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SET THEORY GENERATED BY ABELIAN GROUP THEORY

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Introduction. This survey is intended to introduce to logicians some notions, methods and theorems in set theory which arose—largely through the work of Saharon Shelah—out of (successful) attempts to solve problems in abelian group theory, principally the Whitehead problem and the closely related problem of the existence of almost free abelian groups. While Shelah's first independence result regarding the Whitehead problem used established set-theoretical methods (discussed below), his later work required new ideas; it is on these that we focus. We emphasize the nature of the new ideas and the historical context in which they arose, and we do not attempt to give precise technical definitions in all cases, nor to include a comprehensive survey of the algebraic results.

In fact, very little algebraic background is needed beyond the definitions of group and group homomorphism. Unless otherwise specified, we will use the word "group" to refer to an abelian group, that is, the group operation is commutative. The group operation will be denoted by $+$, the identity element by 0, and the inverse of a by $-a$. We shall use na as an abbreviation for $a + a + \cdots + a$ [n times] if n is positive, and $na = (-n)(-a)$ if n is negative.

A group is called *free* if and only if it has a basis, that is, a linearly independent subset which generates the group. (The notions are the same as for vector spaces except that the scalars are from the ring of integers; so not every group is free.) Equivalently, a group is free if and only if it is isomorphic to a direct sum of copies of $\mathbb{Z}$, the group of integers under addition. A crucial fact is that a subgroup of a free group is free [16, Theorem 14.5].

At a couple of points we will have occasion to refer to groups which are not necessarily commutative. In this context—the variety of all groups—the characterization of "free" is different; in fact, with the exceptions of the trivial group (of cardinality 1) and of $\mathbb{Z}$, free groups in this variety are not commutative. However, it is still the case that subgroups of free groups are free [28, p. 95].

The *Whitehead problem* asks whether a certain necessary condition for a group to be free (defined in Section 1) is also sufficient. The *almost*

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free problem asks whether there exists a non-free group such that all of its subgroups of strictly smaller cardinality are free. Define a group to be λ-free if and only if every subgroup of cardinality $< \lambda$ is free; there is a separate problem for each cardinality λ: is there a λ-free group of cardinality λ which is not free? It turns out that the Whitehead problem and the λ-free problem (for many λ) are not solvable in ZFC.

In the first three sections we deal with results generated by the question of whether the Whitehead problem is settled by the Continuum Hypothesis (CH); these include proper forcing, the weak diamond principle, and consistency results about uniformization of ladder systems. In the last three sections we deal with results generated by the question of whether (it is consistent that) there is a λ-free non-free group of cardinality λ; these include "compactness" and "incompactness" theorems. We also touch on applications to and connections with topology and combinatorics.

I would like to express my appreciation to Saharon Shelah for his patience and care in answering my questions about the events and results discussed here.

§1. Whitehead groups and CH. A necessary and sufficient condition for a group A to be free is that whenever $\varphi\colon B_\varphi \to A$ is a homomorphism onto A (with arbitrary domain B_φ), then the kernel of φ

$$\ker(\varphi) = \{\, b \in B_\varphi : \varphi(b) = 0 \,\}$$

is a direct summand of B_φ, i.e.,

$$B_\varphi = \ker(\varphi) \oplus A'$$

for some subgroup A' (which is necessarily isomorphic to A). In fact, this condition is sufficient if we just allow homomorphisms whose kernel is free (because for any A there is a surjective homomorphism $\varphi\colon B_\varphi \to A$ such that B_φ—and hence also $\ker(\varphi)$—is free). Whitehead's problem asks whether the condition is sufficient when we restrict it to homomorphisms φ whose kernel is isomorphic to $\mathbb{Z}$. In homological terms, this condition is expressed as $\mathrm{Ext}(A, \mathbb{Z}) = 0$, or otherwise said, every short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_\varphi \xrightarrow{\varphi} A \longrightarrow 0$$

splits. A group which satisfies this condition is called a *Whitehead group*, or *W-group* for short; so Whitehead's problem asks whether every W-group is free. It had been proved as early as 1951 [43] that countable Whitehead groups are free. But by 1973 there were still only partial results for uncountable groups; then Shelah proved an independence result.

Shelah's original proof [35] made use of two well-studied extensions of ZFC to show that it is independent of ZFC whether every W-group of cardinality $\aleph_1$ is free. Specifically, he showed that Martin's Axiom plus the negation of the Continuum Hypothesis ($\mathrm{MA} + \neg\mathrm{CH}$) implies that there are non-free W-groups of cardinality $\aleph_1$ and, on the other hand, Gödel's Axiom of Constructibility ($\mathrm{V} = \mathrm{L}$) implies that every W-group of cardinality $\aleph_1$ is free. (See [6] for an exposition.) Later we will discuss the situation for larger cardinality, but, until further notice, all groups referred to are of cardinality at most $\aleph_1$.

Shelah's result left open the question of whether CH was sufficient to imply that W-groups are free. (The proof used the "diamond" prediction principles, described below, which are consequences of $\mathrm{V} = \mathrm{L}$ and which imply CH.) Shelah says "Naturally, it was hoped that [the fact that W-groups are free] was not a consequence of CH alone" [3, p. 240]. But some algebraists hoped otherwise, and, in fact, some false proofs from CH circulated. As a result, an erroneous report appeared in print in 1975 (in *Mathematical Reviews*) that two prominent group-theorists had, separately, found proofs from CH that W-groups are free—one of the proofs described as "an easy proof" that CH implies that all W-groups (of arbitrary cardinality) are free.

Meanwhile Shelah tried to interest some set-theorists in the problem; for that purpose he presented them with a combinatorial analog, which he invented for the purpose, having to do with so-called colorings of ladder systems. This led to a response by Keith Devlin which culminated in the discovery of the weak diamond principle [3], which is a consequence of (weak) CH. However, in the end Shelah took on the problem himself, though he did not consider himself an expert in forcing and had not previously done any original iterated forcing proofs. The outcome, in 1976, was not only an independence proof [37], but the start of a line of thought which led eventually to his important work on proper forcing [41]. (The work on proper forcing was motivated by the question of the number of non-isomorphic models.)

In the next section, we will attempt to explain these set-theoretic ideas in more detail, and indicate the connections with the original algebraic problem. Following that, in Section 3, we shall discuss another application of these ideas, to the normal Moore space problem.

§2. Diamonds, ladders and colorings. We begin with some notions and facts which were well-established by 1973. As usual we identify $\aleph_1$ with the set ω_1 of countable ordinals. A subset S of ω_1 is called *stationary* if and only if it has non-empty intersection with every closed unbounded (or *club*) subset of ω_1—where a subset C of ω_1 is a club if and only if it is closed under sups of countable subsets and $\sup C = \omega_1$. One may think of sets containing a club as analogs of sets of measure 1 and stationary sets as analogs of sets of measure > 0. (There is a sense in which this analogy can be made precise; see, for example, [10, §II.4].) We define two subsets of ω_1 to be equivalent iff their symmetric difference is a set of measure 0, that is, contained in the complement of a club. Let $\tilde{S}$ denote the equivalence class of S; there are $2^{\aleph_1}$ equivalence classes of subsets of $\aleph_1$, one of which is the class $\tilde{\emptyset}$ consisting of all non-stationary subsets, and another is $\tilde{\omega}_1$ consisting of all subsets which contain a club.

Ronald Jensen [23] showed that, for every stationary subset S of ω_1, the following principle, $\diamondsuit(S)$, is a consequence of the Axiom of Constructibility:

> *For each $\delta \in S$ there is a subset X_δ of δ such that for every subset Y of ω_1 there is a stationary subset S_Y of S such that for all $\delta \in S_Y$, $Y \cap \delta = X_\delta$.*

That is, the indexed family $(\, X_\delta : \delta \in S\,)$ "predicts" what $Y \cap \delta$ will be, and no matter the choice of Y, the prediction is right on a set of non-zero measure. By letting Y vary over the $2^{\aleph_0}$ subsets of ω, it is easy to see that $\diamondsuit(S)$ implies CH. It is also easy to see that, in *any* model of ZFC, $\diamondsuit(S)$ implies $\diamondsuit(S')$ whenever S' is equivalent to S or contains S. (However, it was open until Shelah's work on the Whitehead problem and CH [37] whether it was consistent with ZFC that there are two stationary subsets S_1 and S_2 of ω_1 such that $\diamondsuit(S_1)$ is true but $\diamondsuit(S_2)$ is not.)

Shelah's original proof from $\mathrm{V} = \mathrm{L}$ that W-groups are free used all of the principles $\diamondsuit(S)$. Here, in outline, is how. From the classical result that a countable W-group is free and the fact that a subgroup of a W-group is a W-group it follows that every W-group is $\aleph_1$-free, that is, every countable subgroup is free. To each $\aleph_1$-free group A of cardinality $\aleph_1$ can be associated an invariant, denoted $\Gamma(A)$, which is the equivalence class, $\tilde{S}$, of a subset of $\aleph_1$. (See Section 6 for more details.) This is an invariant in the sense that $\Gamma(A) \neq \Gamma(B)$ implies $A \ncong B$; the converse fails except in the case that $\Gamma(A) = \tilde{\emptyset}$, in which case A has to be free. In fact, every class other than $\tilde{\emptyset}$ is the Γ-invariant of $2^{\aleph_1}$ different groups. Shelah's proof shows that $\diamondsuit(S)$ implies that every A with $\Gamma(A) = \tilde{S}$ is *not* a W-group. Hence assuming $\mathrm{V} = \mathrm{L}$, the only possibility for a W-group A is $\Gamma(A) = \tilde{\emptyset}$, so A is free.

The nature of this argument suggested strongly to Shelah that CH was not sufficient to imply the conclusion. In an attempt to interest set-theorists in the problem, he formulated the following combinatorial analog. For any countable limit ordinal δ, a *ladder* on δ is a strictly increasing function $\eta_\delta \colon \omega \to \delta$ whose range is cofinal in δ. Given a subset S of ω_1 consisting of limit ordinals, a *ladder system* on S is an indexed family $(\, \eta_\delta : \delta \in S\,)$ where each η_δ is a ladder on δ. A 2-*coloring* of $(\, \eta_\delta : \delta \in S\,)$ is an indexed family $c = (\, c_\delta : \delta \in S\,)$ of functions $c_\delta \colon \omega \to 2$. (Think of $c_\delta(n)$ as being the color of the "nth rung" $\eta_\delta(n)$ of the ladder η_δ.) A *uniformization* of c is a function $h \colon \omega_1 \to 2$ such that for all $\delta \in S$, $c_\delta(n) = h(\eta_\delta(n))$

for all but finitely many $n \in \omega$. Say that $\langle \eta_\delta : \delta \in S \rangle$ has the 2-*uniformization property* if and only if every 2-coloring of $\langle \eta_\delta : \delta \in S \rangle$ has a uniformization. The problem is: is it consistent with CH that there is a ladder system on a stationary set which has the 2-uniformization property?

It is not too hard to show that the existence of a ladder system on a stationary set S which has the 2-uniformization property implies that there is a non-free W-group A (with $\Gamma(A) = \tilde{S}$—see, for example, [8, pp. 98–101]). Later Shelah showed the converse as well, that is, there is a non-free W-group of cardinality $\aleph_1$ only if some ladder system on some stationary S has the 2-uniformization property. (See [39, Theorem 3.9]; the most accessible proof of this fact is in [12, §6].)

It is easily provable in ZFC that if S is not stationary, then $\langle \eta_\delta : \delta \in S \rangle$ has the 2-uniformization property. However, the Axiom of Constructibility implies that no ladder system on a stationary set has the 2-uniformization property. More precisely, $\diamondsuit(S)$ implies that no ladder system on S has the 2-uniformization property; in fact, the diamond principle gives a prediction of what $h{\restriction}\delta$ will be for $\delta \in S$, and then the coloring c_δ is chosen to prevent h from being a uniformization. On the other hand, $\mathrm{MA} + \neg\mathrm{CH}$ implies that every ladder system has the 2-uniformization property.

Shelah conjectured that it is consistent with CH that *every* ladder system has the 2-uniformization property. But Devlin refuted the conjecture by showing (using the method of inner models) that CH implies that no ladder system on a *club* has the 2-uniformization property. After further independent contributions by Devlin and Jensen, Shelah—to quote Devlin—"took this a step further by 'extracting' from the proof the principle Φ (this extraction was so tricky, that it is somewhat misleading to use the word 'extract' at all)". (See [3, §2] for more on the evolution of Φ.) This "weak diamond principle" Φ is the special case $\Phi(\omega_1)$ of the following weakening, $\Phi(S)$, of $\diamondsuit(S)$:

> *Given for every $\delta \in S$ a function $F_\delta\colon \mathcal{P}(\delta) \to 2$, there is a function $\rho\colon \omega_1 \to 2$ such that for every subset Y of ω_1 there is a stationary subset S_Y of S such that for all $\delta \in S_Y$, $F_\delta(Y \cap \delta) = \rho(\delta)$.*

That is, ρ (which depends on the F_δ, but not on Y) predicts the value of $F_\delta(Y \cap \delta)$ and, no matter the choice of Y, is right on a large set of δ. The principle $\Phi(S)$ implies that no ladder system $\langle \eta_\delta : \delta \in S \rangle$ has the 2-uniformization property. In fact, we can defeat the possibility of uniformization with monochrome colorings, i.e., each c_δ is constantly 0 or 1. For $\delta \in S$ define $F_\delta(Z)$ to be 1 if $\mathrm{ran}(\eta_\delta) - Z$ is infinite, and 0 otherwise; for $\delta \notin S$, let F_δ be arbitrary. If ρ is as in $\Phi(S)$ for these F_δ, define c_δ to be constantly $\rho(\delta)$. Assuming $h\colon \omega_1 \to 2$ is a uniformization of this coloring, we obtain a contradiction by letting $Y = \{\, \alpha : h(\alpha) = 1 \,\}$: for $\delta \in S_Y$ we have $\rho(\delta) = F_\delta(Y \cap \delta)$; but

$$\begin{aligned}\rho(\delta) = 1 &\iff \text{for all } n \in \omega \;\; c_\delta(n) = 1\\ &\iff \text{for almost all } n \;\; h(\eta_\delta(n)) = 1\\ &\iff \text{for almost all } n \;\; \eta_\delta(n) \in Y \cap \delta\\ &\iff F_\delta(Y \cap \delta) = 0,\end{aligned}$$

a contradiction.

The principle Φ is a consequence of—even equivalent to—weak CH ($2^{\aleph_0} < 2^{\aleph_1}$), so weak CH implies that no ladder system on a club has the 2-uniformization property. For the Whitehead problem, the analogous consequence of Φ is that no W-group A of cardinality $\aleph_1$ is such that $\Gamma(A) = \tilde{\omega}_1$. As it turns out, this is the strongest fact about W-groups that can be proved from CH alone, and it is a strengthening of a classical result of Chase [2, Prop. 2.4]—one of the partial results about uncountable W-groups referred to above—that (assuming weak CH) if A is a W-group, then every countable subset is contained in a countable free subgroup B with the property that C/B is free whenever C is a countable subgroup of A containing B. In model-theoretic terms (cf. [5]), A is $L_{\infty\omega_1}$*-free*, that is, it satisfies the same sentences of the infinitary language $L_{\infty\omega_1}$ as does the free group of cardinality $\aleph_1$. (It is a consequence of MA + $\neg$CH that there is a W-group which is not $L_{\infty\omega_1}$-free [38].)

It was then clear that to have any hope of obtaining the desired consistency result, one must begin in the ground model (of ZFC + GCH) with a ladder system $\eta = (\eta_\delta : \delta \in S)$ on some stationary S which is also co-stationary (i.e., $\omega_1 - S$ is stationary). The iterated forcing that Shelah used to construct a generic extension in which η has the 2-uniformization property is a natural one. For any 2-coloring c of the ladder system η, define the basic forcing (poset), Q_c to consist of countable approximations, $h\restriction\beta$, to the desired uniformizing function, h, of c. One then iterates this forcing $2^{\aleph_1} = \aleph_2$ times, making sure to cover all (names for) 2-colorings of the ladder system. The iteration is with countable support, that is, the conditions in the ultimate forcing are essentially elements of the "product" of the different Q_c which are non-trivial in only countably many coordinates. It is then routine (for forcing experts) to check that in the generic extension ω_1 is preserved, GCH holds, and the ladder system $(\eta_\delta : \delta \in S)$ has the 2-uniformization property.

What was non-routine and new was the proof that S remains stationary in the new universe. (In group-theoretic terms, we want a *non-free* Whitehead group A so its invariant, $\Gamma(A) = \tilde{S}$, must be stationary.) Shelah gave a direct argument; the modern approach (cf. [41, Chapters III, V]) is to use his powerful proper forcing theorem. A *proper* forcing is one which preserves stationary sets in a very general context (see [22] or [41] for details);

in particular, every stationary subset of $\aleph_1$ remains stationary in the generic extension. There is an explicit combinatorial characterization of when a forcing (poset) is proper, one which generalizes the countable chain condition. The fundamental theorem is that an iteration of proper posets with countable support is also proper; this applies in the setting described above because the posets Q_c are proper.

From the modern point of view the fact that GCH continues to hold in the new universe is a consequence of another property of the posets Q_c: they are $(\omega_1 - S)$-complete, which means, roughly, that the union of a countable chain of members of Q_c is in Q_c provided that the domain of the union is not in S. (See [41, Chapter V].)

Shelah's paper [37, p. 202] proves the consistency result about uniformization and then derives the desired group-theoretic theorem (that it is consistent with GCH that W-groups are not all free) in a manner which is, as Alan Mekler remarked [29, p. 271], "rather cryptic". After questioning Shelah about it, and being informed of the advances in proper forcing that Shelah had made, Mekler wrote an exposition [29] from the newer point of view, based on what he denoted $\mathrm{Ax}(S)$, which may be thought of as a version of Martin's Axiom consistent with GCH: it says:

If P is a proper poset of cardinality $\aleph_1$ which is $(\omega_1 - S)$-complete and $\{ D_\nu : \nu < \omega_1 \}$ is a family of dense subsets of P, then there is a directed $G \subseteq P$ such that for all $\nu < \omega_1$ $G \cap D_\nu \neq \emptyset$.

This axiom implies that there are non-free W-groups; in fact, it implies that every $\aleph_1$-free group A with $\Gamma(A) \subseteq \tilde{S}$ is Whitehead. Moreover, the existence of a stationary and co-stationary S such that $\mathrm{Ax}(S)$ holds is consistent with the condition that $\diamondsuit(E)$ holds for all E with $E - S$ stationary. Hence it is consistent with GCH that there is a stationary and co-stationary S such that A is a W-group if and only if $\Gamma(A) \subseteq \tilde{S}$.

While the delicate methods used in the original paper [37] can be replaced, they remain essential for the results contained in a second paper [39] of Shelah on the Whitehead problem and CH. There it is proved, for example, that it is consistent with GCH that there are two different ladder systems on the same stationary set so that one of them has the 2-uniformization property and the other does not. Similarly, it is consistent that there are two $\aleph_1$-free groups A and B with $\Gamma(A) = \Gamma(B)$ while A is a W-group and B is not.

Even more delicate methods are needed for the proof in [40] that it is consistent with GCH that $\mathrm{Ext}(A, \mathbb{Z}) \cong \mathbb{Q}$ for some A. The analogous result for uniformization fails: cf. [39, Theorem 6.2]. (Shelah has commented to me that while Whitehead problems are uniformization problems with an "algebraic coating", the $\mathrm{Ext}(A, \mathbb{Z}) \cong \mathbb{Q}$ problem involves a "more intimate mixing" of algebra and set theory.)

The method of uniformization can be extended to larger cardinals than $\aleph_1$ (cf. [32]). In this way one obtains analogs, above the continuum, of consequences of Martin's Axiom. For example, it is consistent with ZFC + GCH that the smallest non-free W-group has cardinality, say, $\aleph_{36}$ (cf. [11]).

§3. Normal Moore spaces and CH. Devlin and Shelah found another significant application of the methods developed for the Whitehead problem: to the normal Moore space problem. A *Moore space* is a regular topological space, X, with a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers such that for all $p \in X$ and all open sets U with $p \in U$, there is an $n \in \omega$ such that $\bigcup\{G \in \mathcal{G}_n : p \in G\}$ is contained in U. Metric spaces are easily seen to be Moore spaces. Jones [24] asked whether every *normal* Moore space is metrizable and showed that weak CH implies that every *separable* normal Moore space is metrizable.

Jones later [25] defined a space now known as a Jones space; it is essentially a special Aronszajn tree with the tree topology, that is, a basis consists of all open intervals. Jones showed that it is a Moore space and not metrizable but was unable to decide if it was normal. In 1975 Fleissner [14] showed that the problem is undecidable in ZFC: MA + ¬CH implies that a Jones space is normal—thus proving the consistency of a negative answer to Jones' question; but, on the other hand, $\diamondsuit(\omega_1)$ implies that a Jones space is not normal. Using the weak diamond principle, Devlin and Shelah [4, §2] improved the latter by showing that even weak CH implies that a Jones space is non-normal.

Devlin and Shelah [4, §3] also used the consistency result about the existence of a ladder system with the 2-uniformization property to prove that a negative answer to Jones' question is consistent with GCH. Specifically a variant of a Jones space is used, where the ladders lie on branches of the tree and final segments of the ladders $(\eta_\delta : \delta \in S)$ determine neighborhoods of the limit points. (Other nodes of the tree are isolated.) The usual arguments show (in ZFC) that the space is a non-metrizable Moore space. The 2-uniformization property of $(\eta_\delta : \delta \in S)$ is used to show that the space is normal.

In fact, the result provides one half of an independence result. A space is said to be *collectionwise Hausdorff* if and only if every discrete subset of the space can be separated by disjoint open sets; that is, the Hausdorff property is extended from two points to an arbitrary discrete set. One defines similarly collectionwise normal. A topological space is metrizable if and only if it is a collectionwise normal Moore space [1], so the normal Moore space problem is equivalent to asking whether every normal Moore space is collectionwise normal. A weaker question is whether every normal Moore space is collectionwise Hausdorff. Fleissner [13] proved that V = L implies that every normal Moore space is collectionwise Hausdorff, while Devlin-Shelah proved that their space is not collectionwise Hausdorff. Therefore it is independent of GCH whether every normal Moore space is collectionwise Hausdorff.

(In the early 80s Nyikos and Kunen showed that the consistency of a positive answer to the normal Moore space question follows from the consistency of a strongly compact cardinal and Fleissner proved that a large cardinal hypothesis is needed for this result. Fleissner also showed that CH implies the negative answer: there is a non-metrizable normal Moore space. See the survey articles by Tall [44] and Fleissner [15] for more details on these and related subjects.)

§4. **Compactness.** We are now going to consider abelian groups of arbitrary cardinality. Recall that a group is λ-free if and only if every subgroup of cardinality $< \lambda$ is free. As we have observed, Whitehead groups of arbitrary cardinality are (provably in ZFC) $\aleph_1$-free, and $V = L$ implies that Whitehead groups of cardinality $\aleph_1$ are free, or equivalently, W-groups of arbitrary cardinality are $\aleph_2$-free. (Remember: a subgroup of a Whitehead group is Whitehead.) Even more is true: $V = L$ implies the diamond principles $\diamondsuit_\lambda(S)$ for all regular uncountable cardinals λ and all stationary subsets S of λ; essentially the same argument used in Shelah's original paper on the Whitehead Problem shows that these principles imply that a λ-free Whitehead group of cardinality λ is free. Therefore, by induction, $V = L$ implies that every W-group is $\aleph_{n+1}$-free for every $n \in \omega$.

The induction reaches a barrier at the singular cardinal $\aleph_\omega$: a W-group of cardinality $\aleph_\omega$ is $\aleph_\omega$-free, but the method does not imply that it is free. However, a theorem (of ZFC) of Paul Hill [19] from 1970 provides the boost needed; it implies that if λ is a singular cardinal of cofinality ω, then "compactness" holds at λ, i.e., every λ-free abelian group of cardinality λ is free. (The terminology is suggested by the model-theoretic notion of compactness, rather than the topological one.) By Shelah's method and a transfinite induction it follows that $V = L$ implies that every W-group of cardinality $< \aleph_{\omega_1}$ is free. About 1972 Paul Hill [20] obtained a proof (in ZFC) that compactness holds also at singular cardinals of cofinality ω_1. It follows that $V = L$ implies that every Whitehead group of cardinality $< \aleph_{\omega_2}$ is free.

In the summer of 1974 Shelah became aware of Hill's cofinality ω_1 result (when I showed him a copy of Hill's preprint) while he was visiting Stanford prior to the International Congress of Mathematicians (ICM) in Vancouver. After studying the paper Shelah was able to prove a very general "Singular Compactness Theorem" which was announced by E. C. Milner (see below) at the ICM. A special case of the theorem says that compactness holds for abelian groups at every singular cardinal λ. As a consequence, $\mathrm{V} = \mathrm{L}$ implies that *every* Whitehead group (of arbitrary cardinality) is free.

The Singular Compactness Theorem is much more general: it applies to an abstract notion of "free"—defined axiomatically—and says, roughly, that if an object of singular cardinality λ has the property that "most" of its subobjects of cardinality $< \lambda$ are "free", then the object is "free". In particular, the theorem applies to the standard notion of "free" in any variety. For varieties with the (Schreier) property that subalgebras of free algebras are free—such as the variety of all (not necessarily commutative) groups—"most" can be taken to mean "all". Thus for the variety of groups, the theorem says that a (non-commutative) group of singular cardinality λ is free if all of its subgroups of cardinality $< \lambda$ are free.

For varieties in general, care must be taken with the definition of "most": it must be strong enough for the proof, yet weak enough that "most" subalgebras of a free algebra are free. It is not hard to see that a free algebra F has the property that for any cardinal $\kappa < |F|$ there is a *closed unbounded* set C_κ of free subalgebras of F each of cardinality κ, that is, every subset of F of cardinality κ is contained in a member of C_κ, and C_κ is closed under unions of well-ordered chains of length at most κ. We shall say that an algebra A is κ^+*-free* in case there is a closed unbounded set, C_κ, of free subalgebras of A each of cardinality κ. One version of the Singular Compactness Theorem, for an arbitrary variety $\mathcal{V}$ (with at most countably many operations, all finitary) is then as follows:

(†) *Suppose that A is an algebra of singular cardinality λ with the property that for all $\kappa < \lambda$, A is κ^+-free. Then A is free.*

Shelah later found simpler proofs of the Singular Compactness Theorem than the original one. A user-friendly version is to be found in a paper of Hodges [21]; there the meaning of "most" is defined by means of a two-person game and is weaker than the hypothesis in (†). Versions for modules can be found in [10, §IV.3].

§5. Transversal theory and incompactness. Another application of the Singular Compactness Theorem is to transversal theory. Transversal theory was the subject of an address [34] by E. C. Milner at the 1974 ICM; it is a branch of combinatorial mathematics which has as its starting point the marriage theorem of Philip Hall [18]. A *transversal* for an indexed family of sets $F = (F_\nu : \nu < \lambda)$ is a *one-one* function $\varphi\colon \lambda \to \bigcup_{\nu<\lambda} F_\nu$ such that for all $\nu < \lambda$, $\varphi(\nu) \in F_\nu$. We will say that F is "free" if and only if F has a transversal and F is λ-"free" if and only if every subfamily of size $< \lambda$ is "free". A version of the marriage theorem, due to Marshall Hall [17], says that if each F_ν is finite, then F is "free" if F is $\aleph_0$-"free".

Modifying somewhat Milner's notation, let $T(\lambda, \kappa, \mu)$ denote the statement that a sufficient condition for a family $F = (F_\nu : \nu < \lambda)$ to be "free" is that each F_ν has cardinality $< \kappa$ and F is μ-"free". So Marshall Hall's theorem says, in this notation, that for all λ, $T(\lambda, \aleph_0, \aleph_0)$ is true.

The Singular Compactness Theorem applies to this setting and says that for all singular λ and all $\kappa < \lambda$, $T(\lambda, \kappa, \lambda)$ is true. Another "compactness" result is that for weakly compact λ, $T(\lambda, \lambda, \lambda)$ is true. A parallel result had been independently observed for abelian groups: if λ is weakly compact, every λ-free group of cardinality λ is free. Moreover, parallel "incompactness" results were obtained independently by people working in the two areas before 1974; for example:

(1) Assuming V = L, for every regular non-weakly compact λ:
(a) $\neg T(\lambda, \aleph_1, \lambda)$;
(b) there is a non-free λ-free abelian group of cardinality λ;

and

(2) in ZFC, for every regular λ:
(a) $\neg T(\lambda, \aleph_1, \lambda)$ implies $\neg T(\lambda^+, \aleph_1, \lambda^+)$;
(b) if there is a non-free λ-free abelian group of cardinality λ, then there is a non-free λ^+-free abelian group of cardinality λ^+.

In his Singular Compactness paper Shelah conjectured that these analogies were not accidental, more precisely that it is provable in ZFC that for every cardinal λ the following are equivalent:

(i) $\neg T(\lambda, \aleph_1, \lambda)$;
(ii) *there is a λ-free abelian group of cardinality λ which is not free.*

He finally proved this so-called "Incompactness Theorem" in a difficult paper [42] published in 1985 (with an appendix by Alan Mekler, which provides a more accessible introduction to the methods; see also [10, Chapter VII]). The proof introduced and made use of the notion of a λ-set, a generalization of the notion of stationary set, which will be described in the next section. The paper also shows that (i) and (ii) are equivalent to:

(iii) *there is an $L_{\infty\lambda}$-free abelian group of cardinality λ which is not free.*

and imply:

(iv) *there is a λ-free non-abelian group of cardinality λ which is not free.*

(Here the notions refer to freeness in the variety of all groups.) Shelah conjectured—and at one point even had a typed manuscript giving a "proof"—that (iv) implies (i), but this remains an open problem. (However, if we assume V = L, then (iv) *is* equivalent to (i), which is equivalent to λ being regular and not weakly compact.)

The only known proof of the algebraic theorem that (ii) implies (iii) is through the combinatorial property (i). Moreover, the existence of a combinatorial equivalent, like (i), to the algebraic statement (ii) makes easier the proof of consistency results which say that (ii) may fail for some regular non-weakly compact λ. (Fine structure results imply that this is a large cardinal property, that is, the failure of (ii) for some regular λ implies the consistency of the existence of a large cardinal; Mekler and Shelah [31] have pinpointed the exact consistency strength.) Magidor and Shelah [27] have proved consistency results of this sort; for example, it is consistent (assuming the consistency of an infinite number of supercompact cardinals) that (ii) fails for $\lambda = \aleph_{\omega^2+1}$. They also show that this is the smallest regular λ for which (ii) is not provable in ZFC. However, open questions remain; for example, is it consistent that there are only countably many λ—or even only countably many $\lambda < \aleph_{\omega_1}$—such that there is a λ-free abelian group of cardinality λ which is not free?

For other varieties $\mathcal{V}$, such as the variety of left R-modules for a given ring R, it is most natural to consider the *essentially non-free spectrum* of $\mathcal{V}$, namely, the class of λ such that there is a λ-free algebra A in $\mathcal{V}$ of cardinality λ such that for any free algebra F, $A * F$ is not free. (See [33]. For modules, this is equivalent to asking that A be non-projective, not just non-free.) If we assume V = L, the essentially non-free spectrum is either empty or consists of all regular non-weakly compact cardinals, depending on algebraic properties of the variety. (For example, for the variety of R-modules, it is empty if and only if R is a left perfect ring: cf. [9].) In ZFC there are many open questions, but a sweeping conjecture of Mekler and Shelah [33] implies that there are classes K_μ of cardinals (for $\mu \in \omega + 1$) such that $K_0 = \phi$, K_ω is the class of cardinals satisfying (i), $K_\mu \subseteq K_\nu$ for all $\mu \leq \nu$, and for any variety $\mathcal{V}$, the essentially non-free spectrum of $\mathcal{V}$ equals K_μ for some $\mu \in \omega + 1$.

§6. λ-sets. Let ${}^{<\omega}\lambda$ denote the tree of all finite sequences of elements of λ. A *λ-set* is a labeled subtree T of ${}^{<\omega}\lambda$, closed under restriction, where to each node η of T is assigned a regular infinite cardinal λ_η. We require that for the root $\langle\,\rangle$ of T, $\lambda_{\langle\,\rangle} = \lambda$; moreover:

(a) $\eta \in T$ is a final node if and only if $\lambda_\eta = \aleph_0$;

and if η is a non-final node, then

(b) $S_\eta = \{\, \beta : \eta \frown \langle\beta\rangle \in T \,\}$ is a stationary subset of λ_η.

To understand the definition, it is helpful to see how a λ-set arises from a λ-free, non-free, abelian group A of cardinality λ. Write A as the union of a continuous chain of (free) subgroups B_α of cardinality $< \lambda$

$$A = \bigcup_{\alpha<\lambda} B_\alpha$$

such that if B_β/B_α is not free for some $\beta > \alpha$, then $B_{\alpha+1}/B_\alpha$ is not free. (Any continuous chain can be thinned so as to satisfy this last condition.) Let

$$S = \{ \alpha < \lambda : B_{\alpha+1}/B_\alpha \text{ is not free} \}.$$

A crucial fact is that if A is not free, then S is a stationary subset of λ. Indeed, if S is not stationary, there is a club $C \subseteq \lambda$ such that $C \cap S = \emptyset$; writing $A = \bigcup_{\alpha\in C} B_\alpha$ we show A is free by constructing inductively a basis of A: for all $\alpha \in C$, if α^+ denotes the smallest element of C larger than α, then B_{α^+}/B_α is free (since $\alpha \notin S$) and hence we can extend a basis of B_α to a basis of B_{α^+}.

The set S defined above is the Γ-invariant of A described in Section 2, that is, $\Gamma(A) = \tilde{S}$. Let $S_{\langle\,\rangle} = S$ and for $\beta \in S_{\langle\,\rangle}$, let $\lambda_{\langle\beta\rangle}(<\lambda)$ be minimal such that $B_{\beta+1}/B_\beta$ has a subgroup of cardinality $\lambda_{\langle\beta\rangle}$ which is not free; $\lambda_{\langle\beta\rangle}$ is regular by the Singular Compactness Theorem. If $\lambda_{\langle\beta\rangle} \neq \aleph_0$, choose $A_\beta \subseteq B_{\beta+1}$ of cardinality $\lambda_{\langle\beta\rangle}$ such that $(A_\beta + B_\beta)/B_\beta$ is not free. Then $(A_\beta + B_\beta)/B_\beta$ is $\lambda_{\langle\beta\rangle}$-free, and essentially by induction, pulling back to A_β, we can erect a $\lambda_{\langle\beta\rangle}$-set at the node $\langle\beta\rangle$.

There is additional data that comes out of this deconstruction of A. To incorporate it, we introduce the notion of a λ-*system* which is a λ-set together with the assignment of a set B_η to each $\eta \in T$. We require that $B_{\langle\,\rangle} = \emptyset$ and for each $\eta \in T$:

(c) $\lambda_{\eta\frown\langle\beta\rangle} \leq |B_{\eta\frown\langle\beta\rangle}| < \lambda_\eta$ and $\{ B_{\eta\frown\langle\beta\rangle} : \beta \in S_\eta \}$ is a continuous chain of sets.

For the λ-system associated to A, we let $B_{\langle\beta\rangle} = B_\beta$.

In a similar fashion, a λ-"free" non-"free" family $F = (F_\nu : \nu < \lambda)$ of countable sets gives rise to a λ-system. In fact, the proof of the Incompactness Theorem (that is the equivalence of (i) and (ii) of the previous section) passes through

(i′) *there is a λ-system $(T, \langle \lambda_\eta, B_\eta : \eta \in T \rangle)$ and a λ-"free" family $(F_\eta : \eta \in T_f)$ indexed by the set T_f of final nodes of T such that for every $\eta \in T_f$, $F_\eta \subseteq \bigcup_{m\leq\ell(\eta)} B_{\eta\restriction m}$.*

We call this a λ-"free" family *based on a λ-system*. Assuming V = L, for any regular non-weakly compact λ, there is a λ-"free" family *based on a λ-system* $(T, \langle \lambda_\eta, B_\eta : \eta \in T \rangle)$ where $T \subseteq {}^1\lambda$, i.e., T is simply a stationary subset, $S_{\langle\,\rangle}$, of λ (non-reflecting and consisting of ordinals of cofinality ω); this is the reason that the incompactness phenomenon is more tractable under V = L.

Combining the uniformization ideas of Section 2 with the incompactness methods of this and the previous section, one is able to prove a combinatorial equivalent to the existence of a λ-free W-group of cardinality λ which is not free; it consists of a λ-"free" family based on a λ-system which has a kind of uniformization property. Though it is a fairly complicated object, it allows the proof of a purely algebraic theorem [12]:

If there is a λ-free W-group of cardinality λ which is not free, then there are 2^λ different $L_{\infty\lambda}$-free Whitehead groups of cardinality λ.

In conclusion, we mention that Mekler and Shelah [30] have used a definable version of the diamond principles for λ-systems to prove that it is independent of ZFC $+ \neg$CH whether every W-group is free. As they remark, it is perhaps a reflection of the psychology of mathematics that this seems less interesting than the corresponding independence result about CH with which we began.

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