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GEOMETRY, CALCULUS AND ZIL'BER'S CONJECTURE

YA'ACOV PETERZIL AND SERGEI STARCHENKO

§1. Introduction. By and large, definitions of a differentiable structure on a set involve two ingredients, topology and algebra. However, in some cases, partial information on one or both of these is sufficient. A very simple example is that of the field $\mathbb{R}$ (or any real closed field) where algebra alone determines the ordering and hence the topology of the field:

$$a < b \Leftrightarrow \exists x \in \mathbb{R}\,(x \neq 0 \;\&\; x^2 = b - a).$$

In the case of the field $\mathbb{C}$, the algebraic structure is insufficient to determine the Euclidean topology; another topology, Zariski, is associated with the field but this will be too coarse to give a differentiable structure.

A celebrated example of how partial algebraic and topological data (G a locally euclidean group) determines a differentiable structure (G is a Lie group) is Hilbert's 5th problem and its solution by Montgomery-Zippin and Gleason.

The main result which we discuss here (see [13] for the full version) is of a similar flavor: we recover an algebraic and later differentiable structure from a topological data. We begin with a linearly ordered set $\langle M, <\rangle$ equipped with the order topology, and its cartesian products with the product topologies. We then consider the collection of definable subsets of M^n, $n = 1, 2, \dots$, in some first order expansion $\mathcal{M}$ of $\langle M, <\rangle$. The main topological assumption is that definable functions in one variable are piecewise continuous, or equivalently that $\mathcal{M}$ is *order minimal* (see definition in the next section). In order to get a differentiable structure we need to assume also the existence of a sufficiently complicated definable subset of M^4. We then conclude that one can develop on these grounds a theory of differentiation for definable sets in $\mathcal{M}$. In particular, one can find two definable functions $\{+, \cdot\}$ such that for some open interval $I \subseteq M$, $\langle I, +, \cdot\rangle$ is a real closed field, call it R. As is already known, under these assumptions every definable subset of R^n can be partitioned into finitely many definable manifolds-like sets with respect to R.

We discuss the theorem from few different angles. In the language of model theory we formulate first a more general result, the Trichotomy Theorem,

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and consider it as part of the classification program set out by Zil'ber. We then treat the main part of the Trichotomy Theorem separately and describe it in a topological language, using only functions on topological spaces.

The main idea behind the proof is "Geometric calculus", a treatment of tangency and differentiability on purely geometric grounds. Instead of sketching the proof of the main theorem we discuss here this idea and show how one can treat basic notions and theorems in Calculus without ever referring to a field structure. Although we do not directly use that in proving the main theorem, almost all ideas of the proofs were inspired by this approach.

§2. The model theoretic approach. To simplify the statements in this survey we will assume that $<$ is a dense linear ordering on M. A function $F : M^n \to M$ is said to be definable in $\mathcal{M}$ if its graph is a definable subset of M^{n+1} (definability is always assumed to be with parameters). A group (ring) is definable in $\mathcal{M}$ if its underlying set and the group (ring) operation(s) are definable.

Definition 2.1. $\mathcal{M}$ is called *order-minimal* or *o-minimal* if for every $D \subseteq M$, if D is definable in $\mathcal{M}$ then D is definable in $\langle M, <\rangle$ (equivalently, D is a finite union of intervals with endpoints in $M \cup \{\pm\infty\}$).

Since a linear ordering is assumed to be definable, o-minimal structures are unstable (see [14] for the definitions). However, they are closely related to *strongly minimal* structures which are stable ($\mathcal{M}$ is strongly minimal if for every $\mathcal{N} \equiv \mathcal{M}$, if $D \subseteq N$ is definable in $\mathcal{N}$ then D is definable in $\langle N, \emptyset\rangle$ alone; equivalently, D is finite or co-finite).

The classes of o-minimal and strongly minimal structures share many model theoretic properties (e.g., the existence and uniqueness of prime models over arbitrary sets). However, one immediate difference is that there is no need for a notion of *strong o-minimality*, i.e., if $\mathcal{M}$ is o-minimal and $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is o-minimal as well (see [8]).

Three basic types of examples of o-minimal structures are

(i) $\langle D, <\rangle$, $<$ a dense linear ordering;
(ii) $\langle V, <, +, d\rangle_{d\in D}$ an ordered vector space over an ordered division ring;
(iii) $\langle R, <, +, \cdot\rangle$ a real closed field, or more generally, o-minimal expansions of real closed fields, such as $\langle \mathbb{R}, <, +, \cdot, e^x\rangle$ (see [18]).

It is of great interest to know which structures are o-minimal since o-minimality forces strong compatibility between definable sets, topology and differentiability (when the latter makes sense). To quote some of these properties (see [15, 8, 17]):

Assume that $\mathcal{M}$ is an o-minimal structure. Then

(A) Every definable subset of M^n has finitely many definably-connected components. (A definable set is definably connected if it has no definable clopen subset.) Moreover, if the underlying ordered set is $\langle \mathbb{R}, < \rangle$ then we can replace 'definably-connected ' with 'connected'.

(B) Every definable $F : M^n \to M$ is piecewise continuous; i.e, M^n can be partitioned into finitely many definable sets, on each of which F is continuous.

(C) If $\mathcal{M} = \langle M, <, *, \cdots \rangle$ is an expansion of an ordered group then the group is divisible and abelian.

(D) If $\mathcal{M} = \langle M, <, +, *, \cdots \rangle$ is an expansion of an ordered ring then the ring is a real closed field. Moreover, every definable function $F : M^n \to M$ is piecewise differentiable (with respect to the ordered field).

For the next property we need one more definition: Given a first order structure $\mathcal{M}$, $a \in M$, $B \subseteq M$, *a is in the algebraic closure of B* if there is a formula $\varphi(x, \bar{b})$, $\bar{b}$ from B, such that $\varphi(a, \bar{b})$ holds and $\varphi(x, \bar{b})$ has only finitely many solutions in M. We denote this by $a \in acl(B)$.

(E) $acl(-)$ satisfies the Exchange principle: If $a, b \in M$, $A \subseteq M$ and $b \in acl(A, a) \setminus acl(A)$ then $a \in acl(A, b)$.

Property (E), which holds also in strongly minimal structures makes $\langle M, acl(-) \rangle$ into a pre-geometry or a matroid. One should not confuse the two notions of geometries that we use. One geometry is a "continuous," Euclidean-like geometry, where the main objects are the definable sets, seen as subsets of the topological space M^n. The other geometry is a "combinatorial" geometry of the algebraic closure, where the main objects are subsets (not definable in general) of M together with the closure operator $acl(-)$. It is exactly the interplay between the two geometries which yields strong structure theorems, similarly perhaps to the way in which the analytic and algebraic structures yield theorems in real or complex algebraic geometry.

Given an o-minimal structure $\mathcal{M}$, we distinguish between two types of points in M. We call $a \in \mathcal{M}$ *nontrivial* if there is an open interval I containing a and a definable continuous function $F : I \times I \to M$ which is strictly monotone in each variable. A point which is not nontrivial is called *trivial*. If $\langle G, <, + \rangle$ is an ordered divisible group then, clearly, every $g \in G$ is nontrivial as is witnessed by $F(x, y) = x + y$. For another example, take $\mathcal{M} = \langle \mathbb{R}, <, P|(0,1)^2 \rangle$, $P(x, y) = \frac{x+e^y}{y}$. $\mathcal{M}$ is a reduct of $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$ hence o-minimal. The nontrivial points in $\mathcal{M}$ are the positive reals.

The trichotomy theorem below says that the three types of o-minimal structures that we have mentioned above exhaust to some extent all such examples. However, because of the special nature of ordered structures, we can analyze the structures only locally. The saturation assumption below is needed only to get a cleaner statement for T2.

THE TRICHOTOMY THEOREM [13]. *Let $\mathcal{M}$ be an o-minimal, ω^+-saturated structure. Given $a \in M$, one and only one of the following hold:*

T1. *a is trivial;*

T2. *The structure that $\mathcal{M}$ induces on some convex neighborhood of a is an ordered vector space over an ordered division ring;*

T3. *The structure that $\mathcal{M}$ induces on some open interval around a is an o-minimal expansion of a real closed field.*

As we will see, each one of the above three cases can be given a purely geometric characterization, involving no algebraic structure whatsoever. The first to suggest the possibility of this phenomenon was Boris Zil'ber in a slightly different model theoretic context. We formulate his ideas in a general framework.

2.1. The Zil'ber principle. We assume here that $\mathcal{M}$ is a first order structure (not necessarily linearly ordered). The following definition is taken from [5].

DEFINITION 2.2. $\mathcal{M}$ is *a geometric structure* if

(i) $acl(-)$ satisfies the Exchange principle and

(ii) for any formula $\varphi(\bar{x}, \bar{y})$ there is an $n \in \mathbb{N}$ such that, for any $\bar{b}$ in M, $\varphi(\bar{x}, \bar{b})$ has either less than n solutions in $\mathcal{M}$ or infinitely many.

O-minimal and strongly minimal structures are geometric structures. The field of p-adics and all pseudo-finite fields (see [5]) are geometric structures as well.

Given a geometric structure $\mathcal{M}$ we assign a dimension to definable sets as follows: $D \subseteq M$ has dimension 1 if D is infinite. By induction, $D \subseteq M^{n+1}$ has dimension $n+1$ if for infinitely many $x_{n+1} \in M$, $\{\bar{x} \in M^n : \langle \bar{x}, x_{n+1} \rangle \in D\}$ has dimension n. Given a definable $D \subseteq M^n$, we let $dim(D)$ be the maximal k such that a projection of D on some k coordinates has dimension k. In all the field cases mentioned above the dimension we get is just the algebro-geometric dimension of the Zariski closure of D. In the strongly minimal case $\dim(D)$ coincides with the Morley Rank of D while in the o-minimal case this dimension has a natural topological interpretation which resembles dimension of manifolds over $\mathbb{R}$.

Since the algebraic closure in a geometric structure satisfies Exchange we also have another notion of dimension which is associated with the *acl*-geometry. This is just a generalization of dimension in vector spaces or fields. The continuous and combinatorial notions of dimensions are compatible once we work in a sufficiently saturated structure; namely, the dimension of a 0-definable set $D \subseteq M^n$ equals the maximal k such that D contains a tuple whose *acl*-dimension is k.

For our purposes, *a curve* is any definable 1-dimensional subset of M^2 and a family of curves $\mathcal{F}$ is said to be *definable* if it can be written as the collection of fibers of a definable subset of $M^2 \times M^k$ (i.e., the parameter set is a definable subset of M^k). $\mathcal{F}$ is called *normal* if any two curves from $\mathcal{F}$ which are given by different parameters intersect at most finitely many times. If $\mathcal{F}$ is normal we can assign to it a dimension, which is just the dimension of the parameter set as a definable subset of M^k.

In many cases definable families of curves have to be replaced by *interpretable* families, i.e., we allow the parameter set to be a subset of M^k/E for a definable equivalence relation E.

Let us consider the normal families of curves which are definable in the three types of o-minimal structures above. In the case of a dense linear ordering any normal family of curves can be shown to arise from the families

$$\{\langle x, b\rangle : b \in M\} \text{ and } \{\langle a, y\rangle : a \in M\}.$$

In particular, every normal family is of dimension at most 1.

In the case of an ordered vector space over a division ring D, normal families of curves arise basically as follows: we fix $d \in D$ and then

$$\mathcal{F} = \{\langle x, d(x) + a\rangle : a \in M\}.$$

Again, every such family has dimension at most 1.

When $\mathcal{M}$ is an expansion of a real closed field R then for every n, the family of graphs of polynomials of degree at most $n-1$ is clearly a normal family of curves of dimension n. Also, if $f : R \to R$ is $\mathcal{M}$-definable and f is not semilinear on any subinterval of R then the family

$$\{\langle x, f(a + x) + b\rangle : a, b \in R\}$$

is a normal family of curves of dimension 2. In fact, this last example can be carried out in any o-minimal expansion of an ordered group which is not (a reduct of) a vector space.

Every geometric structure can be analyzed as the three examples above. Namely, given a geometric structure, one and only one of the following holds.

Z1. Every interpretable normal family of curves $\mathcal{F}$ is of dimension at most 1 and for all but finitely many curves $\mathcal{C} \in \mathcal{F}$, for all but finitely many points $\langle a, b\rangle \in \mathcal{C}$, either $\dim(\mathcal{C} \cap (\{a\} \times M)) = 1$ or $\dim(\mathcal{C} \cap (M \times \{b\})) = 1$.

Z2. Every interpretable normal family of curves is of dimension at most 1, but Z1 does not hold.

Z3. There is an interpretable normal family of curves of dimension greater than 1.

In the early 1980's Boris Zil'ber (see [19]), in his analysis of $\aleph_1$-categorical structures, suggested that the above trichotomy corresponds to the interpretability (or the lack of which) of certain algebraic structures in $\mathcal{M}$. He called Z2 and Z3 the module-like and field-like cases, respectively, and conjectured that if a strongly minimal structure satisfies Z3 then it can interpret a field. We formulate this correspondence as follows:

DEFINITION 2.3. A class $\mathcal{K}$ of geometric structures is said to satisfy the Zil'ber Principle, ZP, if for every $\mathcal{M} \in \mathcal{K}$,

(i) $\mathcal{M}$ satisfies Z1 if and only if no group is interpretable in $\mathcal{M}$.
(ii) $\mathcal{M}$ satisfies Z2 if and only if all definable sets arise from an interpretable vector space (or more generally a module) structure.
(iii) $\mathcal{M}$ satisfies Z3 if and only if a field can be interpreted in $\mathcal{M}$.

Some of the above implications are easy to establish: If Z1 holds then no group is interpretable in $\mathcal{M}$. If a field F is interpretable in $\mathcal{M}$ then, as we pointed out above, Z3 holds. Various formulations of (ii) were established for different classes of structures (see [2] and [4] for stable structures, see [11] and [9] for o-minimal structures). Hence, in many cases in order to establish the Zil'ber Principle it is sufficient to prove that Z3 implies the interpretability of a field.

However, the class of strongly minimal structures proved to be less well behaved than Zil'ber had originally conjectured. Hrushovski (see [3]) disproved the conjecture by constructing a strongly minimal structure satisfying Z3, with no group interpretable in it. Hrushovski and Zil'ber then considered a more restrictive class of strongly minimal structures called Zariski geometries (see [7] and [6]). As they showed, Zariski Geometries do satisfy the Zil'ber Principle. ZP was also established for many strongly minimal structures interpretable in algebraically closed fields (see [16]). In Hrushovski's proof of the Mordell-Lang Conjecture for function fields (see [1]) he makes use of ZP for strongly minimal sets in differentially closed fields and separably closed fields.

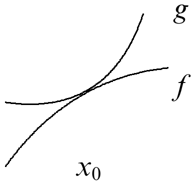
Seen in this light, the Trichotomy Theorem above easily *implies the Zil'ber Principle for the class of o-minimal structures*, with one restriction: In the case that $\mathcal{M}$ satisfy Z2 we can only say that locally, around each nontrivial point, the definable sets arise from an ordered vector space structure. No such global result can be obtained unless there is a definable interaction between different parts of $\mathcal{M}$.

Classification of infinite definable simple groups. One application of the Trichotomy Theorem, in a joint work of the authors with A. Pillay, is the classification of all infinite definable simple groups in o-minimal structures, a fundamental open question in the investigation of stable groups. A definable group is called *definably simple* if it has no definable normal subgroups.

THEOREM 2.4 ([12]). *Let G be an infinite non-abelian definably-simple group definable in an o-minimal structure $\mathcal{M}$. Then there is an algebraic group H* *over a definable real closed field R such that G is definably isomorphic to a semi-algebraic connected component of H.*

§3. The topological approach. At the risk of repeating parts of the discussion from the previous sections we will consider now the same questions from a slightly different angle.

As before, $\langle M, <\rangle$ is a densely ordered set equipped with the order topology. If $\langle M, <\rangle$ supports the structure of an ordered field F then the differentiability of functions with respect to F is a well-defined notion. In some simple cases it can be analyzed using just the ordering of M and the structure it induces on M^n. For example, if f, g are two differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and if their graphs in some neighborhood of x_0 intersect as in the figure below



then it clearly follows that $f'(x_0) = g'(x_0)$. As we show here, if we limit our discussion to a well behaved family of functions on M^n then one can develop a theory of differentiability using only such simple geometric ideas.

Definition 3.1. For each $n \in \mathbb{N}$ let $\mathcal{S}_n$ be a collection of functions $F : M^n \to M$, and let $\mathcal{S} = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$. $\mathcal{S}$ is called *a system of definable functions on* $\langle M, <\rangle$ if it contains all projections, constant functions, the function $\min(x, y)$ and is closed under the following:

(i) Extension: If $f \in \mathcal{S}_n$ and $g(\bar{x}, y) = f(\bar{x})$ then $g \in \mathcal{S}_{n+1}$.

(ii) Implicit definition: If $g \in \mathcal{S}_{n+1}$, $a \in M$ and for every $\bar{x} \in M^n$ there is a unique y such that $g(\bar{x}, y) = a$ then there is $f \in \mathcal{S}_n$ such that $g(\bar{x}, f(\bar{x})) = a$ for all $\bar{x} \in M^n$.

(iii) Existential definition: If $a, b \in M$, $g_1, g_2 \in \mathcal{S}_{n+1}$ and

$$f(\bar{x}) = \begin{cases} a & \exists y\ g_1(\bar{x}, y) = g_2(\bar{x}, y) \\ b & \text{otherwise} \end{cases}$$

then $f \in \mathcal{S}_n$.

If $\mathcal{S}$ is a system of definable functions on M then we can canonically associate to it a first order expansion $\mathcal{M}_{\mathcal{S}}$ of $\langle M, <\rangle$, in which the collection of definable total functions is exactly $\mathcal{S}$. $\mathcal{M}_{\mathcal{S}}$ is just the expansion of $\langle M, <\rangle$ in the language containing a function symbol for every $f \in \mathcal{S}$, interpreted by the corresponding function on M^n. As we pointed out, if $\mathcal{M}_{\mathcal{S}}$ is o-minimal then given any $f \in \mathcal{S}_1$, M can be partitioned into finitely many intervals (with endpoints in $M \cup \{\pm\infty\}$) on each of which f is continuous. We say that $\mathcal{S}$ is an *o-minimal system* if it satisfies this latter property. It is easy to verify that $\mathcal{M}_\mathcal{S}$ is an o-minimal structure if and only if $\mathcal{S}$ is an o-minimal system.

We call $\mathcal{S}$ *rich* if it contains a function $F \in \mathcal{S}_3$ such that for some open interval $I \subseteq M$ and some open set $U \subseteq M^2$, if $\bar{u} \neq \bar{v}$ are both in U then there are at most finitely many $x \in I$ such that $F(x, \bar{u}) = F(x, \bar{v})$. It is easy to verify that $\mathcal{S}$ is rich if and only if $\mathcal{M}_\mathcal{S}$ satisfies Z3.

We say that a group is definable in $\mathcal{S}$ on an interval $I \subseteq M$ if there is a function in $\mathcal{S}_2$ such that, when restricted to $I \times I$, makes I into a group. Similarly for a field. Clearly, if a real closed field is definable in $\mathcal{S}$ on some interval then the function $F(x, u_1, u_2) = xu_1 + u_2$ witnesses the richness of $\mathcal{S}$. The converse also holds, and it is an essential part of the proof of ZP for o-minimal structures. Namely, we have

THEOREM 3.2. *Let M be a densely ordered set, $\mathcal{S}$ an o-minimal system of definable functions on $\langle M, < \rangle$. Then $\mathcal{S}$ is rich if and only if a real closed field is definable in $\mathcal{S}$ on some interval in M.*

3.1. Geometric calculus. As is clear from the statement of Theorem 3.2, there is only one object from which to start the construction of a field, i.e., the function F which witnesses the richness of $\mathcal{M}$. The construction of the field from F follows what by now is a standard approach, with the central notion being tangency of curves at a point. After defining tangency we use a geometric analogue of the Chain rule for the composition of two functions in order to construct the field (for a simple example how this is done in additive reducts of $\langle \mathbb{R}, <, +, \cdots \rangle$ see [10]). The main effort in our proof is devoted to overcomeing certain technichal difficulties which arise but we will not try to describe those here. Instead, we will show how to use F to define a geometric notion of differentiability, in very much the same way that the function $H(x, u_1, u_2) = xu_1 + u_2$ is used to define derivatives over $\mathbb{R}$.

As a first step we want to replace F with a more suitable function for our purpose.

DEFINITION 3.3. Given I an open interval, $U \subseteq M^2$ an open set, we call a continuous function $F : I \times U \to M$ *nice on $I \times U$* if

(i) For every $\bar{u} \neq \bar{v}$, both in U, there is at most one x in I such that $F(x, \bar{u}) = F(x, \bar{v})$.

(ii) If G_F is the graph of F on $I \times U$, namely

$$G_F = \{\langle x, u_1, u_2, y \rangle : \langle x, u_1, u_2 \rangle \in I \times U \,\&\, F(x, u_1, u_2) = y\},$$

then G_F is definably connected and any projection of G_F on three of the four coordinate axes is a one-to-one map.

The example we have in mind for a nice function is still $xu_1 + u_2$ on $\mathbb{R}^3$. However, if $f : \mathbb{R} \to \mathbb{R}$ has continuous nonvanishing first and second derivatives on $\mathbb{R}$, then $F(x, u_1, u_2) = f(x + u_1) + u_2$ is also nice on $\mathbb{R}^3$.

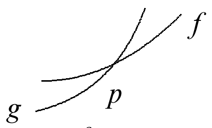
A technical lemma that we prove in [13] states that *every rich o-minimal system contains a nice function on some set* $I \times U$. We now fix such function F. We let $f_{\bar{u}}(x) = F(x, \bar{u})$ and denote by $\mathcal{F}$ the family $\{f_{\bar{u}}(x) : \bar{u} \in U\}$. Notice that by clause (ii) of Definition 3.3,

$$(1)\forall\langle x, y\rangle \in I \times M \ \forall a \in M \exists^{\leq 1} b \text{ such that } \langle a, b\rangle \in U \text{ and } f_{a,b}(x) = y.$$

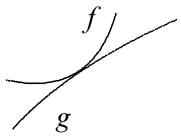
Notation. Given $p \in M$ we say that a property P holds for $x \in (p)$ if P holds for every x in some neighborhood of p. We say that P holds for $x \in (p)^+$ (or $x \in (p)^-$) if it holds for every $x > p$ (or $x < p$), x sufficiently close to p.

Let f, g be in $\mathcal{S}_1$ and assume that $f(p) = g(p)$ for $p \in M$. We write $f \preccurlyeq_p g$ if $f(x) \leq g(x)$ for $x \in (p)^+$ and $g(x) \leq f(x)$ for $x \in (p)^-$ (see figure). We say that $f(x)$ *touches* $g(x)$ *from above at* p if $f(x) \geq g(x)$ for $x \in (p)$ (see figure). f and g are said to *touch each other at* p if f touches g from above or vice-versa.

Notice that by o-minimality if $f(p) = g(p)$ then either $f \preccurlyeq_p g$ or $g \preccurlyeq_p f$ or f and g touch each other at p.

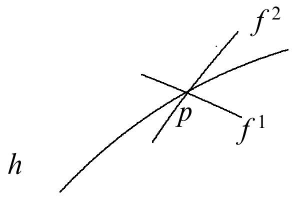


$f \preccurlyeq_p g$



f touches g from above

One difficulty that we face when we want to define differentiability with respect to $\mathcal{F}$ is the possibility that all the "derivatives" which are represented by $\mathcal{F}$ are either too small or too large to analyze a particular function h. We hence restrict ourselves only to $\mathcal{F}$-*bounded functions*: h is $\mathcal{F}$-bounded at p if there are $f^1, f^2 \in \mathcal{F}$ such that $f^1(p) = f^2(p) = h(p)$ and $f^1 \preccurlyeq_p h \preccurlyeq_p f^2$ (see figure).



h is $\mathcal{F}$-bounded at p

We can now define the notion of a derivative with respect to $\mathcal{F}$.

DEFINITION 3.4. Let h be a function from M to M, h $\mathcal{F}$-bounded at $p \in I$. A function $f_{a,b} \in \mathcal{F}$ is called *the $\mathcal{F}$-tangent function to h at p*, denoted by $\tau_{\mathcal{F}}h(p)$, if

(i) $f_{a,b}(p) = h(p)$;
(ii) for every $g \in \mathcal{F}$ such that $g(p) = h(p)$; if $g(x) \leq h(x)$ for $x \in (p)^+$ then $g(x) \leq f_{a,b}(x)$ for $x \in (p)^+$ and if $h(x) \leq g(x)$ for $x \in (p)^+$ then $f_{a,b}(x) \leq g(x)$ for $x \in (p)^+$;
(iii) for every $g \in \mathcal{F}$ such that $g(p) = h(p)$, if $g(x) \leq h(x)$ for $x \in (p)^-$ then $g(x) \leq f_{a,b}(x)$ for $x \in (p)^-$ and if $h(x) \leq g(x)$ for $x \in (p)^-$ then $f_{a,b}(x) \leq g(x)$ for $x \in (p)^-$.

When such $f_{a,b}$ exists we say that h is *$\mathcal{F}$-differentiable at p* and call *a the $\mathcal{F}$-derivative* of h at p; in notation $d_{\mathcal{F}}h(p) = a$. (By (1) there is at most one such a.)

Notice that in the case that $\mathcal{F}$ is the family of linear functions on $\mathbb{R}$, if h and $\mathcal{F}$ are definable in the same o-minimal structure then the $\mathcal{F}$-derivative of h is just the standard derivative. More generally, the $\mathcal{F}$-tangent function to h at a point p can be seen as the best approximation to h among functions from $\mathcal{F}$ at p.

The next theorem is the main justification for the above definitions. As it shows, although clauses (ii) and (iii) of the definition make $\mathcal{F}$-tangency a notion which is very much relative to $\mathcal{F}$, it can be recognized almost everywhere without any reference to other functions in $\mathcal{F}$.

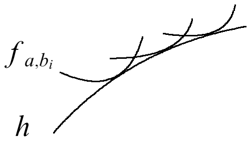
THEOREM 3.5. *Assume that $h \in \mathcal{S}_1$ is $\mathcal{F}$-bounded at every point in I. Then*
(i) *For all but finitely many points $p \in I$, h is $\mathcal{F}$-differentiable at p.*
(ii) *For all but finitely many points $p \in I$, $\tau_{\mathcal{F}}h(p)$ is the unique function in $\mathcal{F}$ which **touches** h at p.*

With Theorem 3.5 at hand we can now develop a large part of basic calculus. For example, we know that a function with constant derivative is linear. The following theorem generalizes this fact to $\mathcal{F}$-derivatives.

THEOREM 3.6. *Let $h \in \mathcal{S}_1$ be continuous and $\mathcal{F}$-differentiable on I. If $d_{\mathcal{F}}h(x)$ is constant on I then $h \in \mathcal{F}$.*

The proof demonstrates the strength of Theorem 3.5 so we sketch it: By theorem 3.5, we can partition I into finitely many open intervals and points such that on each interval $\tau_{\mathcal{F}}h$ touches h either from above or from below, uniformly. By working on each interval seperately, we may assume that, for all $p \in I$, $\tau_{\mathcal{F}}h(p)$ touches h from above or below, uniformly. But then, by continuity arguments, when p_1 and p_2 are close to each other the graphs of

$\tau_{\mathcal{F}} h(p_1)$ and $\tau_{\mathcal{F}} h(p_2)$ must intersect (see figure below). Since $d_{\mathcal{F}} h(p)$ is constant, $\tau_{\mathcal{F}} h(p_1)(x) = f_{a,b_1}(x)$ and $\tau_{\mathcal{F}} h(p_2)(x) = f_{a,b_2}(x)$ for some a, b_1, b_2. It follows that for some $x_0 \in I$ $f_{a,b_1}(x_0) = f_{a,b_2}(x_0)$, hence by (1), $b_1 = b_2$. i.e., there is a single function $f_{a,b}(x) \in \mathcal{F}$ such that $h(x)$ touches $f_{a,b}(x)$ at every point in I. Clearly then $h(x) = f_{a,b}(x)$ on I.



⊣

Similarly one can formulate and prove a generalization of Lagrange's Mean Value Theorem:

THEOREM 3.7. *Let $h \in \mathcal{S}_1$ be continuous on $[p_1, p_2]$ and $\mathcal{F}$-differentiable on (p_1, p_2). Assume that $f_{a,b}(p_1) = h(p_1)$ and $f_{a,b}(p_2) = h(p_2)$. Then there is $p \in (p_1, p_2)$, such that $d_{\mathcal{F}} h(p) = a$.*

A notion of concavity with respect to $\mathcal{F}$ can also be defined by saying that a function $h(x)$ is *$\mathcal{F}$-concave up on I* if $d_{\mathcal{F}} h(p)$ is increasing at p. Some basic properties which are usually related to the second derivative can be proved using the above methods.

Generally speaking, we believe that a large part of basic Calculus, with respect to a nice family $\mathcal{F}$, can now be developed on these purely geometric grounds.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
HAIFA UNIVERSITY
HAIFA, ISRAEL

E-mail: kobi@mathcs2.haifa.ac.il

DEPARTMENT OF MATHEMATICS
VANDERBILT UNIVERSITY
NASHVILLE, TN 37240, USA

E-mail: starchen@math.vanderbilt.edu