting variables for content-terms, whose antecedent is the conjunction of the premises and whose consequent is the conclusion of the given argument-text. In addition, it is clear that Russell only considers argument-texts with a finite number of premises. Tarski's contribution, on the other hand, is the strongest generalized material implication conception. In fact, the enlargement of the generalization obtained by means of his semantic devices is far superior to any of the preceding definitions. In addition, Tarski also takes into account the fact that in many scientific contexts, logical consequence relates infinite premise-sets with individual sentences.

Corcoran, Prawitz and Etchemendy have already pointed out that the content of this kind of account involves a *reductionist* feature since logical consequence is "explained" in terms of (material) truth. Of course, Tarski had already provided an adequate account for truth and hence he comes up with an extraordinarily valuable definition. However, the philosophically inclined logician is left to wonder to what extent the definition provided may also serve as a good analysis of the intended concept of logical consequence.

The pre-Tarskian tradition of logic has continually attempted to provide a necessary and sufficient condition for an argument to be valid. However, both the no expressible countermodels view of validity which also seems to be the main point of Bolzano's contribution[16] and the *fixed-domain variable content* view, shared by Russell and also by Lewis and Langford, are also reductionist generalization accounts in the previous sense. Nevertheless, the significance of Tarski's approach should not be overlooked. He provided an overall correct semantic theory for the logical and extra-logical terms as opposed to the mere elucidations of his predecessors. In addition he made clear that all extra-logical terms are to be varied, whereas none of his predecessors set up clear conditions for the replacement of extra-logical terms by variables[17].

Despite the fact that the information containment conception, the necessity conception and the impossibility conception of logical consequence were widely employed by logicians, both traditionally and mathematically oriented, Tarski's 1936 paper conspicuously avoids explicit mention of them. In fact, there are only a few passages that involve modal expressions and none of these makes reference to any of the three conceptions of consequence. (See specially Tarski [29, p. 411, l. 6d and p. 415, l. 13d and p. 414, l. 10u].)

The fact of the matter is that the semantics of modal words in ordinary English is difficult to elucidate. However most of Tarski's modal wordings in the text seem to serve rather a purely rhetorical purpose and are not indicative of any philosophical commitment. Let us recall that Tarski's aim in his paper is to supersede a "common" notion of consequence in the same fashion as his definition of truth was intended to replace the corresponding pre-formal

[16] Tarski's definition also extends Bolzano's attempt in that Bolzano only considered argument-texts with consistent premise-sets. See Thompson [42].

[17] For a detailed historical account see Corcoran [6, Sections 6 and 7].

vague notion. Modal expressions such as 'it can never happen' or 'must' are used for emphatic purposes and they are simply superimposed over the strict definitional wording, either to take part in Tarski's explications, or to shape the informal way of restating accomplished results. One is inclined to say they are mere *vestiges* of the ordinary way of referring to the pre-formal notion of logical consequence.

However, it is clear that this modal rhetoric reflects upon the level of generalization gained by the formality requirement embodied in his fixed-domain variable content viewpoint. It seems plausible then to conjecture that Tarski saw his definition as a reconstructive refinement of a previously established conception, perhaps common among some mathematically oriented logicians. Logical consequence in Tarski's sense, as opposed to material consequence, requires preservation of truth in *every* sequence in the framework of *Principia* and not logical necessity or any other modality. Surely this seems less philosophically problematic and more clear than the modalities contained in the three conceptions mentioned above.

Tarski [31, pp. 8, 11, and 29] uses the expressions 'possible realization' and 'possible definition'. Again, in my opinion, the context of the discussion indicates no loaded modal meaning attached to the word 'possible'. On the contrary, Tarski seems rather to be referring to a selection of an option among many already existent options, whether known or unknown. Tarski [30, p. 22] explicitly claims that expressions such as 'follows necessarily' reflect a kind of psychological compulsion to assume the consequences of propositions already assumed, reflecting the absence of knowledge of the truth-values of the premises and the conclusion[18]. We also find some passages in the same vein in Carnap's autobiography which suggest that Tarski was similar to Quine in his avoidance of modal notions. Thus, once again, it seems unlikely that Tarski had either the containment conception, or the necessity conception, or the impossibility conception of consequence in mind. Moreover, he makes no special effort to bring this feature of his conception to the attention of the reader. In fact, he never says that logical consequence is a modal notion or that it involves necessity in any sense. One can only conclude that these conceptions were not his focus. In a word, Tarski's no-countermodels definition is non-modal, perhaps even philosophically *de-modalized*[19].

To conclude, I would like briefly to refer to Tarski's more recent account of logical consequence. Tarski's view on logical consequence undergoes a

[18] I am very grateful to John Corcoran for calling my attention to this quite illustrative quotation concerning Tarski's opinions about modal notions.

[19] The reader may consider the following quotations, which are extracted from Tarski [35] "Truth and Proof", which is also written in a colloquial, non-technical style. These examples are self-contained and they exhibit the same kind of figurative, non-literal use of modal expressions next to the corresponding non-modal expression:

change between the thirties and the fifties. In Tarski [31, 32, 33, 34] a set-theoretic viewpoint is exhibited. In this view, it is irrelevant whether the language is interpreted or not. Interpretations are set-theoretic objects, namely, elements of the universe of pure sets. This view presupposes an ontology of sets. Validity of a given argument-text amounts to the non-existence of a certain sort of set that provides for a countermodel, and invalidity amounts to the existence of such a set that provides for a countermodel. We may ask whether the problem of giving a necessary and sufficient condition for logical consequence is solved under this account. Corcoran [4, p. 43] points out that Tarski might have considered a necessary condition for validity to be both necessary and sufficient : " . . . [for Tarski] the invalidity of an argument depends on *the existence* of a suitable domain and there might not be 'enough' domains to provide 'counter interpretations' for all invalid arguments".

The account of the fifties reduces logical consequence to set theory. This is a different and peculiar *reductionist* move since it suggests a kind of ontological circularity. Logic is traditionally understood as the science underlying all the sciences, including set theory. Although the mathematical character of logic is as well established as any other mathematical discipline like physics or chemistry, I believe the reduction suggested involves two problems. The first is whether standard versions of set-theory contain only true principles; particularly whether the axiom of infinity is true. Second, if the principles of set theory are true, then either they are materially true or their truth involves some kind of necessity. In the first case, validity would be dependent on material truth about sets. In the second case, validity would be dependent on a kind of necessary truth, which I believe quite a few logicians would take to be weaker than logical necessity. In other words, the realm of mathematical possiblity seems to be smaller than the realm of logical possibility. In either case, the formality requirement of the concept of logical consequence has become blurred insofar as the full meaning of this condition really makes sense when *all* formally similar situations are accounted for. A way out of this peculiar situation is to interpret that in the fifties Tarski was not giving

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"For several reasons an expression enclosed in quotes must be treated gramatically as a single word having no syntactical parts. Hence the word 'snow', which undoubtedly occurs in the definiendum as a part, does not occur there as a syntactical part." (p. 64).

"However, words which are not syntactical parts of the definiendum cannot create a vicious circle, and the danger of a vicious circle vanishes" (p. 64).

In both passages the first sentence contains a modal expression not intended to indicate any sort of logical necessity, while the second is a non-modal sentence expressing a closely related idea. The next example shows a non-modal sentence followed by a modal one:

"The result shows indeed that in no domain of mathematics is the notion of provability a perfect substitute for the notion of truth [. . .]. The notion of a true sentence functions thus as an ideal limit which can never be reached but which we try to approximate by gradually widening the set of provable sentences."

an explication of logical consequence but rather giving a mathematical analogue or a model of logical phenomena, undoubtly fruitful for mathematical purposes. In other words, what is suggested here is that in his more recent account Tarski is not saying what logical consequence is but rather what logical consequence *is like*[20].

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REFERENCES

[1] R. Carnap, ***The logical syntax of language***, 8th ed., London, Routledge and Kegan Paul, 1937/71.

[2] ———, *Foundations of logic and mathematics.*, ***International encyclopedia of unified science*** (O. Neurath et al., editors), University of Chicago Press, 1938, Section 7, pp. 139–213.

[3] A. Church, ***Introduction to mathematical logic***, 6th ed., Princeton University Press, Princeton, 1956/70.

[4] J. Corcoran, *Conceptual structure of classical logic*, ***Philosophy and Phenomenological Research***, vol. 33 (1972), pp. 25–47.

[5] ———, *Gaps between logical theory and mathematical practice*, ***The methodological unity of science*** (M. Bunge, editor), Reidel Publishing Company, Dordrecht-Holland, 1973, pp. 23–50.

[6] ———, *Meanings of implication*, ***Dialogos***, vol. 9 (1973), pp. 59–76, Spanish translation by José M. Sagüillo in ***Agora***, vol. V (1985), pp. 279–294; reprinted in [15, pp. 85–100].

[7] ———, *Categoricity*, ***History and Philosophy of Logic***, vol. 1 (1980), pp. 187–207.

[8] ———, *Argumentations and logic*, ***Argumentation***, vol. 3 (1989), pp. 17–43, Spanish translation by Rubén Blanco in ***Agora***, vol. XIII/1 (1994), pp. 27–55.

[9] ———, *Information recovery problems*, ***Theoria***, vol. 24 (1995), pp. 55–78.

[10] ———, *Information-theoretic logic*, ***Truth in perspective. Proceedings of the international congress "Truth: logic, representation and world"***, Ashgate Publishing Limited, Aldershot, England, United Kingdom, 1997, forthcoming.

[20] For a detailed philosophical account of the mathematical character of logic, see Corcoran [5].

[11] J. Etchemendy, ***The concept of logical consequence***, Harvard University Press, Cambridge, Massachusetts, 1990.

[12] L. Fernández Moreno, *Neutralidad temática y la delimitación de las constantes lógicas en Carnap, Tarski y Quine*, ***Contextos***, vol. XI (1993), nos. 21–22, pp. 59–75.

[13] M. García-Carpintero, *The grounds for the model-theoretic account of the logical properties*, ***Notre Dame Journal of Formal Logic***, vol. 34 (1993), pp. 107–131.

[14] W. Hodges, *Truth in a structure*, ***Proceedings of the Aristotelian Society***, vol. 86 (1986), pp. 135–151.

[15] R. I. G. Hughes, ***A philosophical companion to first order logic***, Hackett, Indianapolis & Cambridge, 1993.

[16] D. H. Mellor, ***F. P. Ramsey. Philosophical papers***, Cambridge University Press, 1990.

[17] D. Prawitz, *Remarks on some approaches to the concept of logical consequence*, ***Synthese***, vol. 62 (1985), pp. 153–171.

[18] W. v. O. Quine, *The problem of interpreting modal logic*, ***Journal of Symbolic Logic***, vol. 12 (1947), pp. 43–48.

[19] ———, *Immanence and validity*, included in [20, pp. 242–50], 1991.

[20] ———, ***Selected logic papers***, Harvard University Press, Cambridge, Massachusetts & London, England, 1995.

[21] F. Ramsey, *The foundations of mathematics.*, included in [16, pp. 164–224], 1925.

[22] B. Russell, ***Introduction to mathematical philosophy***, 3rd ed., Dover Publications, New York, 1919/93.

[23] J. M. Sagüillo, *Validez y semántica representacional*, ***Theoria***, vol. 24 (1995), pp. 103–120.

[24] G. Sher, ***The bounds of logic***, MIT Press, Cambridge, Massachusetts, 1991.

[25] A. Tarski, *Fundamental concepts of the methodology of the deductive sciences*, included in [36, pp. 60–109], 1930.

[26] ———, *The concept of truth in formalized languages*, included in [36, pp. 152–278], 1933.

[27] ———, *On some fundamental concepts of metamathematics*, included in [36, pp. 30–37], 1933.

[28] ———, *Some observations on the concepts of ω-consistency and ω-completeness*, included in [36, pp. 279–295], 1933.

[29] ———, *On the concept of logical consequence*, included in [36, pp. 409–20], 1936.

[30] ———, ***Introduction to logic***, Oxford University Press, 1941/94, 4th ed. by J. Tarski.

[31] ———, *A general method in proofs of undecidability*, included in [41, pp. 3–35], 1953.

[32] ———, *Contributions to the theory of models I*, included in [37, pp. 515–526], 1954.

[33] ———, *Contributions to the theory of models II*, included in [37, pp. 527–535], 1954.

[34] ———, *Contributions to the theory of models III*, included in [37, pp. 537–547], 1955.

[35] ———, *Truth and proof*, ***Scientific American***, vol. 220 (1969), no. 6, pp. 63–77, reprinted in [15, pp. 101–125].

[36] ———, ***Logic, semantics, metamathematics***, Hackett Publishing Company, Indianapolis, 1983, 2nd ed. edited with an introduction by John Corcoran.

[37] ———, ***Alfred Tarski: Collected papers. Volumen 3, 1945–1957*** (S. R. Givant and R. N. Mckenzie, editors), Birkhäuser, Basel, Boston, Stuttgart, 1986.

[38] ———, *What are logical notions*, (introduced and edited by John Corcoran), ***History and Philosophy of Logic***, vol. l7 (1986), pp. 143–154.

[39] ———, *A philosophical letter of Alfred Tarski*, ***Journal of Philosophy***, vol. 84 (1987), pp. 28–32, from 1944.

[40] A. Tarski and A. Lindenbaum, *On the limitations of the means of expression of deductive theories*, included in [36, pp. 384–392], 1934–35.

[41] A. Tarski, A. Mostowski, and R. Robinson, ***Undecidable theories***, North-Holland Publishing Company, Amsterdam, 1953.

[42] P. Thompson, *Bolzano's deducibility and Tarski's logical consequence*, ***History and Philosophy of Logic***, vol. 2 (1981), pp. 11–20.

[43] L. Vega, *Alfred Tarski* (1936)*: sobre el concepto de consecuencia lógica*, 1984, included in [44].

[44] L. Vega and P. Castrillo (editors), ***Lecturas de lógica II***, UNED, Madrid, 1984.

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