The Bulletin of Symbolic Logic
Volume 2, Number 1, March 1996

TWO APPLICATIONS OF INNER MODEL THEORY TO THE STUDY OF $\underset{\sim}{\Sigma}^1_2$ SETS

GREG HJORTH

§0. Preface. There has been an expectation that the endgame of the more tenacious problems raised by the Los Angeles 'cabal' school of descriptive set theory in the 1970's should ultimately be played out with the use of inner model theory. Questions phrased in the language of descriptive set theory, where both the conclusions and the assumptions are couched in terms that only mention simply definable sets of reals, and which have proved resistant to purely descriptive set theoretic arguments, may at last find their solution through the connection between determinacy and large cardinals.

Perhaps the most striking example was given by [24], where the core model theory was used to analyze the structure of HOD and then show that all regular cardinals below $\Theta^{L(\mathbb{R})}$ are measurable. John Steel's analysis also settled a number of structural questions regarding $HOD^{L(\mathbb{R})}$, such as GCH.

Another illustration is provided by [21]. There an application of large cardinals and inner model theory is used to generalize the Harrington-Martin theorem that Π^1_1 determinacy implies $(< \omega^2 - \Pi^1_1)$ determinacy.

However, it is harder to find examples of theorems regarding the structure of the projective sets whose only known proof from determinacy assumptions uses the link between determinacy and large cardinals. We may equivalently ask whether there are second order statements of number theory that cannot be proved under PD–the axiom of projective determinacy–without appealing to the large cardinal consequences of the PD, such as the existence of certain kinds of inner models that contain given types of large cardinals.

An example of this phenomenon is presented by Section 2 of this paper. It should be mentioned that at one stage it appeared as if the first such example was to be given by

Received November 15, 1995.
I thank John Steel for reading through an earlier draft of this announcement and making many helpful suggestions.

1079-8986/96/0201-0004/$2.40

THEOREM 0.1 (Steel, Woodin). [1] *Assume* $\underset{\sim}{\Pi}^1_2$ *determinacy. Let* $A \subset \mathbb{R}$ *be an uncountable* Δ^1_3 *set. Then there exists a real* y_0 *such that* $\forall x \geq_T y_0 \exists x_0 \in A(x \in \Delta^1_3(x_0) \wedge x_0 \in \Delta^1_3(x))$.

However about a year after the proof of 0.1 was completed, an argument was found that made no use of large cardinals ([5]).

In some ways the results of this announcement are closest in spirit to Hugh Woodin's (unpublished) combinatorial proof of $\mathrm{PWO}(\Pi^1_3)$ from the existence of a Woodin cardinal δ and a measurable cardinal $\kappa > \delta$. This was an old result, in as much as it was previously known how to obtain $\underset{\sim}{\Pi}^1_2$ determinacy from the large cardinals (Martin and Steel), and how to obtain $\mathrm{PWO}(\Pi^1_3)$ from $\underset{\sim}{\Pi}^1_2$ determinacy (Martin, Moschovakis); but the interest here is not in the conclusion as such, but in the method of reasoning, which exploited the inner model theory of [17]. Indeed, Woodin's argument influenced the development of the Harrington type theorem in Section 3.

The first of the two results in this announcement may be viewed as a Σ^1_2 analogue of

THEOREM 0.2 (Louveau). *Let* $A \in \Delta^1_1 \cap \underset{\sim}{\Pi}^0_\alpha$. *Then there is a real* $a \in \Delta^1_1$ *with* $A \in \Pi^0_\alpha(a)$. (*See* [16]).

This theorem has been extremely fertile, giving birth to a great number of corollaries in both classical and effective descriptive set theory. Loosely speaking, it states that if a Borel set happens to be reasonably simple in some effective sense, then it is witnessed to be simple by some easily calculable real; and hence in many contexts one is able to save a whole quantifier by knowing that a code for a given Borel set is close at hand. (See [10] for an example.)

Partial results at the level of Σ^1_2 are given by

THEOREM 0.3 (Kechris). *Assume* $\underset{\sim}{\Pi}^1_2$ *determinacy and that* $A \in \Delta^1_3 \cap \underset{\sim}{\Delta}^1_1$ *with* $A \neq \emptyset$. *Then* $A \cap \Delta^1_3 \neq \emptyset$. ([11]).

THEOREM 0.4 (Louveau). *Assume* $\underset{\sim}{\Pi}^1_2$ *determinacy and that* $A \in \Delta^1_3 \cap \underset{\sim}{\Pi}^1_1$. *Then there is* $a \in \Delta^1_3$ *with* $A \in \Sigma^1_2(a)$. ([15]).

Here it is shown that

THEOREM 0.5. *Assume* $\underset{\sim}{\Pi}^1_2$ *determinacy and that* $A \in \Delta^1_3 \cap \underset{\sim}{\Sigma}^1_2$. *Then there is* $a \in \Delta^1_3$ *with* $A \in \Sigma^1_2(a)$.

This answers a question raised in [11].

[1] The exact history of this theorem is that Woodin showed how to prove the result assuming the "$\underset{\sim}{\Sigma}^1_3$ correctness" conjecture from core model theory, and Steel later proved the conjecture in [23], thereby completing the picture.

COROLLARY 0.6. *Assume* $\underset{\sim}{\Pi}^1_2$ *determinacy and that* $A \in \Delta^1_3 \cap \underset{\sim}{\Sigma}^1_2$ *with* $A \neq \emptyset$. *Then* $A \cap \Delta^1_3 \neq \emptyset$.

This answers a question raised in [15].

A completely different kind of problem for $\underset{\sim}{\Sigma}^1_2$ sets is inspired by

THEOREM 0.7 (Harrington). *Assume* $AD^{L(\mathbb{R})}$. *Then there is no* $\underset{\sim}{\delta}^1_1$ *sequence of distinct Borel sets of bounded rank in* $L(\mathbb{R})$. (*See* [3]).

It is often felt that the size of the continuum *feels* very different to the size of $\aleph_1 (= \underset{\sim}{\delta}^1_1)$, as the kinds of processes that give rise to $2^{\aleph_0}$ are very different to those that give rise to $\aleph_1$. One way to try to turn this vague feeling into a precise theorem is to observe that large cardinals or determinacy imply that there is no *simply definable* $\aleph_1$ sequence of distinct reals. Since there is a natural bijection between closed sets and reals, and since the objects in $L(\mathbb{R})$ would seem to constitute a plausible explication of the notion of *reasonably definable*, we might very well view Harrington's theorem as a kind of generalization of the intuition that $2^{\aleph_0}$ and $\aleph_1$ are somehow qualitatively very different.

Although this is not an issue for the perspective of the present paper, it should also be mentioned that fragments of Harrington's theorem can be proved in just ZFC, where they are useful tools. (See for instance [25] or Section 7 of [1].)

Some partial results in this direction for Σ^1_2 were given by

THEOREM 0.8 (Kechris). *Assume* $AD^{L(\mathbb{R})}$. *Then there is no* $\underset{\sim}{\delta}^1_2$ *sequence of disjoint* $\underset{\sim}{\Delta}^1_2$ *sets in* $L(\mathbb{R})$. (*See* [12].)

and

THEOREM 0.9 (Jackson, Martin). *Assume* $AD^{L(\mathbb{R})}$. *Then there is no* $\underset{\sim}{\delta}^1_2$ *sequence of strictly increasing* $\underset{\sim}{\Sigma}^1_2$ *sets in* $L(\mathbb{R})$. (*See* [7].)

THEOREM 0.10. *Assume* $AD^{L(\mathbb{R})}$. *Then there is no* $\underset{\sim}{\delta}^1_2$ *sequence of distinct* $\underset{\sim}{\Sigma}^1_2$ *sets in* $L(\mathbb{R})$.

This answers a question that has been raised in [9], [6], and [18].

Although some sort of an introduction is given in Section 1 below, the reader may wish to refer to [17] for the precise definitions of Doddages, premouse, extender, iteration strategy and the other concepts from coarse inner model theory; [14] gives the necessary basics of Q-theory and effective descriptive set theory in the context of determinacy; background of a more general nature, including the definition of the projective and Borel hierarchy can be found in [8] or [20].

§1. A bare bones introduction. We will use the *coarse inner model theory* of [17]. The authors of [17] show that if $\vec{E}$ is an appropriately chosen extender sequence in V, then we can form an inner model $L[\vec{E}]$ with certain nice properties; by choosing the sequence to be sufficiently *maximal* we obtain some of the large cardinal strength in V inside the model $L[\vec{E}]$. In general their methods do not necessarily show that $L[\vec{E}]$ has a nice wellordering of the reals, but by iterating the process, forming $L[\vec{E}]^{L[\vec{E}]}$, and so on, we can at last obtain a model of this form with a Δ^1_3 wellordering of the reals. If we begin with a Woodin cardinal δ in V, we will in the end arrive at some $\bar{\delta} \leq \delta$ Woodin in the final $L[\vec{E}]$. An important refinement of this result is due to Woodin and follows from the development of the *free extender algebra* below: in the case when these models are *sufficiently iterable* we can obtain $L[\vec{E}] \cap \mathbb{R} = Q_3$.

The inner model theory constructed in [17] is often called *coarse inner model theory* since no effort is made to build the models from below in the sense of providing them with a *fine structure* that admits a level by level analysis; examples of fine structural models include Gödel's L, the Dodd-Jensen core model of [2], and the later extension given by Mitchell and Steel in [19] and [23].

We will in general work with *expanded premice* of the form $\mathcal{M} = (M, \in, \delta, \mathcal{F}, \vec{v})$, where $M = L((V_\delta)^M)$, or sometimes $M = L_\alpha((V_\delta)^M)$, some α, and $\mathcal{F}$ is the δ-maximal Doddage. $L[\vec{E}]^{\mathcal{M}}$ will denote the model obtained by performing the Martin-Steel construction inside M, with the sequence $\mathcal{F}$. For $\mathcal{M}$ such an expanded premouse, M will always denote the underlying set, and $\delta^{\mathcal{M}}$ will denote the δ above. We will only be interested in the case that for all $\bar{\delta} < \delta$ $L[\vec{E}]^{\mathcal{M}}$ and M satisfy that $\bar{\delta}$ is not Woodin; by the definition of $\mathcal{F}$ being δ-maximal it follows that there are no extenders in $L[\vec{E}]$ above δ, and so that δ is the only possible Woodin cardinal in $L[\vec{E}]$.

A crucial notion from [17] is that of *iteration trees*. This gives a method of *iterating* an expanded premouse $\mathcal{M}$ along a tree of embeddings arising from extenders taken from the image of the $\mathcal{F}$ sequence.

In this paper all iteration trees will be in the coarse structural sense of [17]; they will moreover be $+2$ and *normal*. We will only be concerned with countable objects, and so in particular the reader should assume that all iteration trees are countable and that all expanded premice are constructed from some countable $(V_{\delta^{\mathcal{M}}})^M$, in the sense that $M = L((V_{\delta^{\mathcal{M}}})^M)$ with $\delta^{\mathcal{M}} < \omega_1$.

Here there is a range of subtly different notions of *iterability*.

[17] shows that all (countable) submodels, M, of sufficiently large initial segments of V_θ are Π^1_2 *iterable*, in the sense of there being no illfoundedness forced upon us at successor steps, and for each countable iteration tree $\mathcal{T}$ on M there are branches through $\mathcal{T}$ that are α-wellfounded for each $\alpha < \omega_1$; in

fact they show that it must be *completely iterable* in the sense outlined below. An absoluteness argument thereby implies that each class size expanded premouse $\mathcal{M}$ must necessarily be Π^1_2 iterable. (This notion is needed for the boundedness argument of Section 3.)

I will say that $\mathcal{M}_0$ is *completely* iterable if given any sequence of iteration trees, $(\mathcal{T}_i)$, each $\mathcal{T}_i$ on $\mathcal{M}_i$, with final model $\mathcal{M}_{i+1}$, with branches at limit stages chosen according to some specified *iteration strategy*, we have that (i) there are no illfounded ultrapowers possible at successor steps; (ii) our iteration strategy always chooses wellfounded cofinal branches at limit stages; and finally, (iii) if this sequence of trees is infinite, then $\mathrm{DirLim}_{i\in\omega}\mathcal{M}_i$ is wellfounded. It is implicit in [17] that if δ is Woodin in V, $V_\delta^\sharp$ exists, and we form $\mathcal{M} = (M, \in, \bar{\delta}, \mathcal{F}, \vec{v})$ by taking a countable $\bar{\delta}$ with $M = L((V_{\bar{\delta}})^M)$ an elementary substructure of $L(V_\delta)$, then $\mathcal{M}$ will be completely iterable in this sense.

A slightly more general notion is provided by work of Woodin. For the purpose of this paper I will simply say that an expanded premouse is *iterable* if it satisfies (i) and (ii) above with respect to some suitably chosen iteration strategy. [2] We are now in a position to state the essential theorem that links the inner model theory that we will use and the determinacy assumptions with which we begin.

THEOREM 1.1 (Woodin). *Assuming $\underset{\sim}{\Pi}^1_2$ determinacy, and $y \in \mathbb{R}$, there is an iterable $\mathcal{M} = (M, \in, \delta, \mathcal{F}, \vec{v})$ with δ Woodin in $L[\vec{E}, y]$, $y \in M$, $\mathbb{R} \cap M = Q_3(y)$.*

For such an $L[\vec{E}]$ one obtains that its reals are exactly equal to Q_3. That the reals are included in $L[\vec{E}]$ follows routinely from the comparison process given in [17]. The reverse inclusion, along with the stronger result that all class size models containing a Woodin cardinal are closed under the Q_3 operation, can be shown by appealing to Woodin's *free extender algebra*.

Let $\mathcal{L}_{\delta,0}$ be the infinitary propositional logic formed by closing all basic statements of the form $n \in \dot{x}$ under negation and less than δ disjunction and conjunction. Given $N \subset M$ any two class inner models, $\vec{F}$ a δ sequence of extenders over M, with $\vec{F} \cap N$ in N, we can form the free extender algebra over N, $\mathbb{B}(N, \vec{F})$ by taking $\mathcal{L}^N_{\delta,0}$, and passing to the Lindenbaum algebra induced by the following kinds of identifications: fix $F \in \vec{F}$ of length ρ, critical point κ, and $\nu < \rho$, $j_F : N \to Ult(N, F)$ the ultrapower map, and suppose that $(\phi_\alpha)_{\alpha\in\kappa} \subset \mathcal{L}^N_{\delta,0}$ is a sequence in N with each $(\phi_\alpha) \in H(\kappa)^N$; suppose that $j((\phi_\alpha)_{\alpha\in\kappa}) = (\bar{\phi}_\alpha)_{\alpha\in\bar{\kappa}}$ with each $\bar{\phi}_\alpha \in H(\rho)^{Ult(N,F)}$ for $\alpha < \nu$.

[2] In fact, it is known that in our present context–assuming sharps for reals and considering only countable trees on class sized 1-small mice with $\delta < \omega_1^V$–that any such iteration strategy is determined by the requirement that it always chooses the unique wellfounded branch at limit stages.

Then we introduce the identity

$$|[\bigvee(\phi_\alpha)_{\alpha\in\kappa}]| = |[\bigvee(\bar{\phi}_\alpha)_{\alpha\in\nu}]|.$$

$\mathbb{B}(N, \vec{F}, \alpha)$ is the version using only extenders with critical points above α.

Woodin has shown that if $\vec{F}$ witnesses δ Woodin in N–in the sense that for all $A \subset \delta$ there is a $\kappa < \delta$ that is witnessed to be $< \delta$ A-strong by the extenders on the $\vec{F}$ sequence–then $\mathbb{B}(N, \vec{F})$ is δ-c.c. in N and that every real in M is $\mathbb{B}(N, \vec{F})$-generic over N. He has also adapted the comparison process to show that given any real $y \in V$ we can attempt to iterate N to make y generic over its image. Provided we have mild coherence conditions on the extender sequence, such as arising in [17], this can only fail as a result of a crisis in iterability for N, or the corresponding expanded premouse $\mathcal{M}$ if $N = L[\vec{E}]^{\mathcal{M}}$; even here we still obtain y generic over the corresponding *piece together model*, $L[\vec{E}(\mathcal{T})]$, obtained by taking the limiting value of the $\vec{E}$ sequence along the tree. (Here the reader could compare the proof of 6.13 of [17] for an example of how one can piece together a model along the branches of an iteration tree of limit length.)

LEMMA 1.2. *N as above, $\vec{F}$ witnessing Woodiness, in the sense outlined above. $\mathbb{B}(N, \vec{F}) \times \mathbb{B}(N, \vec{F})$ is δ-c.c. in N.*

PROOF. Instead suppose that $A = (|[\phi_\alpha]|, |[\theta_\alpha]|)_{\alpha\in\delta}$ provides an antichain in N. By the assumption on $\vec{F}$ we find κ that is $< \delta$ A-strong over N, and then some $F \in \vec{F}$ with j_F A-strong past $(\phi_\kappa, \theta_\kappa)$. Let μ be the canonical normal measure on κ obtained from j_F. Then without loss of generality,

$$\forall^\mu \alpha \in \kappa \phi_\alpha \perp \phi_\kappa,$$

whence we can adjust the sequence to obtain $(\phi'_\alpha)_{\alpha\in\kappa}$ with

$$\forall^\mu \alpha \in \kappa \phi_\alpha = \phi'_\alpha,$$

$$\forall \alpha \in \kappa \phi'_\alpha \perp \phi_\kappa.$$

Therefore $(j_F(\phi'_\alpha)_{\alpha\in\kappa}))_\kappa = \phi_\kappa$ and so

$$\bigvee((\phi'_\alpha)_{\alpha\in\kappa}) \vee \phi_\kappa \Leftrightarrow \bigvee((\phi'_\alpha)_{\alpha\in\kappa}),$$

by the definition of $\mathbb{B}(N, \vec{F})$, thereby contradicting the assumption that $|[\phi_\kappa]| \neq 0$ in $\mathbb{B}(N, \vec{F})$. ⊣

A more detailed survey, along with precise definitions and the proofs of some earlier results using similar ideas, should eventually appear in [4]. The proofs below are only intended to be sketches.

§2. The Louveau type theorem. At all stages we work under the assumption of $\utilde{\Pi}^1_2$ determinacy. The existence of sufficiently nice inner models in which some $\delta < \omega_1$ is Woodin follows from the remarks in Section 1.

LEMMA 2.1. *Let M be a class inner model in which some $\delta < \omega_1$ is Woodin, $M \cap \mathbb{R} = Q_3$. Let A be a Π^1_3 set such that $A \cap M^{\mathbb{P}} \neq 0$ for some $\mathbb{P} \in (V_\delta)^M$. Then $\Delta^1_3 \cap A \neq 0$.*

PROOF. By Spector-Gandy we find φ a Π^1_2 formula such that $a \in A \Leftrightarrow \exists b \in Q_3(a)(\varphi(a,b))$. Since all small forcing extensions of M are closed under the Q_3 operation, in light of the Woodin cardinal δ, we have that M can reflect this and find a countable in M $\bar{M}_{\bar{\delta}}$ and $\bar{\mathbb{P}} \in \bar{M}_{\bar{\delta}}$ with $L(\bar{M}_{\bar{\delta}})^{\bar{\mathbb{P}}} \models \exists a, b(\varphi(a,b))$. Then we can find an $L(\bar{M}_{\bar{\delta}})$-generic in M, with the resulting a and b both in Q_3. By searching for the least such (a,b) in a good Δ^1_3 wellordering of Q_3, and applying Spector-Gandy one last time, we obtain $a \in \Delta^1_3$. ⊣

THEOREM 2.2 ($\utilde{\Pi}^1_2$ determinacy). $A \in \Delta^1_3 \cap \utilde{\Sigma}^1_2 \Rightarrow \exists a \in \Delta^1_3(A \in \Sigma^1_2(a))$.

PROOF. Fix an expanded iterable class sized premouse, $\mathcal{M}$, with $M \cap \mathbb{R} = Q_3$, $\delta < \omega_1$ Woodin in $L[\vec{E}]$. Let a be such that $A \in \Sigma^1_2(a)$. We may assume a is $\mathbb{B}(\vec{E}, L[\vec{E}])$-generic over $L[\vec{E}]$. Let $A = p[B_0]$ and $\neg A = p[B_1]$, $B_0, B_1 \in \Pi^1_2$. Choose $|[\psi]| \in \mathbb{B}(\vec{E}, L[\vec{E}])$ forcing that $x(\dot{G})$, the real described by the generic object, codes a $\utilde{\Sigma}^1_2$ set separating $p[B_0]$ and $p[B_1]$ through all future generic extensions and such that $a \models \psi$. (Since a codes such a $\utilde{\Sigma}^1_2$ set in $L[\vec{E}, a]$, this ψ exists by the forcing theorem.)

We then reflect back all pertinent facts to get $\alpha < \beta < \delta$, with $\psi \in L_\beta[\vec{E}|_\alpha] \prec L_\gamma[\vec{E}]$ for some sensible γ. So in particular we have that α is a strong limit cardinal and locally appears Woodin.

Let A_0 be the set of reals x such that for some iteration $i : L[\vec{E}|_\alpha] \to L[\vec{E}_0|_{\alpha_0}]$, $i(\beta) = \beta_0$, we have that x is $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$-generic over $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ below some ϕ with $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ satisfying that $(|[\phi]|, |[i(\psi)]|)$ forces in the product algebra that $x(G_l)$ is in the $\utilde{\Sigma}^1_2$ set coded by $x(G_r)$. (Here G_l and G_r correspond to the left and right copies of $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$, so that $G_l \times G_r$ is $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}]) \times \mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$ generic. Given a generic filter G on the free extender algebra, it will be convenient to let $x(G)$ denote the real described by G, in the sense that $x(G)(n) = 1 \Leftrightarrow$ "$n \in \dot{x}$" $\in G$.) Since for any $x \in A$ with $(x,y) \in B_0$ we can iterate to make (x,y) generic over the resulting $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$, it follows that $A \subset A_0$.

To see the reverse conclusion, suppose $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ is an iterate of $L_\beta[\vec{E}|_\alpha]$ witnessing that $y \in A_0$, and assume also that $(y,z) \in B_1$. Let ψ_0 be the image of ψ under this iteration. Thus we must have some ϕ with $(|[\phi]|, |[\psi_0]|)$ forcing that $x(G_l)$ is in the $\utilde{\Sigma}^1_2$ set coded by $x(G_r)$. By extending the iteration, and drawing on the assumption of iterability, we may assume that $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ is an initial segment of $L[\vec{E}_0] = L[\vec{E}]^{\mathcal{M}_0}$ and (y, z) is $\mathbb{B}(\vec{E}_0, L[\vec{E}_0], \alpha_0)$-generic over $L[E_0]$. Let $g \subset \mathbb{B}(\vec{E}, L[\vec{E}_0])$ be an $L[\vec{E}_0][y, z]$-generic filter below $|[\psi_0]|$. Let G_y be the $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$-generic filter over $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$ generated by y.

Claim. *(g, G_y) is $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$)-generic over $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}]) \times \mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$*

Otherwise we obtain a maximal antichain $\mathcal{A} = (|[\varphi_\alpha]|, |[\theta_\alpha]|)_{\alpha\in\rho}$, some $\rho \in \alpha_0$ by 1.3, and $|[\phi_1]| \in g$ with $\forall\alpha \in \rho(\varphi_\alpha \perp \phi_1 \vee y \models \neg\theta_\alpha)$. Since $\mathcal{A}$ is an antichain, $(|[\bigwedge\{\neg\varphi_\alpha : \varphi_\alpha\perp\phi_1\}]|, |[\bigwedge\{\neg\theta_\alpha : \neg(\varphi_\alpha\perp\phi_1)\}]|)$ cannot be a condition in $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])\times$ $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$. Since $|[\phi_1]|$ is non-zero in $\mathbb{B}(\vec{E}_0, L[\vec{E}_0])$, and hence $|[\bigwedge\{\neg\varphi_\alpha : \varphi_\alpha\perp\phi_1\}]|$ is non-zero in $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$, this requires $|[\bigwedge\{\neg\theta_\alpha : \neg(\varphi_\alpha\perp\phi_1)\}]|$ to be zero in $\mathbb{B}(\vec{E}_0|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$, contradicting the assumption that G_y is a filter on this Boolean algebra. ($\dashv$Claim.)

Then by the upward absoluteness of Σ^1_2 facts, we get y in the $\underset{\sim}{\Sigma}^1_2$ set coded by the real, $x(g)$, described by g. By the product lemma we get that (y, z) is generic over $L[\vec{E}][x(g)]$, and a contradiction.

Thus $A = A_0$, and so in a small forcing extension of M we obtain a $\underset{\sim}{\Sigma}^1_2$ code for A. Now the theorem follows by 2.1. $\dashv$

The argument shows the stronger result that if A_0 and A_1 are two disjoint Σ^1_3 sets that can be separated by a $\underset{\sim}{\Sigma}^1_2$ set, then they can be separated by a $\Sigma^1_2(a)$ set for some $a \in \Delta^1_3$.

Corollary 0.6 now follows by Shoenfield absoluteness and the fact that $\underset{\sim}{\Pi}^1_2$ determinacy is easily sufficient to imply that the Δ^1_3 reals are closed under constructibility.

§3. The Harrington type theorem. There are technical reasons why we will need to keep track of the power set operation; thus it will be convenient to work with a variation of the usual extender sequence.

Definition 3.1. Fix $\mathcal{M}$ an expanded premouse with $L[\vec{E}]$ the derived inner model. Let $R(\vec{E})$ consist of $E_\alpha \in \vec{E}$ such that

(i) ν_α is a Beth fixed point in $L[\vec{E}]$;
(ii) E_α is ν_α strong over $L[\vec{E}]$;
(iii) for $\kappa_\alpha = cp(E_\alpha)$, $H(\kappa_\alpha)^{L[\vec{E}]} = L_{\kappa_\alpha}[\vec{E}]$.

It follows from the definitions that the $R(\vec{E})$ sequence satisfies the coherence condition used in [17] and that $R(\vec{E})$ witnesses δ Woodin in $L[\vec{E}]$ provided $\vec{E}$ does. For our purposes it is important to note that an iteration tree using just the extenders on the $R(\vec{E})$ sequence will have an agreement condition along the branches not just for $(\vec{E}|_{\rho_\alpha})^{\mathcal{M}_\alpha}$ and $(V_{\rho_\alpha})^{\mathcal{M}_\alpha}$ but also $(V_{\rho_\alpha})^{L[\vec{E}]^{\mathcal{M}_\alpha}}$. The $R(\vec{E})$ sequence does not have closure under initial segments; but fortunately, Woodin's iteration to make every real generic does not require this.

LEMMA 3.2 (Essentially Woodin). *Let $\mathcal{M}$ be an expanded premouse, with $\vec{E}$ witnessing δ Woodin in $L[\vec{E}]$. Let x be a real. Then there is an iteration tree $\mathcal{T}$ on $\mathcal{M}$ such that*

(1) *$\mathcal{T}$ uses only extenders on the $\mathcal{F}$ sequence that project to extenders on the $R(\vec{E})$ sequence; and we have either:*

(2a) *$\mathcal{T}$ has a final model $\mathcal{M}_0$ with $L[\vec{E}_0]$ the derived inner model and x $\mathbb{B}(R(\vec{E}_0), L[\vec{E}_0])$-generic over $L[\vec{E}_0]$; or*

(2b) *$\mathcal{T}$ has limit length and for $\delta(\mathcal{T}) = sup\{\rho_\alpha : \alpha < lh(\mathcal{T})\}$*

 (i) *$lim_{\alpha<\lambda}(L[\vec{E}|_{\rho_\alpha}])^{\mathcal{M}_\alpha} =_{df} L[\vec{E}(\mathcal{T})]$ is well defined;*

 (ii) *$lim_{\alpha<\lambda}(V_{\rho_\alpha}^{L[\vec{E}]})^{\mathcal{M}_\alpha}$ is well defined and equals $(V_{\delta(\mathcal{T})})^{L[\vec{E}(\mathcal{T})]}$;*

 (iii) *$lim_{\alpha<\lambda}(R(\vec{E})|_{\rho_\alpha})^{\mathcal{M}_\alpha}$ is well defined and equals $R(\vec{E}(\mathcal{T}))$;*

 (iv) *x is $\mathbb{B}(R(\vec{E}(\mathcal{T})), L[\vec{E}(\mathcal{T})])$-generic over $L[\vec{E}(\mathcal{T})]$.*

3.2 is proved by iterating away the least extender on the sequence which provides an "axiom" not satisfied by x. Either we go out to a tree of length $\delta^+ + 1$, and then obtain the usual type of contradiction by reflection, or the process succeeds as in case (2a), or we are witness to a failure in iterability and (2b) results. The one place we need the $R(\vec{E})$ sequence is in case (2b); here we want to know that for any b cofinal in $\mathcal{T}$ with $\mathcal{M}_b$ defined to be the model obtained by taking the direct limit along b, we must have if $\mathcal{M}_b$ is wellfounded past $\delta(\mathcal{T})$, then $(H(\delta(\mathcal{T}))^{L[\vec{E}]})^{\mathcal{M}_b} \subset L_{\delta(\mathcal{T})}[\vec{E}(\mathcal{T})]$.

THEOREM 3.3. *Assume $AD^{L(\mathbb{R})}$. Then there is no $\underset{\sim}{\delta}^1_2$ sequence of distinct $\underset{\sim}{\Sigma}^1_2$ sets in $L(\mathbb{R})$.*

PROOF. (Under the further assumption that there is a Woodin cardinal and all sets have sharps.) Assume that $(B_\alpha)_{\alpha\in\underset{\sim}{\delta}^1_2}$ witnesses otherwise. In what follows, we will be using sharps to code ordinals less than $\underset{\sim}{\delta}^1_2$; it is important to note that this is absolute to $L[w]$ for any $w \in 2^\omega$. Using the coding lemma for the pointclass $\underset{\sim}{\Sigma}^1_3$, we find a real z that codes the entire sequence in the sense that there is a set $D \in \Pi^1_2(z)$ such that for each $\gamma < \underset{\sim}{\delta}^1_2$ there is a w coding γ and a real $y = (y_0, y_1)$ with $D(w, y)$, and, moreover, for any such (w, y) we have that y_0 codes B_γ as a section of the universal Σ^1_2 set. For ease of presentation, let us assume z is recursive and $D \in \Pi^1_2$. Let $\mathcal{M}$ be a completely iterable expanded premouse, with $\vec{E}$ witnessing $\delta < \omega_1$ Woodin in $L[\vec{E}]$; we obtain $\mathcal{M}$ by [17] and our large cardinal assumptions.

The rough outline of the rest of the proof is as follows. We will find a $\underset{\sim}{\Sigma}^1_2$ set I of objects that arise in the lower parts of iterates of $\mathcal{M}$ such that for all $\gamma < \underset{\sim}{\delta}^1_2$ there is a $N \in I$ such that B_γ is N-generic over some $\mathbb{P} \in (V_{\delta^N})^N$. This is related to proof of 2.2, but with a modification described below that requires a kind of boundedness argument. Given the code y of B_γ, we can describe γ in a $\Pi^1_3(y)$ fashion. So by a theorem of Kechris and Martin we will get that a sharp coding γ is small generic over N. This will produce a contradiction to Kunen-Martin.

The first step is to show that we can obtain a sufficiently big $\underset{\sim}{\Sigma}^1_2$ set. This requires an argument to the effect that there is some easily identifiable stage κ below the first $< \delta$ strong cardinal in $\mathcal{M}$ such that everything of interest occurs below κ, and its images through all iterations. This in turn involves an appeal to boundedness in the sense of descriptive set theory.

CLAIM 1. *Let w code $\gamma < \underset{\sim}{\delta}^1_2$, $x, y \in \mathbb{R}$, and assume $D(w, y)$ and $x \in B_\gamma$. Let $N_0 \subset N$ be two proper class models with $\vec{F}$ an extender sequence in N such that $F|N_0$ witnesses $\delta_0 < \omega_1$ Woodin in N_0; assume $w, x, y \in N$. Then there are non-zero $\|[\varphi]\|, \|[\psi]\| \in \mathbb{B}(\vec{F}, N_0)$ such that $x \models \phi, (w, y) \models \psi$ and $(\|[\phi]\|, \|[\psi]\|)$ forces in $\mathbb{B}(\vec{F}, N_0) \times \mathbb{B}(\vec{F}, N_0)$ that $x(G_l)$ is in the $\underset{\sim}{\Sigma}^1_2$ set coded by $((x(G_r))_1)_0$.*[3]

PROOF OF CLAIM. We can take $\|[\psi]\|$ to be some condition that forces w codes γ and $D(w, y)$. Note that this is absolute to $L[\vec{E}][x, y]$. Given any such generic below $\|[\psi]\|$, we must get that x is in the encoded $\underset{\sim}{\Sigma}^1_2$ set, and hence there is a φ as required by the forcing theorem. ($\dashv$Claim.)

Let A be a Π^1_3 not Σ^1_3 subset of ω. Let A_1 be the set of $n \in \omega$ such that there is some Π^1_2 iterable $\mathcal{M}_0$, with derived inner model $L[\vec{E}_0]$, with the $\vec{E}$ witnessing Woodiness of $\delta_0 < \omega_1$ in $L[\vec{E}_0]$, and there are $x, w, y \in \mathcal{M}_0$ such that:

(i)$D(w, y)$;

(ii) w codes some $\gamma < \underset{\sim}{\delta}^1_2$; and for $\alpha < \beta < \delta_0$ least such that there is some $\|[\phi]\|, \|[\psi]\| \in \mathbb{B}(R(\vec{E}_0|_\alpha), L_\beta[\vec{E}_0|_\alpha])$ with

(iii) $x \models \phi, (w, y) \models \psi$;

(iv)and $L_\beta[\vec{E}_0|_\alpha]$ satisfies ZFC, and that $R(\vec{E}_0|_\alpha)$ witnesses Woodiness of α, and $(\|[\phi]\|, \|[\psi]\|)$ forces that $x(G_l)$ is in the $\underset{\sim}{\Sigma}^1_2$ set coded by $((x(G_r))_1)_0$: we have that there is some $\bar{\alpha} < \bar{\beta} < \alpha$;

(v) $L_{\bar{\beta}}[\vec{E}_0|_{\bar{\alpha}}]$ satisfies ZFC, and that $R(\vec{E}_0|_{\bar{\alpha}})$ witnesses Woodiness of $\bar{\alpha}$, and that $\mathbb{B}(R(\vec{E}_0|_{\bar{\alpha}}), L_{\bar{\beta}}[\vec{E}_0|_{\bar{\alpha}}]) \Vdash n \in A$.

It is transparent from the definitions that $A_1 \in \Sigma^1_3$.

CLAIM 2. $A_1 \subset A$.

[3]As a clarification: if v is a real with $v = (w, y)$ and $y = (y_0, y_1)$, then $((v)_1)_0 = y_0$.

If n not in A and $\mathcal{M}_0$, α, β, and so on satisfy (i)–(iv) above, then for any real u witnessing n not being in A we can attempt to iterate $\mathcal{M}_0$ so that u is generic over the image of $L[\vec{E}_0]$. If we end up in case (2b) of 3.2 then there can be no $\bar{\alpha}$ and $\bar{\beta}$ as in (v) by elementarity of the iteration, and we are done. If instead we arrive at case (2b) then it follows easily from u being generic over the final piece together model that there is no $\bar{\alpha}_0$ and $\bar{\beta}_0$ less than $\delta(\mathcal{T})$ satisfying (v). However from Claim 1, there are many $\beta_0 > \delta(\mathcal{T})$ such that β_0 and $\delta(\mathcal{T})$ satisfy (iv) for some suitable ϕ and ψ; it then follows from the existence of a β_0-wellfounded branch and the observation that (iv) is Σ_1 in the power set operation that there can be no such $\bar{\alpha} < \bar{\beta} < \alpha$ in the original $L[\vec{E}_0]$. ($\dashv$Claim.)

Since A is not Σ^1_3, we can fix some $n_0 \in A \setminus A_1$. This will give a bound on how far up the $\vec{E}$ sequence we will need to search in any arbitrary Doddage.

The second step in the proof is to obtain the $\underset{\sim}{\Sigma}{}^1_2$ sets in the some small generic extension of some iterate of our original $\mathcal{M}$. Fix γ. By iterating $\mathcal{M}$ finitely many times, and appealing to the iterability assumption on $\mathcal{M}$, we may assume that in all further iterations γ is a fixed point. Choose $|[\psi]| \in \mathbb{B}(R(\vec{E}, L[\vec{E}])$ the maximal condition forcing that $(x(G))_0$ codes γ and that $D((x(G))_0, (x(G))_1)$ holds.

Choose $\bar{\alpha} < \bar{\beta} < \delta$ be least such that $L_{\bar{\beta}}[\vec{E}|_{\bar{\alpha}}]$ satisfies ZFC, $R(\vec{E})|_{\bar{\alpha}}$ witnesses Woodiness of $\bar{\alpha}$, and hence that $H(\bar{\alpha})^{L[\vec{E}]} \subset L_{\bar{\alpha}}[\vec{E}]$, and that $L_{\bar{\beta}}[\vec{E}|_{\bar{\alpha}}] \models \mathbb{B}(R(\vec{E})|_{\bar{\alpha}}, L_{\bar{\beta}}[\vec{E}|_{\bar{\alpha}}]) \Vdash n_0 \in A$. By reflection we get $\bar{\beta}$ less than the least $< \delta$ strong cardinal in $\mathcal{M}$, and therefore $L[\vec{E}]$.

A similar reflection argument gives some $\kappa \in (\bar{\beta}, \delta)$ strongly inaccessible in M, with no extenders on $\vec{E}$ overlapping κ, and in particular κ less than the least $< \delta$ strong. Let S be the set of $(\alpha_0, \beta_0, \phi_0, \psi_0)$ such that

(vi) $\alpha_0 \leq \bar{\alpha}$, $\beta_0 < \kappa$, $\phi_0, \psi_0 \in H(\bar{\alpha})^{L[\vec{E}]}$;

(vii) $|[\psi \wedge \psi_0]| \neq 0$ in $\mathbb{B}(R(\vec{E}), L[\vec{E}])$;

(viii) $L_{\beta_0}[\vec{E}|_{\alpha_0}]$ satisfies ZFC, $R(\vec{E})|_{\alpha_0}$ witnesses Woodiness of α_0, and that in $\mathbb{B}(R(\vec{E})|_{\alpha_0}, L_{\beta_0}[\vec{E}|_{\alpha_0}]) \times \mathbb{B}(R(\vec{E})|_{\alpha_0}, L_{\beta_0}[\vec{E}|_{\alpha_0}])$ the condition $(|[\phi_0]|, |[\psi_0]|)$ forces that $x(\dot{G}_l)$ is in the $\underset{\sim}{\Sigma}{}^1_2$ set coded by $((x(\dot{G}_r))_1)_0$.

Note that $S \in (V_\kappa)^M$. We then let B^*_γ consist of all x such that there is an iteration tree $\mathcal{T}$ on $L((V_\kappa)^M)$, using only extenders on the $\mathcal{F}$ sequence, $i_{\mathcal{T}} : L((V_\kappa)^M) \to N = L((V_{\tilde{\kappa}})^N)$, such that for some $(\alpha_0, \beta_0, \phi_0, \psi_0) \in i_{\mathcal{T}}(S)$ we have that x is $\mathbb{B}(R(\vec{E}_0)^N|_{\alpha_0}, L_{\beta_0}[\vec{E}^N|_{\alpha_0}])$ generic, and $x \models \phi_0$. B^*_γ is $\underset{\sim}{\Sigma}{}^1_2$ in any parameter coding $(V_\kappa)^M$.

Claim 3. $B^*_\gamma \subset B_\gamma$.

Given an iteration $\mathcal{T}$ on $L((V_\kappa)^M)$, and $(\alpha_0, \beta_0, \phi_0, \psi_0) \in i_{\mathcal{T}}(S)$ witnessing that $x \in B^*_\gamma$, we can canonically copy $\mathcal{T}$ to a tree $\mathcal{T}^*$ on $\mathcal{M}$, with $i_{\mathcal{T}}|_{(V_\kappa)^M} =$

$i_{\mathcal{T}^*}|_{(V_\kappa)^M}$. Let $\psi^* = i_{\mathcal{T}^*}(\psi)$, $\vec{E}_0 = i_{\mathcal{T}^*}(\vec{E})$, $S^* = i_{\mathcal{T}^*}(S) = i_{\mathcal{T}}(S)$. Then $|[\psi^*]|$ and $|[\psi_0]|$ are compatible conditions in the free extender algebra, x is $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ generic over $\mathbb{B}(R(\vec{E}_0)|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$ below the condition $|[\phi_0]|$. We then let g be a sufficiently generic filter on $\mathbb{B}(\vec{E}_0, L[\vec{E}_0])$ below $|[\psi^* \wedge \psi_0]|$. As in the proof of 2.2, we obtain that (G_x, g) is a $L_{\beta_0}[\vec{E}_0|_{\alpha_0}]$ generic filter on $\mathbb{B}(R(\vec{E}_0)|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}]) \times \mathbb{B}(R(\vec{E}_0)|_{\alpha_0}, L_{\beta_0}[\vec{E}_0|_{\alpha_0}])$. It then follows by the assumption $(\alpha_0, \beta_0, \phi_0, \psi_0) \in S^*$ that x is in the $\utilde{\Sigma}^1_2$ set coded by $((x(g))_1)_0$, and hence, since γ is a fixed point of $i_{\mathcal{T}^*}$ and $|[\psi^*]| \in g$, that x is in B_γ as required. (⊣Claim)

Claim 4. $B_\gamma \subset B^*_\gamma$.

Fix $x \in B_\gamma$, w coding γ, and $(w, y) \in D$, so y_0 codes B_γ. Let $\mathcal{N}$ be a completely iterable expanded premouse with $\mathcal{N} = (N, \in, \delta_0, \mathcal{F}_0, \vec{v}_0)$, and $L[\vec{E}_0]$ the derived inner model, $x, y, w \in N$. We then iterate $\mathcal{M}$ and $\mathcal{N}$, by $\mathcal{T}$ and $\mathcal{U}$, to expanded premice $\mathcal{M}_0$ and $\mathcal{N}_0$ so that $L[\vec{E}]^{\mathcal{M}_0} = L[\vec{E}]^{\mathcal{N}_0}$.

Note that we can find an iteration tree $\mathcal{T}_0$ on $L((V_\kappa)^M)$, which lifts to an iteration tree on $\mathcal{T}^*_0$ on $\mathcal{M}$, as in Claim 3, such that there is a second iteration tree, $\mathcal{T}^*_1$ on $i_{\mathcal{T}^*_0}(\mathcal{M})$, using only extenders whose critical points are above the image of κ, with $\mathcal{T}$ equal to $\mathcal{T}^*_0$ followed by $\mathcal{T}^*_1$; the point here is that the comparison between $\mathcal{M}|_\kappa = (L((V_\kappa)^M), \in, \delta, \mathcal{F}|_\kappa, \vec{v}|_\kappa)$ and $\mathcal{N}$ forms an initial segment of the comparison between $\mathcal{M}$ and $\mathcal{N}$, in light of the fact that no extender overlaps κ. Thus we have that $i_{\mathcal{T}_0}(S) = i_{\mathcal{T}^*_0}(S) =_{df} S^*$.

By Claim 2 and the definition of S, we have that there are some $(\alpha_0, \beta_0, \phi_0, \psi_0) \in S^*$ such that $x \models \phi_0$ and $(w, y) \models \psi_0$. Thus the iteration tree $\mathcal{T}_0$ on $L((V_\kappa)^M)$ witnesses $x \in B^*_\gamma$. (⊣Claim.)

In other words, we have now shown that $\mathcal{M}$ can be iterated finitely many times to make a $\utilde{\Sigma}^1_2$ code for B_γ small generic over the image of M, for any $\gamma < \utilde{\delta}^1_2$.

The final step of the argument is to note that this gives us a $\utilde{\Sigma}^1_2$ in the codes subset C of $\{X^\sharp : X \in HC\}$ with $\{\text{Next}(\omega_1, X) : X^\sharp \in C\} = \utilde{\delta}^1_2$ and a consequent contradiction to the Kunen-Martin theorem.

Claim 5. *$\forall \gamma < \utilde{\delta}^1_2$ there are finitely many iteration trees, $\mathcal{T}_0, \dots, \mathcal{T}_n$, with $\mathcal{T}_0$ on $\mathcal{M} =_{df} \mathcal{M}_0$, $\mathcal{T}_i$ on $\mathcal{M}_i$ and final model $\mathcal{M}_{i+1}$, and some $X \in H(\delta^{\mathcal{M}_n})^{M_n}$ with Next$(\omega_1, X) \geq \gamma$.*

With finitely many such iteration trees we eventually obtain that there is some $\mathcal{M}_n$ with a parameter b coding B_γ small generic over M_n, by the remark after Claim 4. It suffices to show that $M_n[b]$ has some real x with $\text{Next}(\omega_1, x) \geq \gamma$. However, the set of w such that

(i) w codes an ordinal less than $\utilde{\delta}^1_2$; and,

(ii) for all y with $D(w, y)$ we have that y_0 codes the same $\underset{\sim}{\Sigma}^1_2$ set as b; is $\Pi^1_3(b)$. So by [13], there is some $w \in \Delta^1_3(b)$ that codes γ. However, since $M_n[b]$ still has the Woodin cardinal, $w \in M_n$ by the applications of the free extender algebra discussed in Section 1. ($\dashv$Claim.)

(Remark: This is why we needed to obtain the set S bounded below the first $< \delta$ strong. Otherwise we only find that B_γ has a code as a $\underset{\sim}{\Sigma}^1_2$ set in $M_n[G]$, for $G \subset \mathrm{Coll}(\omega, \delta^{\mathcal{M}_n})$ generic over M_n, which would fall short since $M_n[G]$ is no longer closed downwards under Δ^1_3 reducibility.)

A short calulation as in the proof of a similar point in [4] shows that the set of $\{X^\sharp : X$ is small generic over some iteration of $\mathcal{M}\}$ is $\underset{\sim}{\Sigma}^1_2$ in the codes, and this is our set C that provides a counterexample to Kunen-Martin. $\dashv$

It remains to make some comments on how one removes the large cardinal assumption used in the above proof. One method can be found in [22]. In these notes Woodin shows under $AD^{L(\mathbb{R})}$ that for any $A \in L(\mathbb{R})$, $A \subset Ord$, there is a cone of reals x such that $HOD_A^{L[x,A]}$ has a Woodin cardinal. Therefore if we let z be the parameter for the sequence $(B_\gamma)_{\gamma \in \underset{\sim}{\delta}^1_2}$ from the start of the proof, and we let A be a subset of Ord coding z and the tree for the complete Σ^2_1 set in $L(\mathbb{R})$, Woodin's theorem gives us that there is some inner model, of the form $HOD_A^{L[x,A]}$, containing a Woodin cardinal $\delta < \omega_1$ and the real z. A routine absoluteness argument gives that $HOD_A^{L[x]} \cap (HC)^V$ is closed under sharps. The rest of the proof proceeds as before.

REFERENCES

[1] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, to appear in the ***London Mathematical Society Lecture Note Series***.

[2] A. Dodd, ***The core model***, London Mathematical Society Lecture Note Series, vol. 61, Cambridge University Press, Cambridge, 1982.

[3] L. A. Harrington, *Analytic determinacy and* $0^\sharp$, ***Journal of Symbolic Logic***, vol. 43 (1978), pp. 685–693.

[4] G. Hjorth, *Some applications of coarse inner model theory*, preprint, Caltech, 1994.

[5] ———, *Uncountable* Δ^1_3 *sets of reals*, handwritten note, 1994.

[6] S. Jackson, *Partition properties and well-ordered unions*, ***Annals of Pure and Applied Logic***, vol. 48 (1990), pp. 81–101.

[7] S. Jackson and D. A. Martin, *Pointclasses and well-ordered unions*, ***Cabal seminar 79–81*** (A. S. Kechris, D. A. Martin, and Y. N. Moschovakis, editors), Lecture Notes in Mathematics 1019, Springer-Verlag, Berlin and New York, 1983, pp. 56–66.

[8] T. Jech, ***Set theory***, Academic Press, London, 1978.

[9] A. Kanamori, ***The higher infinite***, Springer-Verlag, Berlin and New York, 1994.

[10] V. Kanovei, *Undecidable and decidable propertiesof constituents*, ***Mathematics of USSR Sbornik***, vol. 124 (1984), pp. 491–519.

[11] A. S. Kechris, *A basis theorem for* Δ^1_3 *Borel sets*, handwritten note, 1978.

[12] ———, *On transfinite sequences of projective sets with an application to* Σ^1_2 *sets*, ***Logic Colloquium '77*** (A. Macintyre, L. Pacholski, and J. Paris, editors), North-Holland, Amsterdam, 1978, pp. 155–160.

[13] A. S. Kechris and D. A. Martin, *On the theory of* Π^1_3 *sets of real*, ***Bulletin of the American Mathematical Society***, vol. 260 (1980), pp. 363–378.

[14] A. S. Kechris, D. A. Martin, and R.M. Solovay, *Q-theory*, ***Cabal seminar 79–81*** (A. S. Kechris, D. A. Martin, and Y. N. Moschovakis, editors), Lecture Notes in Mathematics 1019, Springer-Verlag, Berlin and New York, 1983, pp. 199–281.

[15] A. Louveau, *A basis theorem for* Δ^1_{2n+1}, handwritten note, 1978.

[16] ———, *A separation theorem for* Σ^1_1 *sets*, ***Transactions of the American Mathematical Society***, vol. 260 (1980), pp. 363–378.

[17] D. A. Martin and J. R. Steel, *Iteration trees*, ***Journal of the American Mathematical Society***, vol. 7 (1994), pp. 1–72.

[18] A. W. Miller, *Arnie Miller's problem list*, ***Israel Mathematics Conference Proceedings***, vol. 6 (1993), pp. 645–654.

[19] W. Mitchell and J. R. Steel, ***The fine structure of iteration trees***, Lecture notes in Logic, no. 3, Springer-Verlag, Berlin, 1994.

[20] Y. N. Moschovakis, ***Descriptive set theory***, North-Holland, Amsterdam, 1980.

[21] I. Neeman, *Optimal proofs of determinacy*, this Bulletin, vol. 1 (1995), pp. 327–339.

[22] E. Schimmerling, *Notes on some lectures by Woodin*, Berkeley, 1990.

[23] J. R. Steel, *The core model iterability problem*, to appear in the ***Lecture Notes in Logic Series***.

[24] ———, $HOD^{L(\mathbb{R})}$ *is a core model*, this Bulletin, vol. 1 (1995), pp. 75–84.

[25] J. Stern, *On Lusin's restricted continuum problem*, ***Annals of Mathematics***, vol. 120 (1984), pp. 7–37.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095-1555, USA

E-mail: greg@math.ucla.edu