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A MODEL-THEORETIC APPROACH TO ORDINAL ANALYSIS

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Abstract. We describe a model-theoretic approach to ordinal analysis via the finite combinatorial notion of an α-large set of natural numbers. In contrast to syntactic approaches that use cut elimination, this approach involves constructing finite sets of numbers with combinatorial properties that, in nonstandard instances, give rise to models of the theory being analyzed. This method is applied to obtain ordinal analyses of a number of interesting subsystems of first- and second-order arithmetic.

§1. Introduction. Two of proof theory's defining goals are the justification of classical theories on constructive grounds, and the extraction of constructive information from classical proofs. Since Gentzen, ordinal analysis has been a major component in these pursuits, and the assignment of recursive ordinals to theories has proven to be an illuminating way of measuring their constructive strength. The traditional approach to ordinal analysis, which uses cut-elimination procedures to transform proofs in various deductive calculi, has a very syntactic flavor. The goal of this paper is to describe an alternative, model-theoretic approach, one that we hope will find favor with mathematicians of a more semantic bent. Basically these techniques are modifications of known ones, but new here is the adaptation of these techniques to second-order theories.

The origins of our approach can be found in the 1970's, in which Paris and others [13, 15] explored the use of finite combinatorial principles that, in nonstandard instances, give rise to models of arithmetic. A crowning achievement of this pursuit is the Paris-Harrington statement [16, 10, 11], a slight variant of the Ramsey's theorem for finite sets that is equivalent to the 1-consistency of Peano Arithmetic (PA). Ketonen and Solovay [12] later developed the notion of an α-large set for ordinal notations α, and used it to determine effective bounds on the numbers asserted to exist by Paris and Harrington's combinatorial statement. In [15] Paris showed how to use the notion of α-large interval to build models of the theories $I\Sigma_n$ for $n \geq 1$ (i.e., the theories obtained by restricting the induction axioms of PA to Σ^0_n-formulas). In [30, 32] the second author extended these methods to apply

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to fragments of PA based on transfinite induction, yielding sharp upper bounds on their proof-theoretic strength; similar work has been carried out by Kotlarski and Ratajczyk (see [14, 19]). As a by-product, these methods also provide natural "indicators" (see [10, 11, 15, 13]) for the theories in question.

In this paper we hope to convince the reader that such combinatorial methods provide an important alternative to cut elimination when it comes to ordinal analysis. The complementarity of the approach can be summarized as follows: with cut-elimination, one unwinds proofs to obtain cut-free proof trees with height bounded by an ordinal; in our constructions, one *starts* with an ordinally-large interval and uses that to construct a model of the theory in question.

The outline of this paper is as follows. In Sections 2 through 6 we present an overview of our methods, set down preliminary definitions and conventions, and discuss the way in which our constructions yield ordinal analyses. In Section 7 we construct a model of the first-order theory $I\Sigma_1$, and in Section 8 we discuss the construction of second-order objects like Turing jumps. In Section 9 we use these ideas to build a model of WKL_0. A suitable iteration of the Turing jump construction gives us finite jump hierarchies, which are useful in constructing models of $I\Sigma_n$, PA, ACA_0, and Σ^1_1- AC_0 in Sections 10 to 12.

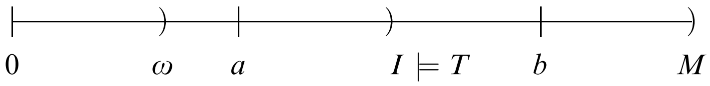
All the theories analyzed in this paper have proof-theoretic strength at most that of Peano Arithmetic. These methods, however, are extended to stronger "predicative" theories in [4], where we use appropriately large nonstandard intervals to build models of theories of strength up to Γ_0 (and, in fact, just a little bit beyond). These constructions employ a transfinite jump lemma that extends the finite jump lemma introduced here.

For more information on the traditional ordinal analyses of the theories discussed here and in [4], as well as proofs that the bounds we give are sharp (obtained by proving instances of transfinite induction within the theories themselves), see, for example, [17, 18, 20, 24, 6, 25]. For information on theories of first-order arithmetic, see [10, 11], and for more information on the relevant theories of second-order arithmetic, see, for example, [26, 27, 28, 6, 2, 3].

§**2. Overview.** In this section we give an informal introduction to the model-theoretic techniques we will use below. Given a theory T, our goal is to determine an ordinal notation α that provides an upper bound to its proof-theoretic strength. If T is a theory in the language of first-order arithmetic, we will use the combinatorial notion of an α-large interval of natural numbers $[a, b]$ to construct a model of T. In fact, we'll show that if $\mathcal{M}$ is a nonstandard model of arithmetic with universe M, and a and b are nonstandard elements such that

$$\mathcal{M} \models [a, b] \text{ is } \alpha\text{-large}$$

then there is an initial segment I of M containing a but not b, such that I is a model of T. This situation is described by the diagram below.



If b is taken to be the least element of M such that $[a, b]$ is α-large, this yields a model I of T in which the combinatorial assertion

$$\forall x \, \exists y \, ([x, y] \text{ is } \alpha\text{-large}) \tag{1}$$

is false, and hence this assertion is not provable in T. On the other hand, in a weak base theory one can prove (1) using transfinite induction up to α, and so as a corollary we obtain an instance of transfinite induction up to α that cannot be proven in T. This is one sense in which the strength of T is bounded by α, and is one of the usual consequences of a traditional ordinal analysis. We will discuss some of the other consequences of an ordinal analysis, and the way they can be obtained from our model-theoretic constructions, in Section 6.

In order to construct the initial segment I that models T, we will use the combinatorial properties of $[a, b]$ to construct a set

$$A = \{a_0, a_1, \dots, a_k\} \subseteq [a, b]$$

with further combinatorial properties that guarantee that if I is any "limit" of A, I will satisfy the axioms of T. For example, if the elements a_i have been listed in increasing order and k is nonstandard, then the set

$$I = \bigcup_{i \in \omega} \{x \mid x < a_i\}$$

will be an initial segment of $\mathcal{M}$ in which elements of A occur cofinally, and will therefore serve our purposes.

In this paper we will be concerned moreover with theories T in the language of second-order arithmetic, which include variables that range over sets of natural numbers. To construct a model of T, we will have to specify the universe of sets that are to interpret these variables as well. Given a set S coded by a single element of the first-order model $\mathcal{M}$ in some reasonable way, we will write

$$S^I = S \cap I,$$

so that S^I is a subset of I. All the sets in our second-order models will be of this form, and so our task will be to come up with combinatorial conditions on sets S that guarantee that S^I will satisfy appropriate axioms of T in any limit I of A. When such is the case we will say that S "approximates" the desired property in A.

In short, then, our constructions take the following form:

(1) we show that if $[a, b]$ is α-large, we can construct sets A, S, and so on, having certain combinatorial properties; and

(2) we show that if I is any limit of A, these combinatorial properties enable us to extract a model of T in which the first-order universe is I.

After fixing some conventions in the next section, we will discuss the relevant ordinals and notation systems in Section 4, define the notion of an α-large interval in Section 5, and describe the proof-theoretic consequences of our techniques in Section 6. The first application of the techniques themselves will appear in Section 7, when we begin by building a model of $I\Sigma_1$.

§**3. Preliminaries.** Though the following list of preliminaries regarding models and theories of arithmetic is long, most of the definitions and conventions are either standard or easily deduced from context. As a result, the reader may want to just skim this section and return to it as necessary. For more detail on the topics of this section the reader is referred to [10, 11].

The language of first-order arithmetic includes a constant zero symbol, function symbols for the operations successor, plus, and times, and a less-than relation. When we speak of **true arithmetic**, we mean the set of sentences in this language that are true in the standard model. As described in the previous section, we need to construct objects in a nonstandard model of true arithmetic, so we fix a particular such model $\mathcal{M}$ from the outset. M will be used to denote the universe of $\mathcal{M}$.

An **initial segment** of $\mathcal{M}$ is a subset of I of M that is closed downwards, that is, if $a \in I$ and $b < a$, then $b \in I$. An initial segment I is a **cut** if it is closed under the successor operation, and **proper** if it is not equal to all of M. Note that if I is a proper cut of $\mathcal{M}$ then I is not definable in $\mathcal{M}$, since any first-order formula defining it would represent a failure of induction in the model.

Though the variables of the language of first-order arithmetic range over natural numbers, modulo coding we can take the universe to include finitary objects like ordered pairs, finite sets, and finite sequences as well. For example, if S is an element of M, we can interpret $i \in S$ to represent the assertion that "the ith bit of the binary expansion of S is equal to 1." In this situation we will say that the set S is **coded in** $\mathcal{M}$, or even more concisely, S is **in** $\mathcal{M}$. Similar conventions hold for other finitary objects as well.

If A is a subset of M and we write

$$A = \{a_0, a_1, \dots, a_k\}$$

it is to be assumed that we have listed the elements of a_i in increasing order. A cut I is a **limit** of A if elements of A occur cofinally in I. For example, if k is nonstandard and we take

$$I =_{\mathrm{def}} \bigcup_{i \in \omega} \{x | x < a_i\}$$

then I will be a limit of A. If I is a cut containing a but not b, we can indicate this fact by writing

$$a < I < b.$$

If I is also closed under plus and times, we will also use I to denote the model for the language of first-order arithmetic, with the operations that are derived from $\mathcal{M}$.

We will use the notation $[a, b]$ to denote the set $\{a, a+1, \dots, b\}$, with similar conventions for open and half-open intervals.

The symbols Σ^0_i, Π^0_i, and Δ^0_i represent the usual classes in the arithmetic hierarchy. In particular, a formula φ is Δ^0_0, or **bounded**, if all its quantifiers are of the form $\forall x \leq t$ or $\exists x \leq t$, where t is a term which doesn't involve the variable x. If φ is Δ^0_0 in an additional set parameter S, we will write that φ is $\Delta^0_0(S)$. An easy induction on formula complexity shows that if I is a cut of $\mathcal{M}$ that is closed under plus and times, and φ is a Δ^0_0 formula, then φ is **absolute** between I and $\mathcal{M}$ in the following sense.

LEMMA 3.1. *If $\varphi(\vec{x})$ is Δ^0_0, I is a cut of $\mathcal{M}$ closed under plus and times, and $\vec{a}$ are parameters from I, then*

$$(I \models \varphi(\vec{a})) \Leftrightarrow (\mathcal{M} \models \varphi(\vec{a})).$$

We will use $\langle a, b\rangle$ to denote an ordered pair with elements a and b, and if s is a sequence in $\mathcal{M}$ we will let s_i denote the ith element of s. If S is a set in $\mathcal{M}$, we use the notation S_i to denote the ith slice of S, namely

$$S_i =_{\mathrm{def}} \{a \mid \langle i, a\rangle \in S\}.$$

Though this introduces some ambiguity between sequences and sets, we trust that our intentions will be clear in context. In fact, if $\langle T_i\rangle_{i \in I}$ is a sequence of sets indexed by elements of another set I, we can define

$$\bigoplus_{i \in I} T_i =_{\mathrm{def}} \{\langle i, x\rangle \mid x \in T_i\} = \bigcup_{i \in I} (\{i\} \times T_i),$$

so that if $S = \bigoplus_{i \in I} T_i$ then the projection S_i is equal to T_i whenever $i \in I$, and $\emptyset$ otherwise. Finally, if $\prec$ is some ordering definable in $\mathcal{M}$ and S is a sequence of sets, we define

$$S_{\prec a} =_{\mathrm{def}} \bigoplus_{b \prec a} S_b.$$

It is important that we choose a coding scheme with definitions whose basic properties can be verified in a weak theory like $I\Delta_0 + (\exp)$ (see below), and such that if I is a cut of $\mathcal{M}$ that satisfies this theory, the definitions are absolute between I and $\mathcal{M}$. Finding coding schemata with these properties is not difficult; see, for example, [10].

If S is a set in $\mathcal{M}$ and I is a cut of $\mathcal{M}$, we define

$$S^I =_{\text{def}} S \cap I,$$

namely, the part of S "seen" by I. A slight extension of Lemma 3.1 yields

LEMMA 3.2. *If $\varphi(\vec{x}, Y)$ is $\Delta_0^0(Y)$, I is a cut of $\mathcal{M}$ closed under plus and times, $\vec{a}$ are parameters in I, and S is a set in $\mathcal{M}$, then*

$$(I \models \varphi(\vec{a}, S^I)) \Leftrightarrow (\mathcal{M} \models \varphi(\vec{a}, S)).$$

It will often be convenient to drop the $\cdot^I$ superscript in S^I, as discussed in Section 8. For S and a in $\mathcal{M}$, we define

$$S^a =_{\text{def}} \{x < a \mid x \in S\}.$$

Note that this agrees with the definition for S^I if we identify a with the set of all elements less than a. We use μ to denote the least-number operator, so that

$$\mu x\ \theta(x)$$

denotes the least x such that $\theta(x)$, if such an x exists, and 0 otherwise. The bounded least-number operator

$$\mu x \leq a\ \theta(x)$$

does not look past a for a witness.

Below we adopt the practice of naming axioms and axiom schemata with parentheses, so for example if Γ is a class of formulas we will use (Γ-IND) to denote the schema of induction

$$\varphi(0) \wedge \forall x\ (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\ \varphi(x)$$

for formulas φ in Γ. The theory $I\Delta_0 + (\exp)$ is a weak base theory which includes quantifier-free defining equations for successor, plus, times, and less-than, the schema (Δ_0^0-IND) of induction for Δ_0^0, and an axiom asserting that exponentiation is total. To express this latter axiom one needs to use a Δ_0^0 formula defining the graph of exponentiation in a reasonable way; see [10] for details.

In $I\Delta_0 + (\exp)$ formulas ψ can be coded as elements $\ulcorner\psi\urcorner$. If ψ contains numeric parameters we can code them using their binary (or dyadic) representations, so that $\ulcorner\psi\urcorner$ becomes an $I\Delta_0 + (\exp)$-definable function of the parameters in ψ.

We fix a universal Σ^0_1 truth predicate

$$\mathrm{Tr}_{\Sigma^0_1}(x) \equiv_{\mathrm{def}} \exists y\ \Theta(x, y),$$

where Θ is Δ^0_0, so that for any Δ^0_0 formula $\psi(y)$ the equivalence

$$\exists y\ \psi(y) \leftrightarrow \mathrm{Tr}_{\Sigma^0_1}(\ulcorner \exists y\ \psi(y) \urcorner) \tag{2}$$

holds in any model of $I\Delta_0 + (\exp)$. After Section 7 we will need to use a formula

$$\mathrm{Tr}_{\Sigma^0_1}(x, Z) \equiv_{\mathrm{def}} \exists y\ \Theta(x, y, Z),$$

which is a Σ^0_1 truth predicate relative to the set parameter Z. Such truth predicates are defined, for example, in [10, 11].

We are also concerned with building models of theories in the language of second-order arithmetic, which includes variables that range over sets of numbers and a binary "element-of" relation $\in$. In this language we take only first-order equality to be basic, defining $X = Y$ to mean

$$\forall z\ (z \in X \leftrightarrow z \in Y).$$

We can specify a structure $\mathcal{N}$ for the language of second-order arithmetic by presenting a first-order part $\mathcal{K}$, and a collection $\mathcal{S}$ of subsets of the universe of $\mathcal{K}$ to interpret the second-order variables. In such a situation we will write

$$\mathcal{N} = \langle \mathcal{K}, \mathcal{S} \rangle.$$

In the second-order setting we allow Σ^i_j, Π^i_j, and Δ^i_j formulas to contain second-order parameters, so that when we say that $\mathcal{N}$ satisfies $I\Delta_0 + (\exp)$ we mean to include induction for Δ^0_0 formulas in the expanded language.

If $\varphi(x)$ is a formula the least-element principle for φ is the assertion

$$\exists x\ \varphi(x) \rightarrow \exists x\ (\varphi(x) \wedge \forall y < x\ \neg\varphi(y)), \tag{LEP}$$

which asserts that if there is any x satisfying φ, then there is a least such x. By (Γ-LEP) we denote the schema in which this principle is applied for formulas φ in Γ. The relationship between induction and the least-element principle is given by the following.

Lemma 3.3. *Over* $I\Delta_0 + (\exp)$, *for fixed i and k, the following schemata are pairwise equivalent*:

(1) Σ^i_k-IND.
(2) Π^i_k-IND.
(3) Σ^i_k-LEP.
(4) Π^i_k-LEP.

For a proof of this lemma, see [10, 11].

§4. Ordinals and ordinal notations. In discussing the role of ordinals in proof theory, it is useful to distinguish between the following concepts:

(1) A **countable ordinal** is an isomorphism class of countable well-orderings. As usual, we can identify countable ordinals with their von Neumann representations as transitive sets that are linearly ordered by the membership relation.

(2) An **ordinal notation system** is a pre-well-ordering $\preceq$ on a set U of terms in a specified language. Intuitively speaking, the elements of U are **notations** that denote countable ordinals. Each notation α is identified with its Gödel number $\ulcorner\alpha\urcorner$. Typically it will be clear from the context whether α refers to a notation or its Gödel number; for example, if we refer to α as an element of M then we mean the Gödel number of the notation, and if we refer to the "form of α" (in terms of the function and constant symbols used to build the notations in U) then α denotes the notation itself.

Saying that the relation $\preceq$ is a *pre*-well-ordering means that it is transitive and reflexive, and that there are no infinite descending $\preceq$-chains. What distinguishes a pre-well-ordering from a well-ordering is that in the former there may be more than one notation for a given ordinal, since

$$\alpha \preceq \beta \wedge \beta \preceq \alpha$$

may hold for distinct α and β. From such sets of equivalent notations, it is often useful to identify a canonical **normal form** representative, which we will write as $\overline{\alpha}$. We will use $|\alpha|$ to denote the order-type of α in the associated well-ordering on equivalence classes of notations, and define $\alpha \prec \beta$ to mean that $\alpha \preceq \beta$ but $\beta \not\preceq \alpha$.

If the set (of Gödel numbers of the elements of) U, the ordering $\preceq$ (as an ordering on Gödel numbers), and the map $\alpha \mapsto \overline{\alpha}$ (as a function on Gödel numbers) are primitive recursive (or elementary recursive, polynomial-time computable, etc.), it makes sense to call the ordinal notation system primitive recursive (resp. elementary recursive, polynomial-time computable, etc.).

(3) An **ordinal notation system with limit sequences** is an ordinal notation system together with an assignment of a cofinal sequence

$$\alpha[0], \alpha[1], \alpha[2], \ldots$$

to every notation α that denotes a limit ordinal. We do not require that equivalent notations have equivalent limit sequences, or that the elements of a limit sequence assigned to a notation in normal form are again in normal form.

If the ordinal notation system is primitive recursive, and the assignment $\langle \lambda, n \rangle \mapsto \lambda[n]$ is primitive recursive, we can say that the ordinal notation system with limit sequences is primitive recursive, etc.

It is usually desirable that basic ordinal functions such as successor, addition, multiplication, exponentiation, and so on are also easily computable in the context of the notations. For the ordinals dealt with in this paper and in [4] it is fairly easy to see that there are elementary recursive notation systems such that the functions in this list are also elementary recursive. Further, it is not difficult to prove simple properties about these functions (algebraic properties and defining equations) in $I\Delta_0$. In [31] it is shown that we can find such notation systems for any fixed recursive ordinal.

In this section we will begin by defining a series of ordinals and functions on ordinals from a classical (set-theoretic) point of view, and then point out that in a very standard way, one can use these functions to define terms which denote ordinals. Finally, we will define limit sequences for these notations. The combinatorial properties of α-large sets, defined in the next section, will depend very heavily on our choice of limit sequences, so we will be careful to choose sequences that facilitate our later constructions. On the other hand, the point of these constructions is to obtain the types of proof-theoretic results described in Section 6, which, in contrast, refer only to countable ordinals and ordinal notation systems. One can therefore think of our choice of limit sequences, and the resulting definition of an α-large set, as a convenient stepping-stone to these end results.

The following “refresher course” on ordinals is somewhat brisk. A more detailed development can be found in [17, 18, 24].

Ordinal addition, multiplication, and exponentiation are defined by transfinite recursion in the usual way. For example, the function $\alpha \mapsto \beta^\alpha$ is defined by the equations

- $\beta^0 =_{\text{def}} 1$
- $\beta^{\alpha+1} =_{\text{def}} \beta^\alpha \cdot \beta$
- $\beta^\lambda =_{\text{def}} \sup \{\beta^\gamma \mid 0 < \gamma < \lambda\}$, for limit ordinals λ.

Note that every ordinal α satisfies one of the following three criteria:

- $\alpha = 0$
- $\alpha = \beta + 1$, in which case we say that α is a **successor** ordinal
- α is not equal to 0 and there is no greatest $\beta < \alpha$, in which we say that α is a **limit ordinal**.

The ordinal ω corresponds to the order-type of the natural numbers.

The sequence of ordinals ω_n is defined inductively by the following equations:

- $\omega_0 =_{\text{def}} 1$
- $\omega_{n+1} =_{\text{def}} \omega^{\omega_n}$.

The ordinal ε_0 is defined to be the limit of this sequence. One can show that ε_0 is the least fixed point of the function $\alpha \mapsto \omega^\alpha$, i.e., the least ordinal β such that $\omega^\beta = \beta$. Furthermore, ε_0 is the least ordinal closed under addition and the map $\alpha \mapsto \omega^\alpha$, and any ordinal less than ε_0 can obtained from 0 and finitely many applications of these two operations.

The ordinal ε_0 is sufficient for the analysis of all the theories dealt with in this paper. However, for the theories analyzed in [4], we will need ordinals that are larger. To go beyond ε_0, define the sequence ω_n^α by

- $\omega_0(\alpha) =_{\text{def}} \alpha$
- $\omega_{n+1}^\alpha =_{\text{def}} \omega^{\omega_n^\alpha}$,

and let ε_1 be the limit of the sequence $\omega_n^{\varepsilon_0+1}$. Then ε_1 is the second fixed-point of the operation $\alpha \mapsto \omega^\alpha$, and equal to the set of ordinals that can be obtained from 0 and ε_0 from addition and the preceding operation. More generally, define

- $\varepsilon_{\alpha+1} = \lim_n \omega_n^{\varepsilon_\alpha+1}$
- $\varepsilon_\lambda = \sup \{\varepsilon_\gamma \mid \gamma < \lambda\}$, for limit ordinals λ,

so that ε_α denotes the αth fixed point of the map $\alpha \mapsto \omega^\alpha$, as well as the set of ordinals that are obtainable from $\{\varepsilon_\beta \mid \beta < \alpha\}$ using finitely many applications of addition and the map $\alpha \mapsto \omega^\alpha$.

This process can be iterated transfinitely to give the Veblen hierarchy of functions, φ_α, defined as follows:

- $\varphi_0(\alpha) = \omega^\alpha$
- $\varphi_{\alpha+1}$ enumerates the fixed points of φ_α
- φ_λ enumerates the simultaneous fixed points of $\{\varphi_\gamma \mid \gamma < \lambda\}$, when λ is a limit.

Note that $\varphi_1(\alpha)$ is just the ordinal ε_α defined in the previous paragraph. It will often be convenient to write $\varphi(\alpha, \beta)$ instead of $\varphi_\alpha(\beta)$.

We can continue the process even further, this time diagonalizing across the first argument of φ. In analogy to the ordinals ω_n, define

- $\gamma_0 =_{\text{def}} \varepsilon_0$
- $\gamma_{n+1} =_{\text{def}} \varphi(\gamma_n, 0)$.

Then the Feferman-Schütte ordinal Γ_0 is defined to be the limit of the sequence γ_n. Alternatively, Γ_0 can be characterized as the least fixed point of the function $\alpha \mapsto \varphi(\alpha, 0)$, the smallest ordinal closed under the map $\alpha, \beta \mapsto \varphi(\alpha, \beta)$, and the set of ordinals that can be obtained from 0 using finitely many applications of addition and the function $\alpha, \beta \mapsto \varphi(\alpha, \beta)$. Also define

- $\gamma_0^\alpha =_{\text{def}} \alpha$
- $\gamma_{n+1}^\alpha =_{\text{def}} \varphi(\gamma_n^\alpha, 0)$.

The ordinals Γ_α are then defined analogously to the ordinals ε_α. More precisely, if we set $\psi(\alpha) = \varphi(\alpha, 0)$, the ordinals Γ_α enumerate the fixed points of ψ.

Of course, one can continue this process of generating ordinals indefinitely. Obtaining ordinals that suffice for the analysis of stronger theories requires new conceptual methods, some even motivated by large cardinal hypotheses. (See, for example, [17, 18, 20, 21, 22].) The ordinals and functions just described, however, suffice for the theories we analyze below and in [4]. In fact, for the theories analyzed in this paper it is enough to have notations for ordinals through ε_0, and so for now we restrict our attention to these.

DEFINITION 4.1. Our set of **ordinal notations** is defined inductively, as follows:

- 0 is an ordinal notation.
- If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are ordinal notations other than 0, then so is

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

- If α is an ordinal notation, so is ω^α.
- ε_0 is an ordinal notation.

Notations of the form $\alpha + 1$ (that is, $\alpha + \omega^0$) are called **successor** notations. A notation that is neither 0 nor a successor notation is called a **limit** notation.

Our treatment of ordinal addition violates unique readability, since, for example, the term $\alpha + \beta + \gamma$ can be interpreted by associating to the left or to the right. As it turns out, blurring this distinction is convenient, and one can check that the definitions and proofs below are insensitive to the way such a term is parsed. When we refer to notations such as 1, 2, ω_n, and so on, these are to be taken as abbreviations for their representations using 0, $+$, and $\omega^{\cdot}$. In particular, $\alpha \cdot n$ denotes the term

$$\alpha + \alpha + \cdots + \alpha$$

in which there are n terms in the sum.

Terms α denote ordinals $|\alpha|$ under the intended interpretation. We would like to emphasize that these representations are not unique, so that, for example, $|1 + \omega| = |\omega|$ and $|\omega^{\varepsilon_0}| = |\varepsilon_0|$. As described in item 2 at the beginning of this section, we identify a normal form $\overline{\alpha}$ with each ordinal notation α. When we use $\overline{\alpha}$ as a variable, we mean that this variable is intended to range over notations in normal form.

Both the mapping of notations to their normal forms and the ordering $\prec$ on notations induced by the intended interpretation, can be described in a very effective way [31, 32]. Oddly enough, the details of the ordering are irrelevant to our model-theoretic constructions, which rely instead on the limit sequences we define below. In contrast, the proof-theoretic results described in Section 6 refer only to the ordering of notations, and not the limit sequences. The bridge between these two aspects of the analysis is given by the following fact: if λ denotes a limit ordinal and $\lambda[n]$ is an element of the limit sequence assigned to λ, then $\lambda[n] \prec \lambda$.

We now assign sequences of notations $\lambda[n]$ to those notations λ that denote limit ordinals. In each case the corresponding sequence of ordinals $|\lambda[n]|$ is increasing and cofinal in the ordinal $|\lambda|$.

DEFINITION 4.2. Sequences are assigned to limit notations as follows. (Here λ denotes a limit ordinal.)

(1) $(\alpha + \beta)[n] =_{\text{def}} \alpha + (\beta[n])$.
(2) $\omega^{\alpha+1}[n] =_{\text{def}} \omega^{\alpha} \cdot (n+2)$.
(3) $\omega^{\lambda}[n] =_{\text{def}} \omega^{\lambda[n]+1}$.
(4) $\varepsilon_0[n] =_{\text{def}} \omega_{n+1}$.

Aside from the occasional "+2" or "+1" which we have added to make our constructions easier, these limit sequences are standard. We feel obligated to point out that equivalent notations might not have equivalent limit sequences; for example $\omega[n] = n + 2$, whereas $(1 + \omega)[n] = n + 3$. Furthermore, elements of a limit sequence $\lambda[n]$ are not required to be in normal form, even if λ is.

For reference, a list of the theories that have so far been analyzed with these methods appears below, together with their proof-theoretic ordinals. The first five lines are dealt with in this paper, while the remaining theories are treated in [4].

Theory	Ordinal
$I\Sigma_1$	ω^{ω}
$\mathrm{RCA}_0, \mathrm{WKL}_0$	ω^{ω}
$I\Sigma_n$	ω_{n+1}
PA	ε_0
$\mathrm{ACA}_0, \Sigma^1_1\text{-}\mathrm{AC}_0$	ε_0
$(\Pi^0_1\text{-CA})_{\prec\omega^{\alpha}}$	$\varphi(\alpha, 0)$
ACA	$\varepsilon_{\varepsilon_0}$
$\Sigma^1_1\text{-}\mathrm{AC}$	$\varphi(\varepsilon_0, 0)$
$\widehat{\mathrm{ID}}_{\mathrm{n}}$	γ_n
$\mathrm{ATR}_0, \widehat{\mathrm{ID}}_{<\omega}$	Γ_0
ATR	Γ_{ε_0}

§5. Ordinal largeness properties. In order to define the notion of an α-large set of numbers, we first need to extend the limit sequence function to define

$$\alpha[a_0, a_1, \ldots, a_k]$$

for arbitrary notations α and sequences $a_1, \ldots, a_k$. Intuitively, the definition allows us to "count down" $k+1$ steps from α, taking predecessors at successor stages, and taking an appropriate element of the limit sequence $\lambda[a_i]$ at limit stages λ.

DEFINITION 5.1. We extend the limit sequence function to successor ordinals by

$$(\alpha + 1)[n] =_{\text{def}} \alpha$$

and set

$$0[n] =_{\text{def}} 0.$$

The limit sequence function is defined on finite sequences by

$$\alpha[a_0, \ldots, a_k] =_{\text{def}} (\cdots((\alpha[a_0])[a_1])\cdots)[a_k].$$

Remember that whenever we write

$$A = \{a_0, a_1, \ldots, a_k\}$$

it is to be assumed that the elements a_i are listed in increasing order. In that case, we define

$$\alpha[A] =_{\text{def}} \alpha[a_0, a_1, \ldots, a_k].$$

With this definition in place, we can finally come to the central combinatorial notion in this paper.

DEFINITION 5.2. A set A is said to be **α-large** if $\alpha[A] = 0$. A is said to be **exactly α-large** if it is α-large, but no initial segment of it is α-large.

So a set A is exactly α-large if the "counting down" procedure hits 0 on the last element of A. The following lemma gives an alternative characterization of the α-large sets.

LEMMA 5.3. *Every set is* 0*-large. A set* $A = \{a_0, a_1, \ldots, a_k\}$ *is*

- $(\alpha + 1)$*-large iff* $A - \{a_0\}$ *is* α*-large, and*
- λ*-large* (*where* λ *is a limit*) *iff* $A - \{a_0\}$ *is* $\lambda[a_0]$*-large.*

To take a concrete example, the reader can verify that a set is n-large iff it has at least n elements. Given the definition of limit sequences in the previous section, the set

$$\{3, 4, 5, 34, 48, 96, 432, 521, 1000\}$$

is $(\omega+2)$-large (in fact, exactly $(\omega+2)$-large; recall that $\omega[5]=6$), whereas the set

$$\{38, 84, 85, 86, 100\}$$

is not. In [32] it is shown that the notion of α-largeness is $\Delta^0_0(\exp)$-definable, and hence absolute between $\mathcal{M}$ and any initial segment I closed under exponentiation.

We now provide some basic combinatorial properties of α-large intervals.

DEFINITION 5.4. An **increasing partition** of a set A is a sequence of sets P_0, $\dots$, P_k such that

- $A = P_0 \cup \dots \cup P_k$, and
- for $i < k$, $\max(P_i) < \min(P_{i+1})$.

LEMMA 5.5. *Suppose $\alpha = \alpha_k + \alpha_{k-1} + \dots + \alpha_0$ and A is α-large. Then there is an increasing partition $P_0, \dots, P_k$ of A, where for each $i \le k$, P_i is α_i-large.*

The lemma follows easily from the definitions and induction on the cardinality of A.

Given α, define the function f_α by

$$f_\alpha(a) = \mu b\ ([a,b] \text{ is } \alpha\text{-large}).$$

Using Lemma 5.5 the reader can verify that $f_\omega(a) > 2a$, $f_{\omega^2}(a) > a^2$, and $f_{\omega^3} > a^a$. In fact, the sequence f_α is closely related to the Wainer-Schwichtenberg hierarchy of fast-growing functions (see [12, 23, 32, 33]).

In order to get a better sense of the functions f_α, define the set of β-recursive functions to be the smallest set that has the usual closure properties of the primitive recursive functions and, additionally, is closed under the scheme of **β-descent recursion**:

$$g(\vec{a}) = \begin{cases} 0 & \text{if } h(0,\vec{a}) \succeq \beta \\ \mu n\ (h(n+1,\vec{a}) \succeq h(n,\vec{a})) & \text{otherwise.} \end{cases}$$

The idea is that $g(\vec{a})$ bounds the length of the descending sequence of notations less than β that is generated by h with parameters $\vec{a}$. A function is $\prec\beta$-recursive if it is γ-recursive for some $\gamma \prec \beta$. (For further discussion see [29]; for other characterizations of the $\prec\beta$-recursive functions see [23, 25].)

It is not difficult to verify that each function f_α is α-recursive, so the construction described in Section 2 provides a method of showing that a theory T doesn't prove a certain α-recursive function to be total. In fact, our constructions yield an even stronger result, as described in Section 6.

In Section 7 we will need the following rather technical lemma.

LEMMA 5.6. *Suppose $\alpha \succ \omega^3$, $\min(A) > 3$, and A is α-large. Then* $\max(A) > 2^{\min(A)}$.

PROOF. A straightforward induction on the cardinality of A, using the limit sequence definitions from the previous section. $\dashv$

§6. How this provides an ordinal analysis. We would like say a few words about how our constructions provide an ordinal analysis. Saying that the proof-theoretic ordinal of a theory T is less than or equal to α usually entails all of the following results:

(1) There is some formula $\varphi(y)$ such that T doesn't prove $\mathrm{TI}(\alpha, \varphi(y))$, where $\mathrm{TI}(\alpha, \varphi(y))$ formalizes transfinite induction up to α for the formula $\varphi(y)$.

(2) Over a weak base theory, $\mathrm{PRWO}(\alpha)$ proves the 1-consistency of T. Here $\mathrm{PRWO}(\alpha)$ is a scheme which asserts that there are no primitive recursive descending sequences beneath α, and "the 1-consistency of T" is the formalized Π^0_2 assertion that if T proves any Σ^0_1-formula (possibly with parameters) then that formula is true.

(3) If T proves a recursive function f to be total, then f is $\prec\alpha$-recursive. By "T proves the recursive function f to be total" we mean that T proves

$$\forall x \, \exists! y \, \varphi(x, y)$$

for some Σ^0_1 formula φ that defines the graph of f in the standard model.

(4) If $\prec$ is any recursive ordering and T proves

$$\forall X \;\; \mathrm{TI}(\prec, y \in X) \tag{3}$$

then the order-type of $\prec$ in the standard model is less than $|\alpha|$. (If T doesn't allow for quantification over sets of numbers, we replace (3) by

$$\mathrm{TI}(\prec, X(y)),$$

where X is a new predicate symbol that we allow to appear in the axiom schemata of T.)

Note that the first three results refer to an ordinal *notation* α, whereas the last result refers to a countable *ordinal* $|\alpha|$, in the "real world." Note also that none of these results refer to limit sequences, directly or indirectly.

Suppose we've carried out the program described in Section 2, and built a model of the theory T from an α-large interval. As we've already pointed out, this gives us a model of T in which the assertion

$$\forall x \; \exists y \; ([x, y] \text{ is } \alpha\text{-large})$$

is false, providing an instance of transfinite induction up to α that fails in this model of T. This yields the first result above.

Showing that over a weak base theory $\mathrm{PRWO}(\alpha)$ proves the 1-consistency of T requires more effort. As it turns out, we don't need to assume that the underlying model $\mathcal{M}$ satisfies all the true statements of arithmetic; in fact, the theory $I\Delta_0 +$ (exp) is sufficient. With some work we can use this fact to carry out the construction in $\mathrm{WKL}_0 + \mathrm{PRWO}(\alpha)$ and prove the 1-consistency of T there. (Since WKL_0 can prove the completeness and compactness of first-order logic, it is strong enough to formalize a good deal of model theory. WKL_0 will be discussed further in Section 9 below.) Since WKL_0 is conservative over $I\Sigma_1$ for arithmetic formulas, we can conclude that the consistency of T is provable in $I\Sigma_1 + \mathrm{PRWO}(\alpha)$.

An even nicer approach, which avoids the use of nonstandard model theory and yields a stronger result, can be found in [30]. Working in primitive recursive arithmetic PRA and using the notion of "Herbrand provability," one can show that "for every a there exists a b such that $[a, b]$ is α-large" implies that "there are arbitrarily large finite approximations to a model of T." Since $\mathrm{PRA} + \mathrm{PRWO}(\alpha)$ proves the hypothesis of this statement, it proves the conclusion as well.

To obtain results of the third type, we need to point out that in fact our constructions typically allow us to build a model of T from an $\alpha[c]$-large interval, for any nonstandard c. Now suppose T proves the function f to be total. In order to show that f is $\prec\alpha$-recursive, it suffices to show that whenever

$$T \vdash \forall x\ \exists y\ \theta(x, y) \tag{4}$$

for some Δ^0_0 formula θ, then the function

$$g(x) = \mu y\ \theta(x, y)$$

is bounded by some $\prec\alpha$-recursive function.

Aiming for a contradiction, then, assume that (4) holds but g is not bounded by any $\prec\alpha$-recursive function. Then for every n, the standard model satisfies

$$\forall n\ \forall x\ \exists y\ ([x, y] \text{ is } \alpha[n]\text{-large} \wedge g(x) > y), \tag{5}$$

since otherwise g would be dominated by the $\prec\alpha$-recursive function $f_{\alpha[n]}$ defined in the previous section. Since (5) is a true statement of arithmetic we can find nonstandard a, b, and c such that

$$[a, b] \text{ is } \alpha[c]\text{-large} \wedge g(a) > b$$

is true in $\mathcal{M}$. But then our construction enables us to build a model of T in which g is not total, violating the assumption (4). (For similar arguments, see [11, 9].)

Finally, one can use our model-theoretic methods to obtain the last type of proof-theoretic result as well. If $\prec$ has order-type greater than $|\alpha|$ in the standard model, then there is an isomorphism between the set of notations less than α and an initial segment of the ordering $\prec$. Using compactness we can find a nonstandard model $\mathcal{M}$ of true arithmetic that comes equipped with such an isomorphism f. Then we "relativize" the constructions with respect to an increasing function g that dominates both f and f^{-1}; for example, take g to be

$$g(x) =_{\text{def}} \max(\{f(y) : y \leq x\} \cup \{f^{-1}(y) : y \leq x\}) + x,$$

and, rather than starting with an exactly α-large interval, we begin with the set $S = \{a, g(a), g^2(a), g^3(a), \dots, g^l(a)\}$ with l chosen so that S is exactly α-large. Much of the construction proceeds as it did originally, leading to a subset $A = \{a_0, a_1, \cdots a_k\}$ of S; then we take a limit point of A to get our desired cut I. Clearly, I will be closed under g and hence I will be closed under both f and f^{-1}. Furthermore, the sequence of notations

$$\alpha, \alpha[a], \alpha[a, g(a)], \alpha[a, g(a), g^2(a)], \dots$$

has no least element in I. Using the isomorphism f we are able to get an infinite $\prec$-descending sequence in I.

§7. Constructing a model of $I\Sigma_1$.

We are finally ready to begin the model-theoretic constructions. The main construction of this section is due to Paris and Kirby (see [13]). Much of what is presented in this section is worked out in greater detail in [10, 11, 30, 32]. We repeat this well-known construction here since new constructions that come later in this paper, as well as in [4], are modeled after this one.

As described in Section 2, our goal is to construct sets $A = \{a_0, a_1, \dots, a_k\}$ such that if k is nonstandard, various axioms are guaranteed to hold in any limit I of A. The theory $I\Delta_0 + (\exp)$ is the weakest theory we want to consider in this regard. Fortunately, it is not difficult to make the axioms of this theory hold in the limit I.

Definition 7.1. Say that the set $A = \{a_0, a_1, \dots, a_k\}$ is **spread out** if for all $i < k - 3$, $2^{a_i} < a_{i+1}$.

The technical condition $i < k - 3$ will facilitate our constructions. Recall that $\mathcal{M}$ is the nonstandard model of true arithmetic that we fixed in Section 3.

Lemma 7.2. *Suppose $A = \{a_0, a_1, \dots, a_k\}$ is a set in $\mathcal{M}$. If A is spread out and I is any limit of A, then*

$$I \models I\Delta_0 + (\exp).$$

In fact, if S is any set in $\mathcal{M}$ then

$$I \models I\Delta_0(S^I) + (\exp).$$

Proof. If I is a limit of A then I cannot contain the last 3 (in fact, n, for any standard n) points of A. I clearly satisfies the quantifier-free defining equations for successor, plus, times, and less-than. Since the formula defining exponentiation is Δ_0^0, I and $\mathcal{M}$ agree as to which elements c are equal to a^b, and it is easy to verify that the conditions on A then guarantee that I will be closed under exponentiation as well. Finally, we need to handle Δ_0^0 induction. By Lemma 3.3 it suffices to verify the Δ_0^0-least element principle. Suppose $\varphi(x)$ is a Δ_0^0 formula and

$$I \models \varphi(e)$$

for some e in I. Find the least such e in $\mathcal{M}$; by Lemma 3.1 this will also be the least such e in I. $\dashv$

Although Δ_0^0 formulas are absolute between I and $\mathcal{M}$, Σ_1^0 formulas might not be, since a witness to the fact that

$$\mathcal{M} \models \exists y\ \varphi(y)$$

might not appear in I. As a result, extending the above result to Σ_1^0 formulas will require some work.

The theory $I\Sigma_1$ is a fragment of Peano Arithmetic that adds the scheme of induction for Σ_1^0 formulas to $I\Delta_0 + (\exp)$. The following lemma gives conditions on A that guarantee that the axioms of $I\Sigma_1$ will hold in any limit. The main idea behind the construction is from [13], and has appeared in various forms in [10, 15, 14, 30, 32]. Recall that

$$\mathrm{Tr}_{\Sigma_1^0}(x) \equiv_{\mathrm{def}} \exists y\ \Theta(x, y)$$

is a complete Σ_1^0 truth definition, in which $\Theta(x, y)$ is Δ_0^0.

LEMMA 7.3. *Suppose $[a, b]$ is ω^c-large. Then there is a set*

$$A = \{a_0, a_1, \dots, a_c\} \subset [a, b]$$

such that A is spread out, and for every $i < c$ the following holds: whenever $e < a_i$,

$$\exists y \leq a_c\ \Theta(e, y) \leftrightarrow \exists y \leq a_{i+1}\ \Theta(e, y).$$

Intuitively speaking, the conclusion of the lemma asserts that if $e < a_i$, then any witness to the truth of $\mathrm{Tr}_{\Sigma_1^0}(e)$ that appears at or below a_c in fact appears at or below a_{i+1}. We defer the proof of Lemma 7.3 so that we can first show how this in fact gives us a model of $I\Sigma_1$.

Suppose the set A satisfies the conclusion of the lemma in our nonstandard model of true arithmetic $\mathcal{M}$. If c is nonstandard, we can find a cut I that is a limit of A. By Lemma 7.2, I will be a model of $I\Delta_0 + (\exp)$, and so the equivalence

$$\exists y\ \psi(y) \leftrightarrow \mathrm{Tr}_{\Sigma_1^0}(\ulcorner \exists y\ \psi(y) \urcorner)$$

from Section 3 will hold in I.

Now if e is any element of I, e will be less than a_{i-1} for some i. The conclusion of the lemma guarantees that the equivalence

$$\exists y \leq a_i\ \Theta(e, y) \ \leftrightarrow\ \exists y \leq a_c\ \Theta(e, y) \tag{6}$$

holds in $\mathcal{M}$. Since I is an initial segment of $\mathcal{M}$ containing a_i but not a_c, we have the chain of implications

$$\begin{aligned} \exists y \leq a_i\ \Theta(e, y) \ &\rightarrow\ \exists y \in I\ \Theta(e, y) \\ &\rightarrow\ \exists y \leq a_c\ \Theta(e, y), \end{aligned}$$

and so (6) implies that these are all in fact equivalent.

The net effect is that if e is any element of I, we have

$$(I \models \mathrm{Tr}_{\Sigma_1^0}(e)) \Longleftrightarrow (\mathcal{M} \models \exists y \leq a_c\ \Theta(e, y)). \tag{7}$$

Ordinarily we wouldn't expect the left-hand side of (7) to be definable in $\mathcal{M}$, since I is not definable there. Equivalence (7) shows that thanks to our construction, it is in fact definable by a Δ_0^0 formula.

Finally, suppose $\varphi(y, \vec{z})$ is any (standard) Δ_0^0 formula and $\vec{p}$ are parameters in I. Since I is a model of $I\Delta_0 + (\exp)$, the code $\ulcorner \exists y\ \varphi(y, \vec{p}) \urcorner$ will appear in I. Equation (7) and the universality of $\mathrm{Tr}_{\Sigma_1^0}$ then imply

$$(I \models \exists y\ \varphi(y, \vec{p})) \Longleftrightarrow (\mathcal{M} \models \exists y \leq a_c\ \Theta(\ulcorner \exists y\ \varphi(y, \vec{p}) \urcorner, y)). \tag{8}$$

THEOREM 7.4. *Suppose a and b are nonstandard elements of $\mathcal{M}$ and*

$$\mathcal{M} \models [a, b] \textit{ is } \omega^{\omega}\textit{-large.}$$

Then there is a cut $a < I < b$ such that $I \models I\Sigma_1$.

PROOF. Since $[a, b]$ is ω^{ω}-large, $[a + 1, b]$ is ω^{a+2}-large. Let

$$A = \{a_0, a_1, \dots, a_c\}$$

be the set satisfying the conclusion of Lemma 7.3 with $a + 1$ in place of a and $c = a + 2$. Lemma 7.2 insures that I is a model $I\Delta_0 + (\exp)$, so we only need to verify that Σ_1^0 induction also holds in I. By Lemma 3.3, it suffices to check the Σ_1^0 least-element principle. Suppose $\varphi(x, y)$ is a Δ_0^0 formula with parameters in I and for some e,

$$I \models \exists y\ \varphi(e, y).$$

We want to find a least such e. Since

$$\mathcal{M} \models \exists y \leq a_c\ \Theta(\ulcorner \exists y\ \varphi(e, y) \urcorner, y)$$

and $\mathcal{M}$ is a model of true arithmetic we can find the least such e in $\mathcal{M}$; but by equivalence (8) this is also the least such e in I. ⊣

We still owe the reader the following.

Proof of Lemma 7.3. To obtain the set A having the desired properties we will construct sequences

$$a_0 < a_1 < \cdots < a_c \leq b_c \leq b_{c-1} \leq \cdots \leq b_0$$

such that for each $i \leq c$ the following hold:

(1) If $i > 0$, then $a_{i-1} < a_i \leq b_i \leq b_{i-1}$.

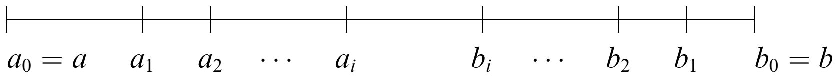
(2) Whenever $i > 0$ and $e < a_{i-1}$ we have

$$\exists y \leq a_i \ \Theta(e, y) \leftrightarrow \exists y \leq b_i \ \Theta(e, y).$$

(3) $[a_i, b_i]$ is ω^{c-i}-large.

(4) If $0 < i < c - 3$, then $2^{a_{i-1}} < a_i$.

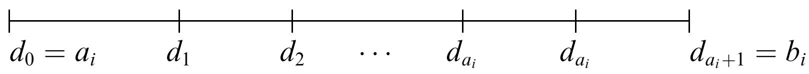
Set $a_0 = a$ and $b_0 = b$; conditions (1), (2), and (4) are immediate in the case $i = 0$, and (3) follows from the hypothesis that $[a, b]$ is ω^c-large. The construction will continue for c steps. In the end, clauses (2) and (4) guarantee that the set A satisfies the conclusion of the lemma. In following the construction, the reader might find it helpful to keep the following picture in mind:



Suppose we've already constructed $a_0, \ldots, a_i$ and $b_0, \ldots, b_i$ satisfying clauses (1–4). We need to show how to construct a_{i+1} and b_{i+1} satisfying (1–4) with $i + 1$ in place of i. Since $[a_i, b_i]$ is ω^{c-i}-large, we have $[a_i + 1, b_i]$ is $\omega^{c-(i+1)} \cdot (a_i + 2)$-large, and so by Lemma 5.5 we can find a sequence

$$d_0 = a_i, d_1, d_2, \ldots, d_{a_i+2} = b_i$$

such that each interval $(d_j, d_{j+1}]$ is $\omega^{c-(i+1)}$-large.



Recall that for any e,

$$\mu y \leq b_i \ \Theta(e, y)$$

denotes the least f less than or equal to b_i such that $\Theta(e, f)$ holds, or 0 if there is no such f. Since there are $a_i + 2$ many intervals $(d_j, d_{j+1}]$ and only a_i values of e less than a_i, by the pigeonhole principle we can find a $j \geq 1$ so that

$$\mu y \leq b_i \ \Theta(e, y) \notin (d_j, d_{j+1}]$$

for any such e. In other words, if

$$\exists y \leq d_{j+1} \ \Theta(e, y)$$

for some $e < a_i$, then in fact

$$\exists y \leq d_j \ \Theta(e, y).$$

If we then take $a_{i+1} = d_j + 1$ and $b_{i+1} = d_{j+1}$, clauses (1–3) are satisfied with $i + 1$ in place of i.

Notice that the condition $j \geq 1$ excludes the first interval, $(d_0, d_1]$. Lemma 5.6 shows that the largeness property on this interval implies that, as long as $i + 1 < c - 3$ (i.e., $c - (i + 1) > 3$), we'll have $2^{d_0} < d_1$. This implies that clause (4) is satisfied as well. ⊣

The brunt of Lemma 7.3 is that if we start from an ω^c-large interval $[a, b]$, we can find a c-large set A with the useful properties described above. (Recall that since c is finite, a c-large set is just a set of cardinality greater than or equal to c.) For later sections we will need to generalize Lemma 7.3 in three different ways. First of all, notice that there was nothing special about the fact that $[a, b]$ was an ω^c-large *interval*; if we started with an ω^c-large *set* C, we could get a c-large $A \subset C$ having the same properties. Second, using the truth predicate $\mathrm{Tr}_{\Sigma^0_1}(x, Z)$, we could easily have relativized the construction to any given set T. Finally, the interesting part: using the definition of a λ-large interval for limits λ, we can iterate the construction "transfinitely" and show that if we start with an ω^α-large set C, we can get an exactly α-large $A \subset C$ having the desired properties.

These extensions are summarized in the following lemma. The proof is very similar to that of Lemma 7.3, with a bit more bookkeeping at limit stages. Similar iterations play an important role in the sequel of this paper [4].

Lemma 7.5. *Suppose C is ω^α-large. Then there is a set*

$$A = \{a_0, a_1, \ldots, a_k\} \subset C$$

such that A is α-large and spread out, and for every $i < k$ the following holds: whenever $e < a_i$,

$$\exists y \leq a_k \ \Theta(e, y) \leftrightarrow \exists y \leq a_{i+1} \ \Theta(e, y).$$

Proof. We will construct a sequence of sets

$$C \supset C_0 \supset C_1 \supset \cdots \supset C_k$$

so that if we set

$$a_i = \min(C_i)$$

and

$$b_i = \max(C_i),$$

for each $i \leq k$ the following hold:

(1) If $i > 0$ then $C_i \subset C_{i-1}$.

(2) Whenever $i > 0$ and $e < a_{i-1}$ we have

$$\exists y \leq a_i\ \Theta(e, y) \leftrightarrow \exists y \leq b_i\ \Theta(e, y).$$

(3) C_0 is ω^α-large and if $i > 0$ then C_i is ω^{α_i}-large, where

$$\alpha_i =_{\text{def}} \alpha[a_0, a_1, \dots, a_{i-1}].$$

(4) If $0 < i < k - 3$ then $2^{a_{i-1}} < a_i$.

We start by setting $C_0 = C$; conditions (1), (2), and (4) are immediate in the case $i = 0$, and (3) follows from the hypothesis that C is ω^α-large. We will construct C_1, C_2, etc., until $\alpha_{i+1} = 0$. If we then set $k =_{\text{def}} i$ and

$$A =_{\text{def}} \{a_0, a_1, \dots, a_k\},$$

the fact that

$$\alpha_{k+1} = \alpha[a_0, a_1, \dots, a_k] = 0$$

implies that A is α-large. Clauses (2) and (4) then guarantee that A satisfies the conclusion of the lemma.

Assuming $C_0, C_1, \dots, C_i$ have been constructed satisfying (1–4), we show how to construct C_{i+1} satisfying (1–4) for $i + 1$ instead of i. Note that $\alpha_{i+1} = \alpha_i[a_i]$. If α_i is a successor notation then $\alpha_i = \alpha_{i+1} + 1$, and so C_i is $\omega^{\alpha_{i+1}+1}$-large. If α_i is a limit notation then by the definitions of large and limit sequence we have $C_i - \{\min C_i\}$ is $\omega^{\alpha_{i+1}+1}$-large. By letting

$$C' =_{\text{def}} \begin{cases} C_i & \text{if } \alpha_i \text{ is a successor notation, and} \\ C_i - \{\min C_i\} & \text{if } \alpha_i \text{ is a limit notation,} \end{cases}$$

then we have that C' is $\omega^{\alpha_{i+1}+1}$-large. If $c_0 =_{\text{def}} \min C'$, then $C' - \{c_0\}$ is $\omega^{\alpha_{i+1}} \cdot (c_0 + 2)$-large. By Lemma 5.5, $C' - \{c_0\}$ can be partitioned into $c_0 + 2$ sets $P_0, P_1, \dots, P_{c_0+1}$, each of which is $\omega^{\alpha_{i+1}}$-large and such that for each $i \leq c_0$, $\max(P_i) < \min(P_{i+1})$. As in the proof of Lemma 7.3 the pigeonhole principle implies that we can find a $j \geq 1$ so that

$$(\mu y \leq \max(C_i)\ \Theta(e, y)) \notin (\min(P_j), \max(P_j)]$$

for any $e < a_i$. Taking

$$C_{i+1} = P_j,$$

and applying Lemma 5.6 to the set P_0, we see that clauses (1–4) now hold with $i + 1$ in place of i. $\dashv$

§8. **Approximating the Turing jump.** In this section we address the issue of constructing sets that approximate second-order objects in the limit I, using the Turing jump as our first example.

Recall that if S is a set in our nonstandard model of arithmetic $\mathcal{M}$ and I is a cut, we've defined S^I to be $S \cap I$. Lemma 3.2 tells us that if $\varphi(X)$ is a $\Delta_0^0(X)$ formula, then

$$(I \models \varphi(S^I)) \iff (\mathcal{M} \models \varphi(S)).$$

In particular, as long as the ordered pair $\langle a, b \rangle$ is in I,

$$(I \models a \in (S^I)_b) \iff (\mathcal{M} \models a \in S_b),$$

so that whenever I is closed under pairing, $(S_b)^I$ and $(S^I)_b$ are equal. Similar properties will hold for the other set formation conventions described in Section 3. As a result, we can safely write $I \models \varphi(S)$ and leave the restriction of the set to I implicit. In practice, we will use the notation S^I when we want to emphasize that the set S has "taken on a new life" in I, and leave the $\cdot^I$ out otherwise.

Returning to the construction in Section 7, suppose A satisfies the conclusion of Lemma 7.3, and set

$$S =_{\mathrm{def}} \{x < a_{c-1} \mid \exists y \leq a_c\ \Theta(x, y)\}.$$

If c is nonstandard and I is any limit of A, we will then have that

$$I \models \forall x\ (x \in S \leftrightarrow \mathrm{Tr}_{\Sigma_1^0}(x)).$$

In other words, from I's point of view, S is a complete Σ_1^0 set. According to Lemma 7.2, I will satisfy $I\Delta_0(S) + (\exp)$. But it is straightforward to show that Δ_0^0 induction relative to a complete Σ_1^0 predicate yields Σ_1^0 induction for ordinary formulas, so that I is necessarily a model of $I\Sigma_1$ as well. These considerations provide another perspective from which to view the construction of a model of $I\Sigma_1$, and motivate the following definitions.

Definition 8.1. Say that S is the **Turing jump** of T, written $S = T'$, if

$$S = \{x \mid \mathrm{Tr}_{\Sigma_1^0}(x, T)\}.$$

The idea is that in any model of $I\Delta_0 + (\exp)$ the Turing jump of T codes the truth of formulas that are Σ_1^0 definable from T (and possibly other numeric parameters), via the universality of $\mathrm{Tr}_{\Sigma_1^0}$. If $T = \emptyset$, we will denote S by $0'$. The set S discussed just before Definition 8.1 gave us a "finite approximation" to $0'$, that is, a set guaranteed *to be* $0'$ in an appropriate limit I. Relativizing this notion gives us the next definition. Recall that

$$S^d =_{\mathrm{def}} S \cap [0, d),$$

and that

$$\mathrm{Tr}_{\Sigma^0_1}(x, Z) \equiv_{\mathrm{def}} \exists y\ \Theta(x, y, Z).$$

DEFINITION 8.2. Let

$$A = \{a_0, a_1, \dots , a_k\},$$

S, and T be finite sets. Say that S **approximates the Turing jump of T in A** if for every $i < k$ and $e < a_i$ we have

$$\exists y \le a_{i+1}\ \Theta(e, y, T) \leftrightarrow \exists y \le a_k\ \Theta(e, y, T);$$

and

$$S^{a_{k-1}} = \{e < a_{k-1} \mid \exists y \le a_k\ \Theta(e, y, T)\}.$$

The following lemma asserts that Definition 8.2 serves our purposes.

LEMMA 8.3. *Let A, S, and T be finite sets in $\mathcal{M}$ such that*

$$\mathcal{M} \models S \textit{ approximates the Turing jump of } T \textit{ in } A.$$

Then in any limit I of A,

$$I \models S \textit{ is the Turing jump of } T.$$

PROOF. Suppose A, S, T, and I are as in the statement of the lemma, and suppose $A = \{a_0, a_1, \dots , a_k\}$. Since I is a limit of A we have $I < a_k$, and if e is any element of I we have that $e < a_i$ for some a_i in I, and so $i < k$. If

$$I \models \exists y\ \Theta(e, y, T)$$

then

$$\mathcal{M} \models \exists y \le a_k\ \Theta(e, y, T),$$

and hence e is in S. Conversely, if

$$I \models e \in S$$

then

$$\mathcal{M} \models \exists y \le a_k\ \Theta(e, y, T)$$

and since $e < a_i$, and $i < k$, we have that

$$\mathcal{M} \models \exists y \le a_{i+1}\ \Theta(e, y, T).$$

Since a_{i+1} is also in I, this implies that

$$I \models \exists y\ \Theta(e, y, T).$$

We've shown that

$$I \models \forall x\ (x \in S \leftrightarrow \mathrm{Tr}_{\Sigma^0_1}(x, T)),$$

completing the proof. ⊣

Lemma 7.5 can now be rephrased as follows.

LEMMA 8.4. *Suppose C is ω^α-large, and T is an arbitrary set. Then there are sets A and S, such that*

$$A = \{a_0, a_1, \dots , a_k\} \subset C$$

is α-large and spread out, and S approximates the jump of T in A.

PROOF. Just take A as in the conclusion of Lemma 7.5 and define

$$S = \{x < a_{k-1} \mid \exists y \leq a_k \; \Theta(x, y, T)\}.$$

This S works. ⊣

We will need the following simple fact later on.

LEMMA 8.5. *Suppose S approximates the jump of T in A and $B \subseteq A$. Then S approximates the jump of T in B.*

The proof follows easily from the definitions.

§9. Constructing models of $\mathbf{RCA}_0$ and $\mathbf{WKL}_0$. RCA_0 is a theory in the language of second-order arithmetic which contains $I\Sigma_1$ (set parameters are now allowed to appear in the induction axioms), and a recursive comprehension scheme,

$$\text{(RCA)}\quad \forall x\ (\exists u\ \varphi(x, u) \leftrightarrow \forall v\ \psi(x, v)) \rightarrow \exists Y\ \forall x\ (x \in Y \leftrightarrow \exists u\ \varphi(x, u))$$

where $\varphi(x, u)$ and $\psi(x, v)$ are Δ^0_0 (again, possibly with number and set parameters). In words, (RCA) asserts that if one has equivalent r.e. and co-r.e. descriptions of a class of numbers, then there is a set corresponding to that class. WKL_0 is the theory RCA_0 together with a weak version of König's lemma,

$$\begin{aligned}\text{(WKL)}\quad \forall T\ (&\text{“}T \text{ is an infinite binary tree”}\\ &\rightarrow \exists P\ (\text{“}P \text{ is a path through } T\text{”}))\end{aligned}$$

where a binary tree T is a set of binary sequences closed under initial segments, and P is a path through T if every initial segment of the characteristic function of P is in T. (The subscripted “0” in the names RCA_0 and WKL_0 indicates that rather than allowing full second-order induction, the induction scheme is restricted to Σ^0_1 formulas, as above.)

Let Σ^0_1-separation be the scheme

$$\begin{aligned}(\Sigma^0_1\text{-SEP})\quad &\forall x\ \neg(\exists u\ \varphi(x, u) \wedge \exists v\ \psi(x, v))\\ &\rightarrow \exists Y\ \forall x\ ((\exists u\ \varphi(x, u) \rightarrow x \in Y) \wedge (\exists v\ \psi(x, v) \rightarrow x \notin Y))\end{aligned}$$

where the formulas φ and ψ are Δ^0_0, possibly with parameters. This scheme asserts that disjoint Σ^0_1 classes of numbers can be separated by a set.

Lemma 9.1. *Over* $I\Delta_0 + (\exp)$ *the axiom schema* (RCA) + (WKL) *is equivalent to* (Σ^0_1-SEP).

Proof. Below we will only need the right-to-left direction, which we consider first. Clearly (Σ^0_1-SEP) implies (RCA), since if $\hat{\varphi}(x) \equiv \exists u\ \varphi(x,u)$ and $\hat{\psi}(x) \equiv \forall v\ \psi(x,v)$ satisfy the hypothesis of (RCA) then $\hat{\varphi}(x)$ and $\neg\hat{\psi}(x)$ define disjoint classes, and a set Y separating $\hat{\varphi}$ from $\neg\hat{\psi}$ satisfies the conclusion of (RCA). To show that (Σ^0_1-SEP) implies (WKL), given any tree T and $\sigma \in T$ define

$$\hat{\varphi}(\sigma) \equiv \text{“there is a level of } T \text{ in which } \sigma\hat{\ }0 \text{ has extensions but } \sigma\hat{\ }1 \text{ does not”}$$

and define $\hat{\psi}(\sigma)$ by switching the 0 and 1 in the definition of $\hat{\varphi}$. Since we can bound the size of the codes of the binary sequences in any fixed level of T, the formulas $\hat{\varphi}$ and $\hat{\psi}$ are Σ^0_1 in T. Assuming T is infinite, it is easy to verify that one can find a path through T recursive in any separation Y of $\hat{\varphi}$ and $\hat{\psi}$ by traveling through the tree and using Y to pick an infinite branch at each stage.

Conversely, to derive (Σ^0_1-SEP) from (RCA) and (WKL), given $\hat{\varphi}$ and $\hat{\psi}$ define T to be the tree of binary sequences σ, such that as far as witnesses less than length(σ) are concerned, σ is consistent with a separation of $\hat{\varphi}$ and $\hat{\psi}$. A path through T yields the desired separation. See [26] for details. $\dashv$

In Section 7 we showed how to construct a model of $I\Sigma_1$ starting from an ω^ω-large interval. We now show that we can in fact do better, and obtain a model of WKL_0. Constructions similar to the one below can be found in [26, 13]. For more information on WKL_0, see, for example, [26, 27, 2].

To state the following lemma we temporarily expand the language of first-order arithmetic to include set parameters P_i for $i \in \omega$.

Lemma 9.2. *Suppose* $[a,b]$ *is* ω^c*-large. Then there are sequences*

$$a = a_0 < a_1 < \cdots < a_c \le b_c \le b_{c-1} \cdots \le b_0 = b$$

and a sequence of sets

$$D_0, D_1, \ldots, D_c,$$

such that $A =_{\mathrm{def}} \{a_0, a_1, \ldots, a_c\}$ *is spread out,* $D_0 = \emptyset$*, and for every* $i < c$ *the following hold*:

(1) *Whenever* $\ulcorner \exists y\ \varphi(y, P_{j_1}, P_{j_2}, \ldots, P_{j_k}) \urcorner < a_i$ *codes a* Σ^0_1 *formula with the set parameters shown, and each* $j_l \le i$*, then we have*

$$\exists y \le b_{i+1}\ \varphi(y, D_{j_1}, D_{j_2}, \ldots, D_{j_k}) \leftrightarrow \exists y \le a_{i+1}\ \varphi(y, D_{j_1}, D_{j_2}, \ldots, D_{j_k}).$$

(2) *If* $\vec{P}$ *is as above and* $\exists y\ \psi(x,y,\vec{P})$ *is coded below* b_i*, then*

$$(D_{i+1})_{\ulcorner \exists y\ \psi(x,y,\vec{P}) \urcorner} = \{x < b_{i+1} \mid \exists y \le b_{i+1}\ \varphi(y, \vec{D})\}.$$

If $f < b_i$ does not code a formula of this form, then $(D_{i+1})_f = \emptyset$.

PROOF. Note that we use the formula $\mathrm{Tr}_{\Sigma_1^0}(x, Z)$ to express the two properties above. The proof requires only a slight modification to the proof of Lemma 7.3: at each stage i we pick an interval $[a_{i+1}, b_{i+1}]$ to satisfy clause (1), and define

$$(D_{i+1})_{\ulcorner \exists y\ \psi(x,y,\vec{P}) \urcorner} = \{x < b_{i+1} \mid \exists y \leq b_{i+1}\ \varphi(y, \vec{D})\}$$

to satisfy clause (2). ⊣

Now we consider what happens when the conclusion of the lemma holds in our nonstandard model of arithmetic $\mathcal{M}$.

LEMMA 9.3. *Suppose c and A satisfy the conclusion of the previous lemma in $\mathcal{M}$. If c is nonstandard and I is any limit of A, let*

$$J = \{j \mid a_j \in I\},$$

so that J is a limit of the set $\{0, 1, \dots, c\}$, and let

$$\mathcal{D} = \{(D_j)_f^I \mid j \in J, f \in I\}.$$

Then

$$\mathcal{N} = \langle I, \mathcal{D} \rangle$$

is a model of $I\Sigma_1 + (\Sigma_1^0\text{-SEP})$, and hence WKL_0.

PROOF. One can verify that I is a model of $I\Delta_0 + (\exp)$ and that Σ_1^0 induction with parameters holds in $\mathcal{N}$ as in the proof of Theorem 7.4, so we focus on $(\Sigma_1^0\text{-SEP})$.

Suppose

$$\mathcal{N} \models \forall x\ \neg(\exists u\ \varphi(x, u) \wedge \exists v\ \psi(x, v))$$

where φ and ψ are Δ_0^0 with parameters from $\mathcal{N}$. Then

$$I \models \forall x\ \neg(\exists u\ \varphi'(x, u, D_{j_1}, \dots, D_{j_k}) \wedge \exists v\ \psi'(x, v, D_{j_1}, \dots, D_{j_k}))$$

where D_{j_1} to D_{j_k} include all the sets from which the parameters of φ and ψ are defined, and φ' and ψ' are obtained from φ and ψ by replacing the parameters $(D_j)_f$ by the corresponding $(P_j)_f$. By reorganizing quantifiers, we have

$$I \models \forall z\ (\forall x \leq z, u \leq z, v \leq z\ \neg(\varphi'(x, u, \vec{D}) \wedge \psi'(x, v, \vec{D}))).$$

Choose i so that each $j_l < i$ and the above formula is coded below a_i. Then the conclusion of the previous lemma tells us that

$$\mathcal{M} \models \forall z \leq b_{i+1}\ (\forall x \leq z, u \leq z, v \leq z\ \neg(\varphi'(x, u, \vec{D}) \wedge \psi'(x, v, \vec{D}))),$$

that is,

$$\mathcal{M} \models \forall x \leq b_{i+1}\ \neg(\exists u \leq a_c\ \varphi'(x, u, \vec{D}) \wedge \exists v \leq a_c\ \psi'(x, v, \vec{D})). \tag{9}$$

Now let

$$C =_{\mathrm{def}} (D_{i+1})_{\ulcorner \exists u\ \varphi'(x,u,\vec{P}) \urcorner}$$

so that

$$C = \{x < b_{i+1} \mid \exists u \leq b_{i+1}\ \varphi'(x, u, \vec{D})\}$$

is in $\mathcal{M}$. We claim that C^I, which is in the second-order universe of $\mathcal{N}$, is the desired separation. If

$$\mathcal{N} \models \exists u\ \varphi(e, u)$$

then

$$I \models \exists u\ \varphi'(e, u, \vec{D})$$

and

$$\mathcal{M} \models \exists u \leq b_{i+1}\ \varphi'(e, u, \vec{D}),$$

so e is in C. On the other hand, if

$$\mathcal{N} \models \exists v\ \psi(e, v)$$

then

$$I \models \exists v\ \psi'(e, v, \vec{D})$$

and

$$\mathcal{M} \models \exists v \leq b_{i+1}\ \psi'(e, v, \vec{D}).$$

Equation (9) then guarantees that

$$\mathcal{M} \models \neg\exists u \leq b_{i+1}\ \varphi'(e, u, \vec{D}),$$

and so e is not in C. ⊣

We'd like point out that in the proof above C^I is *not* necessarily the set

$$\{e \mid \mathcal{N} \models \exists u\ \varphi(e, u)\},$$

since $\mathcal{M}$ may see some witnesses $u \leq b_{i+1}$ that are not in $\mathcal{N}$.

Lemmas 9.2 and 9.3 are sufficient to obtain the ordinal analysis described in Section 6. To provide a more attractive statement of the net result, though, we will use the following trick to code a second-order universe satisfying WKL_0 into a single set: if T_d is a sequence of sets indexed by elements of a set D,

$$S = \bigoplus_{d \in D} T_d,$$

and I is a cut that is closed under the pairing operation, then

$$S^I = \bigoplus_{d \in D^I} T_d^I,$$

so that the only sets T_d^I "seen" by I are those indexed by elements d in $D \cap I$.

THEOREM 9.4. *Suppose a and b are nonstandard elements of $\mathcal{M}$ such that*

$$\mathcal{M} \models [a, b] \text{ is } \omega^\omega\text{-large}.$$

Then there are a cut $a < I < b$ and a finite set S in $\mathcal{M}$ such that

$$\mathcal{N} = \langle I, \{S_i^I \mid i \in I\}\rangle$$

is a model of WKL_0.

PROOF. Let A and D be as in the conclusion of Lemma 9.2, define

$$T_{\langle a_i, f\rangle} = (D_i)_f,$$

and define

$$S = \bigoplus_{i \le c,\ f < a_{i-1}} T_{\langle a_i, f\rangle}.$$

Let I be any limit of A and let

$$J = \{j \mid a_j \in I\}.$$

Since I is a model of $I\Delta_0 + (\exp)$ it is closed under pairing, and so it is easy to verify that

$$\{S_i^I \mid i \in I\} = \{(D_j)_f^I \mid j \in J, f \in I\}.$$

We can now apply Lemma 9.3. ⊣

§**10. Approximating finite jump hierarchies.** In Section 8 we discussed the Turing jump. We now extend the discussion to include finite jump hierarchies.

DEFINITION 10.1. Say H is a **c-level jump hierarchy** if $H_0 = \emptyset$ and for each $i, 0 < i \le c$,

$$H_i = (H_{<i})'.$$

DEFINITION 10.2. Let A and H be finite sets. Say H **approximates a c-level jump hierarchy in** A if for each i, $0 < i \le c$, H_i approximates the Turing jump of $H_{<i}$ in A.

If in Definition 10.1 (10.2) H_0 is equal to some set T, we will say that H is (approximates) a c-level jump hierarchy **from** T. As in the case of Lemma 8.3, proving the following lemma is now straightforward.

LEMMA 10.3. *Suppose A, H, and T are sets in $\mathcal{M}$ such that*

$$\mathcal{M} \models H \text{ approximates a } c\text{-level jump hierarchy from } T \text{ in } A.$$

If I is any limit of A, then

$$I \models H \text{ is a } c\text{-level jump hierarchy from } T.$$

The following lemma, analogous to Lemma 8.4, asserts that we can build an approximation to a finite jump hierarchy if we start from a suitably large interval.

Lemma 10.4. *Suppose $[a, b]$ is ω_c^α-large, and T is any finite set. Then there are sets A and H such that A is an α-large subset of $[a, b]$ and spread out, and H approximates a c-level jump hierarchy from T in A.*

Proof. This is just an iteration of Lemma 8.4. ⊣

What makes a jump hierarchy useful is that it codes the truth of arithmetic formulas of low complexity. Suppose H is a c-level jump hierarchy in some model $\mathcal{N}$ of $I\Delta_0 + (\exp)$ and expand the language of arithmetic so that it includes constants to denote the sets H_l for $l < c$. Say that a formula φ is $\Sigma_m^0(H_l)$ if it is Σ_m^0 definable from H_l. Roughly speaking, if φ is $\Sigma_m^0(H_l)$ then the truth of φ will be coded in H_k, whenever $m + l \leq k \leq c$. The following lemma makes this precise.

Lemma 10.5. *There is an $I\Delta_0 + (\exp)$-definable function*

$$\mathrm{TruthCode}(\ulcorner\psi\urcorner, x)$$

with the following property: whenever $\mathcal{N}$ is a model of $I\Delta_0 + (\exp)$, H is a c-level jump hierarchy in $\mathcal{N}$, ψ is a $\Sigma_m^0(H_l)$ formula, and

$$m + l \leq d \leq c,$$

then the equivalence

$$\psi \leftrightarrow \mathrm{TruthCode}(\ulcorner\psi\urcorner, d) \in H_d$$

holds in $\mathcal{N}$.

Proof. Straightforward, once a reasonable coding scheme for formulas has been defined. ⊣

In Lemma 10.5 we allow for the possibility that ψ has numeric parameters, in which case $\ulcorner\psi\urcorner$ is really a function of these parameters. In words the lemma asserts that we can determine the truth of ψ from any level H_d of the jump hierarchy where d is greater than or equal to $m + l$.

We now introduce some notation. If H is a c-level jump hierarchy in $\mathcal{N}$, $\psi(x)$ is a $\Sigma_m^0(H_l)$ arithmetic formula with only the free variable shown, and $m + l \leq d \leq c$,

$$H_d^{[\psi(x)]} =_{\mathrm{def}} \{e \mid \mathrm{TruthCode}(\ulcorner\psi(e)\urcorner, d) \in H_d\}. \tag{10}$$

If f does not code a formula of the required form, we can just set $H_d^{[f]} = \emptyset$.

Now suppose H is a jump hierarchy in the model $\mathcal{N}$. By definition we have

$$\mathcal{N} \models \forall x\ (x \in H_d^{[\psi(x)]} \leftrightarrow \text{TruthCode}(\ulcorner \psi(x) \urcorner, d) \in H_d).$$

If in addition $\langle \mathcal{N}, H \rangle$ is a model of $I\Delta_0 + (\exp)$ and d is sufficiently big, we have that

$$\mathcal{N} \models \forall x\ (x \in H_d^{[\psi(x)]} \leftrightarrow \psi(x)),$$

or, in other words,

$$\mathcal{N} \models H_d^{[\psi(x)]} = \{x \mid \psi(x)\}.$$

On the other hand, suppose instead that H merely approximates a jump hierarchy in some set A, where H and A are in our nonstandard model $\mathcal{M}$. If A is spread out and I is any limit of A, we have that

$$(H^I)_d^{[\psi(x)]} = (H_d^{[\psi(x)]})^I,$$

so that the set $H_d^{[\psi(x)]}$ *approximates* the set $\{x \mid \psi(x)\}$ in A.

§11. Constructing models of $I\Sigma_n$, PA, and $\mathbf{ACA}_0$. The theory $I\Sigma_n$ extends $I\Sigma_1$ by adding induction for Σ_n^0 formulas, that is, the schema Σ_n^0- IND. Ordinal analyses of Peano Arithmetic and the theories $I\Sigma_n$, based on Schütte's use of infinitary logic to obtain Gentzen's seminal results, can be found in [17, 25, 24].

THEOREM 11.1. *Suppose a and b are nonstandard elements of $\mathcal{M}$ such that*

$$\mathcal{M} \models [a, b] \text{ is } \omega_{n+1}\text{-large.}$$

Then there are a cut $a < I < b$ such that I is a model of $I\Sigma_n$.

PROOF. Since $\omega_{n+1} = \omega_n^\omega$, by Lemma 10.4 we can find sets A and H in $\mathcal{M}$, so that A is ω-large and spread out, and that H approximates an n-level jump hierarchy in A. Since $A \subseteq [a, b]$ is ω-large, $A - \{\min(A)\}$ is c-large for some $c \geq a$, so we can find a cut I that is a limit of A. By Lemma 7.2 I is a model of $I\Delta_0 + (\exp)$, so we only need to verify that the Σ_n^0 least element principle holds in I.

Suppose $\psi(x)$ is a Σ_n^0 formula and

$$I \models \psi(e). \tag{11}$$

Then

$$I \models \text{TruthCode}(\ulcorner \psi(e) \urcorner, n) \in H_n \tag{12}$$

and

$$\mathcal{M} \models \text{TruthCode}(\ulcorner \psi(e) \urcorner, n) \in H_n. \tag{13}$$

By the least-element principle in $\mathcal{M}$ we can find the least $e \in \mathcal{M}$ such that (13) holds. This e is then the least $e \in I$ such that (12), and hence (11), also hold.

(Alternatively, we can use the fact that H^I is an n-level jump hierarchy in I, and reduce Σ^0_n induction to $\Delta^0_0(H)$ induction there.) ⊣

THEOREM 11.2. *Suppose a and b are nonstandard elements of $\mathcal{M}$ such that*

$$\mathcal{M} \models [a, b] \textit{ is } \varepsilon_0\textit{-large}.$$

Then there is a cut $a < I < b$ such that I is a model of PA.

PROOF. Since $[a, b]$ is ε_0-large, $[a + 1, b]$ is ω_{a+1}-large, so we can find an ω-large $A \subseteq [a + 1, b]$ and a set H in $\mathcal{M}$ such that A is spread out and H approximates an a-level jump hierarchy in A. Since A is ω large and $\min(A) \geq a + 1$, as in the previous proof we can find a cut I that is a limit of A. Since a is nonstandard and H^I is an a-level jump hierarchy in I, as in the previous proof we can verify that Σ^0_n induction holds for every standard n. ⊣

The theory ACA_0 consists of RCA_0 together with an arithmetic comprehension scheme

$$\text{(ACA)} \qquad \exists Y\, \forall x\, (x \in Y \leftrightarrow \varphi(x)),$$

where φ is arithmetic, possibly with parameters. Since this includes Σ^0_1 comprehension, we can now take induction to consist of the single axiom

$$0 \in Y \wedge \forall x\, (x \in Y \rightarrow x + 1 \in Y) \rightarrow \forall x\, (x \in Y).$$

LEMMA 11.3. *Suppose $\langle \mathcal{K}, \{H\}\rangle$ is a model of $I\Delta_0 + (\exp)$, c is a nonstandard element of $\mathcal{K}$, and H is a c-level jump hierarchy in $\mathcal{K}$. Let J be any limit of the set $\{0, 1, \ldots, c\}$, and let*

$$\mathcal{H} = \{H^{[k]}_i \mid i \in J, k \in \mathcal{K}\}.$$

Then

$$\mathcal{N} = \langle \mathcal{K}, \mathcal{H}\rangle$$

is a model of ACA_0.

PROOF. The induction axiom for sets follows from Δ^0_0 induction in $\langle \mathcal{K}, \{H\}\rangle$ and the definition of $H^{[k]}_i$ given by equation (10) in Section 10. To verify arithmetic comprehension, let $\varphi(x)$ be a (standard) arithmetic formula with parameters in $\mathcal{N}$. We can assume that $\varphi(x)$ is $\Sigma^0_m(H_l)$ for some $l \in J$ and standard m, since each level of a jump hierarchy codes all the ones that have come before. Since J is a limit, $m + l$ is also in J. But then

$$\forall x\, (x \in H^{[\varphi(x)]}_{m+l} \leftrightarrow \varphi(x))$$

holds in $\mathcal{N}$, and so $H_{m+l}^{[\varphi(x)]}$ is the set that (ACA) asserts to exist. ⊣

We can now use the same trick that we used in the proof of Theorem 9.4 to code the universe of ACA_0 into a single set.

THEOREM 11.4. *Suppose a and b are nonstandard elements of $\mathcal{M}$ such that*

$$\mathcal{M} \models [a, b] \text{ is } \varepsilon_0\text{-large}.$$

Then there are a cut $a < I < b$ and a finite set S in $\mathcal{M}$ such that

$$\mathcal{N} = \langle I, \{S_i^I \mid i \in I\}\rangle$$

is a model of ACA_0.

PROOF. As in the proof of Theorem 11.2 we can obtain sets $A = \{a_0, a_1, \dots, a_c\}$ and H such that c is nonstandard, A is spread out, and H approximates an c-level jump hierarchy in A. Let I be any limit of A, and let

$$J = \{j \mid a_j \in I\},$$

so that J is a limit of $\{0, 1, \dots, c\}$. The fact that A is spread out guarantees that $\langle I, \{H\}\rangle$ is a model of $I\Delta_0+(\exp)$, and so the previous lemma guarantees that

$$\mathcal{N} = \langle I, \{H_i^{[k]} \mid i \in J, k \in I\}\rangle$$

is a model of ACA_0. We can code the second-order universe of $\mathcal{N}$ into a single set S as in the proof of Theorem 9.4. ⊣

§12. Constructing a model of Σ_1^1-AC_0. The theory Σ_1^1-AC_0 adds to ACA_0 a Σ_1^1 axiom of choice,

$$(\Sigma_1^1\text{-}\mathrm{AC}) \qquad \forall x\, \exists Y\, \varphi(x, Y) \to \exists Y\, \forall x\, \varphi(x, Y_x)$$

where φ is Σ_1^1 (i.e., φ is either arithmetic or obtained from an arithmetic formula by prepending existential set quantifiers). This is often useful in that it allows one to code sequences of sets as a single set and bring second-order quantifiers to the outside of a formula. By "absorbing" an existential set quantifier if necessary we can safely assume that the formula φ in (Σ_1^1-AC) is arithmetic.

It is well-known that if one starts with a recursively saturated model of Peano arithmetic $\mathcal{K}$, then the second-order structure

$$\mathcal{N} = \langle \mathcal{K}, \mathrm{Arith}(\mathcal{K})\rangle$$

is a model of Σ_1^1-AC_0, where $\mathrm{Arith}(\mathcal{K})$ denotes the set of subsets of $\mathcal{K}$ that are arithmetically definable from parameters (see, for example, [11, 26]). And, in fact, the model constructed in the proof of Theorem 11.2 *is* recursively saturated, because we have a "truth definition" for formulas of nonstandard complexity (see Theorem 11.5 of [11]). The following theorem draws on this fact.

THEOREM 12.1. *Suppose a and b are nonstandard elements of $\mathcal{M}$ such that*

$$\mathcal{M} \models [a, b] \text{ is } \varepsilon_0\text{-large}.$$

Then there are a cut $a < I < b$ and a finite set S coded in $\mathcal{M}$ such that

$$\langle I, \{S^I_j \mid j \in I\}\rangle$$

is a model of Σ^1_1- AC_0.

PROOF. In fact, we show that the model constructed in the proof of Theorem 11.4 suffices. Let $\mathcal{N}$, H, I, and J be as in the proof of that theorem, and suppose $\mathcal{N}$ satisfies

$$\forall x \; \exists Y \; \varphi(x, Y)$$

for some (standard) arithmetic φ with parameters from $\mathcal{N}$. As in the proof of Lemma 11.3 we can assume that φ is $\Sigma^0_m(H_l)$ for some $l \in J$ and standard m. Then I satisfies

$$\forall x \; \exists y \; \varphi(x, H_e^{[y]}) \tag{14}$$

for any $e > J$, since H_e codes all the sets in the second-order universe of $\mathcal{N}$. If furthermore $e < c - (m + 2)$ (which we can assume since m is standard) we can use the function TruthCode to express (14) as a Δ^0_0 predicate of e in $\langle I, \{H\}\rangle$.

By Δ^0_0 induction in $\langle I, \{H\}\rangle$, find the least e such that (14) holds. This e must be in J (if it weren't, then (14) would fail for $e - 1 > J$). By arithmetic comprehension in $\mathcal{N}$, define S so that for each x

$$S_x = H_e^{[f_x]}$$

where f_x is the least f such that

$$\varphi(x, H_e^{[f]}).$$

Then this S witnesses the conclusion of (Σ^1_1- AC). ⊣

§13. Final comments. Our goal in this paper is not to replace cut elimination as the primary means to an ordinal analysis, but to provide a supplementary approach that helps round out our understanding of the theories involved. We hope the reader feels that this “hands-on” approach to constructing models of the theories described here and in [4] adds to his or her understanding of their axioms.

We expect that these methods will extend, in some form or another, to theories that are significantly stronger. Beyond the predicative theories treated in [4], the next hurdle, of course, is the analyses of impredicative theories like ID_1 and Π^1_1-CA. It is our hope that such analyses will ultimately provide us with a better understanding of their models, and perhaps point the way to obtaining some interesting combinatorial independences as well.

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