THE BULLETIN OF SYMBOLIC LOGIC
Volume 2, Number 1, March 1996

THE MATHEMATICAL DEVELOPMENT OF SET THEORY FROM CANTOR TO COHEN

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Set theory is an autonomous and sophisticated field of mathematics, enormously successful not only at its continuing development of its historical heritage but also at analyzing mathematical propositions cast in set-theoretic terms and gauging their consistency strength. But set theory is also distinguished by having begun intertwined with pronounced metaphysical attitudes, and these have even been regarded as crucial by some of its great developers. This has encouraged the exaggeration of crises in foundations and of metaphysical doctrines in general. However, set theory has proceeded in the opposite direction, from a web of intensions to a theory of extension *par excellence*, and like other fields of mathematics its vitality and progress have depended on a steadily growing core of mathematical structures and methods, problems and results. There is also the stronger contention that from the beginning set theory actually developed through a progression of *mathematical* moves, whatever and sometimes in spite of what has been claimed on its behalf.

What follows is an account of the development of set theory from its beginnings through the creation of forcing based on these contentions, with an avowedly Whiggish emphasis on the heritage that has been retained and developed by current set theory.[1] The whole transfinite landscape can be viewed as the result of Cantor's attempt to articulate and solve the Continuum Problem. Zermelo's axioms can be construed as clarifying the set existence commitments of a single proof, a proof of his Well-Ordering Theorem, the success of the axioms seen as due to their schematic open-endedness and resonance with emerging mathematical practice. And with the later cumulative hierarchy set theory can be regarded as emerging as a theory

Received December 8, 1995, and in revised form, January 31, 1996.

This article is dedicated to Professor Burton Dreben on his coming of age. I owe him particular thanks for his careful reading and numerous suggestions for improvement. My thanks go also to Jose Ruiz and the referee for their helpful comments. Parts of this account were given at the 1995 summer meeting of the Association for Symbolic Logic at Haifa, in the Massachusetts Institute of Technology logic seminar, and to the Paris Logic Group. The author would like to express his thanks to the various organizers, as well as his gratitude to the Hebrew University of Jerusalem for its hospitality during the preparation of this article in the autumn of 1995.

1079-8986/96/0201-0001/$8.10

of well-foundedness, later to expand to a study of consistency strength. Throughout, the subject has not only been sustained by the axiomatic tradition through Gödel and Cohen but also fueled by Cantor's two legacies, the extension of number into the transfinite as transmuted into the theory of large cardinals and the investigation of definable sets of reals as transmuted into descriptive set theory. All this can be viewed as having a historical and mathematical logic internal to set theory, one that is often misrepresented at critical junctures in textbooks (as will be pointed out). This view, from inside set theory and about itself, serves to shift the focus to those tensions and strategies familiar to mathematicians as well as to those moves, often made without much fanfare and sometimes merely linguistic, that have led to the crucial advances.

§1. Cantor.

1.1. Real numbers and countability. Set theory had its beginnings in the great 19th Century transformation of mathematics, a transformation beginning in analysis. Since the creation of the calculus by Newton and Leibniz the function concept had been steadily extended from analytic expressions toward arbitrary correspondences. The first major expansion had been inspired by the explorations of Euler in the 18th Century and featured the infusion of infinite series methods and the analysis of physical phenomena, particularly the vibrating string. In the 19th Century the stress brought on by the unbridled use of series of functions led first Cauchy and then Weierstrass to articulate convergence and continuity. With infinitesimals replaced by the limit concept and that cast in the ε-δ language, a level of deductive rigor was incorporated into mathematics that had been absent for two millennia. Sense for the new functions given in terms of infinite series could only be developed through carefully specified deductive procedures, and proof reemerged as an extension of algebraic calculation and become basic to mathematics in general, promoting new abstractions and generalization.

Working out of this tradition Georg Cantor[2] (1845–1918) in 1870 established a basic uniqueness theorem for trigonometric series: If such a series converges to zero everywhere, then all of its coefficients are zero. To generalize Cantor [1872] started to allow points at which convergence fails, getting to the following formulation: For a collection P of real numbers, let P' be the collection of limit points of P, and $P^{(n)}$ the result of n iterations of this operation. *If a trigonometric series converges to zero everywhere except on a P where $P^{(n)}$ is empty for some n, then all of its coefficients are zero.*[3] Here Cantor was already breaking new ground in considering collections of real numbers *defined* through an operation.

It was in [1872] that Cantor provided his formulation of the real numbers in terms of fundamental sequences of rational numbers, and significantly, this was for the specific purpose of articulating his proof. With the new results of analysis to be secured by proof and proof in turn to be based on prior principles, the regress led in the early 1870's to the appearance of several independent formulations of the real numbers in terms of the rational numbers. It is at first quite striking that the real numbers came to be developed so late, but this can be viewed as part of the expansion of the function concept that shifted the emphasis from the continuum taken as a whole to its extensional construal as a collection of objects. In mathematics objects have been traditionally introduced only with reluctance, but a more arithmetical rather than geometrical approach to the continuum became necessary for the articulation of proofs.

The other well-known formulation of the real numbers is due to Richard Dedekind [1872], through his cuts. Cantor and Dedekind maintained a fruitful correspondence, especially during the 1870's, in which Cantor aired many of his results and speculations.[4] The formulations of the real numbers advanced three important predispositions for set theory: the consideration of infinite collections, their construal as unitary objects, and the encompassing of arbitrary such possibilities. Dedekind [1871] had in fact made these moves in his creation of ideals, infinite collections of algebraic real numbers, and there is an evident similarity between ideals and cuts in the creation of new numbers out of old.[5] The algebraic numbers would soon be the focus of a major breakthrough by Cantor. Although both Cantor and Dedekind carried out an arithmetical reduction of the continuum, they each accommodated its antecedent geometric sense by asserting that each of their real numbers actually corresponds to a point on the line. Neither theft nor honest toil sufficed; Cantor [1872, 128] and Dedekind [1872, III] recognized the need for an *axiom* to this effect, a sort of Church's Thesis of adequacy for the new construal of the continuum as a collection of objects.

Cantor recalled (in his [1880, 358]) that around this time he was already considering infinite iterations of his P' operation using "symbols of infinity":

$$P^{(\infty)} = \bigcap_{n}^{\infty} P^{(n)}, P^{(\infty+1)} = P^{(\infty)\prime}, P^{(\infty+2)}, \ldots$$

$$P^{(\infty\cdot 2)}, \ldots P^{(\infty^2)}, \ldots P^{(\infty^\infty)}, \ldots P^{(\infty^{\infty^\infty})}, \ldots$$

In a crucial conceptual move he began to investigate infinite collections of real numbers and infinitary enumerations for their own sake, and this led first to basic concepts in the study of the continuum and then to the formulation of the transfinite numbers. Set theory was born on that December 1873 day[6] when Cantor established that the collection of real numbers is uncountable, and in the next decades the subject was to blossom through the prodigious progress made by him in the theory of ordinal and cardinal numbers.

The uncountability of the reals was established, of course, via *reductio ad absurdum* as with the irrationality of $\sqrt{2}$. Both impossibility results epitomized that peculiar capability of *reductio* to compel a larger mathematical context allowing for the deniability of hitherto implicit properties. However, there was a crucial constructive component in both Cantor's original and the later explicitly diagonal argument. His first published account [1874] was entitled "On a property of the totality of all real algebraic numbers." After first establishing that property, the countability of the algebraic numbers, he then established: *For any* (*countable*) *sequence of reals, every interval contains a real not in the sequence.* The following is Cantor's argument, in brief:

Suppose that s is a sequence of reals and I an interval. Let $a < b$ be the first two reals of s, if any, in I. Then let $a' < b'$ be the first two reals of s, if any, in the open interval (a, b); $a'' < b''$ the first two reals of s, if any, in (a', b'); and so forth. Then however long this process continues, the (non-empty) intersection of these nested intervals cannot contain any member of s.

By this means Cantor provided a new proof of Joseph Liouville's result [1844, 1851] that there are transcendental numbers, and only afterwards did he impredicatively point out the uncountability of the reals altogether. This presentation reflects Cantor's natural caution in overstepping mathematical sense at the time.[7]

Accounts of Cantor's work have mostly reversed the order for deducing the existence of transcendental numbers, first giving the diagonal argument for the uncountability of the reals and only then drawing the existential conclusion from the countability of the algebraic numbers. Indeed, this is where Ludwig Wittgenstein [1956, I, Appendix II,1–3] located the problematic aspects of the talk of uncountability. In textbooks the inversion may be inevitable, but a historical misrepresentation has been perpetuated that associates diagonalization with non-constructivity.[8] In terms of decimal expansions Cantor's arguments can be implemented to generate the successive digits of a new real.[9] The original Liouville argument depended on an elegant observation about fast convergence, and the digits of the Liouville numbers can be generated much faster. In terms of 2.3 below, the later Baire Category Theorem can be viewed as a direct generalization of Cantor's [1874] result, and the collection of Liouville numbers provides an explicit example of a co-meager yet measure zero set of reals.[10]

1.2. Continuum Hypothesis and ordinal numbers. By his next publication [1878] Cantor had shifted the emphasis to bijective correspondence, stipulating that two sets have the same *power* (Mächtigkeit) *iff* there is such a correspondence between them, and established that the reals $\mathbb{R}$ and n-dimensional space $\mathbb{R}^n$ have the same power. Having made the initial breach in [1874] with a negative result about the *lack* of a bijective correspondence, Cantor secured the new ground with a positive investigation of the *possibilities* for such correspondences.[11] With "sequence" tied traditionally to countability through the indexing, Cantor used "correspondence [Beziehung]." Just as the discovery of the irrational numbers had led to one of the great achievements of Greek mathematics, Eudoxus's theory of geometrical proportions presented in Book V of Euclid's *Elements* and thematically antecedent to Dedekind's [1872] cuts, Cantor began his move toward a full-blown mathematical theory of the infinite.

Although holding the promise of a rewarding investigation Cantor could only find two powers for infinite sets; Cantor conjectured at the end of [1878, 257]:

Every infinite set of reals either is countable
or has the power of the continuum.

This of course is the *Continuum Hypothesis* (CH) in the nascent context. The conjecture viewed as a primordial question about the continuum would stimulate Cantor not only to approach the reals *qua* extensionalized continuum in an increasingly arithmetical fashion but also to grapple with fundamental questions of set existence. His triumphs across a new mathematical context would be like a brilliant light to entice others into the study of the infinite, but his inability to establish CH would also cast a long shadow. Set theory had its beginnings not as some abstract foundation for mathematics but rather as a setting for the articulation and solution of the *Continuum Problem*: to determine whether there are more than two powers embedded in the continuum.

In his next major publication, the *Grundlagen* [1883], Cantor introduced the ordinal numbers and the concept of *well-ordering*. No longer was the infinitary indexing of his trigonometric series investigation mere contrivance. The indices became autonomous as the ordinal numbers, the emergence signified by the notational switch from the ∞ of potentiality to the ω of completion as the last letter of the Greek alphabet. A corresponding transition from subsets of $\mathbb{R}^n$ to a broader concept of set was signaled by the shift in terminology from "point-manifold [Punktmannigfaltigkeit]" to "set [Menge]." The key was well-ordering, and Cantor propounded a basic principle that was to drive the development of set theory [1883, 550]:

"It is always possible to bring every *well-defined*
set into the form of a *well-ordered* set."

He regarded this as a "law of thought [Denkgesetz]" which is "fundamental, rich in consequences, and particularly remarkable for its general validity."

The well-ordering principle was consistent with Cantor's basic view in the *Grundlagen* that the finite and the transfinite are all of a piece and uniformly comprehendable in mathematics,[12] a view bolstered by his systematic development of the arithmetic of ordinal numbers seamlessly encompassing the finite numbers. Cantor also devoted several sections of the *Grundlagen* to a justificatory philosophy of the infinite, and while this metaphysics can be separated from the mathematical development, one concept was to suggest ultimate delimitations for set theory: Beyond the transfinite was the "Absolute," which Cantor eventually associated mathematically with the collection of all ordinal numbers and metaphysically with the transcendence of God.

The Continuum Problem was never far from this development, and was arguably its underlying motivation. The transfinite ordinal numbers were to provide the framework for Cantor's two approaches to the problem, the approach through power and the more direct approach through definable sets of reals.

As for the approach through power, Cantor in the *Grundlagen* defined the first number class (I) to be the set of natural numbers and the second number class (II) to be the set of countably infinite ordinal numbers, and (only) indicated the continuation to the third number class (III) and beyond. He then established that (II) is uncountable, yet *any infinite subset of* (II) *is either countable or has the same power as* (II). Hence, (II) has exactly the property that Cantor sought for the reals, and he had reduced CH to the assertion that the reals and (II) have the same power. The following is Cantor's argument that (II) is uncountable, in brief:

Suppose that s is a countable sequence of countable ordinal numbers, say with initial element a. Let a' be a member of s, if any, such that $a < a'$; let a'' be a member of s, if any, such that $a' < a''$; and so forth. Then however long this process continues, the supremum of these numbers, or its successor, is not a member of s.

This argument was reminiscent of his [1874] argument that the reals are uncountable and strongly suggested a correlation of the reals through their fundamental sequence represention with the members of (II) through associated cofinal sequences.[13] However, despite several announcements Cantor could never develop a workable correlation, the emerging problem being that he could not define a well-ordering of the reals. Cantor suggested successive number classes and infinitely many corresponding powers in the *Grundlagen* [1883, 588], but with the Continuum Problem unresolved he did not venture a systematic treatment.

As for the approach through definable sets of reals, this evolved directly from Cantor's original work on trigonometric series, the "symbols of infinity" used in the analysis of the P' operation becoming the ordinal numbers of the second number class.[14] In the *Grundlagen* Cantor studied P' for uncountable P and defined the key concept of a *perfect* set of reals (non-empty, closed, and containing no isolated points). Incorporating an observation of Ivar Bendixson [1883], Cantor showed in the succeeding [1884] that *any uncountable closed set of reals is the union of a perfect set and a countable set.* For a set A of reals, A has the *perfect set property iff* A is countable or else has a perfect subset. Cantor had shown in particular that *closed sets have the perfect set property.*

Since Cantor [1884, 1884a] had been able to show that any perfect set has the power of the continuum, he had established that "CH holds for closed sets": every closed set either is countable or has the power of the continuum. Or from his new vantage point, he had reduced the Continuum Problem to determining whether there is a closed set of reals of the power of the second number class. He was unable to do so, but he had initiated a program for attacking the Continuum Problem that was to be vigorously pursued (see 2.3 and 2.5).

1.3. Diagonalization and cardinal numbers. In the ensuing years, unable to resolve the Continuum Problem through direct correlations with ordinal numbers Cantor approached size and order from a larger perspective that *would* incorporate the continuum. He identified power with *cardinal number*, an autonomous concept beyond being *une façon de parler* about bijective correspondence, and he went beyond well-orderings to the study of linear *order types*. Cantor embraced a structured view of sets, when "well-defined," as being given together with a linear ordering of their members. Order types and cardinal numbers resulted from successive abstraction, from a set M to its order type $\overline{M}$ and then to its cardinality $\overline{\overline{M}}$.

In [1891] Cantor gave his famous diagonal argument, showing in effect that for any set M the collection of functions from M into a two-element set is of a strictly higher power than that of M. In retrospect the argument can be drawn out from his original [1874] proof of the uncountability of the reals,[15] but now he was able to dispense with their topological properties. On the other hand, the full impredicativity of the diagonal argument had to be faced by allowing arbitrary functions, or what is now regarded as co-extensive, arbitrary subsets. To assert that Cantor had the power *set* operation would be an exaggeration, but he had essentially established how it leads to higher powers.[16] He specifically argued for the first time that there is a power higher than that of the continuum,[17] and pointed out how his argument shows that there is no maximum power. This was a new triumph for Cantor, but it would also embed a central tension into his expanding context.[18] The simplicity of the diagonal argument presumably *demanded* that it become a basic pillar of the structure of the cardinal numbers. But with his view of well-defined sets as being inherently well-ordered he now had to confront head-on a general problem: *From a well-ordering of a set a well-ordering of its power set is not necessarily definable.* He had arrived at the crux of the Continuum Problem, and it was more basic than how reals and ordinal numbers are to be represented.

The two parts of the *Beiträge* [1895, 1897] were to be Cantor's last major publications. They summed up his progress, but their omissions also revealed the great gap left by the Continuum Problem. In the first part he described his post-*Grundlagen* work on cardinal number and the continuum, but soon posed *Cardinal Comparability*, whether

$$\text{for cardinal numbers } \mathfrak{a} \text{ and } \mathfrak{b},\ \mathfrak{a} = \mathfrak{b},\ \mathfrak{a} < \mathfrak{b}, \text{ or } \mathfrak{b} < \mathfrak{a}\,,$$

as a property "by no means self-evident" and which will be established later "when we shall have gained a survey over the ascending sequence of transfinite cardinal numbers and an insight into their connection." He went on to define the addition, multiplication, and exponentiation of cardinal numbers in the now familiar way. With respect to exponentiation, the audacity of considering the collection of all arbitrary functions from a set N into a set M was encased in a terminology that reflected both its novelty and the old view of function as given by an explicit rule.[19] After defining $\aleph_0$ as the cardinal number of the natural numbers, Cantor pointed out that $2^{\aleph_0}$ is the cardinal number of the continuum, and how the labor of [1878] associating $\mathbb{R}^n$ with $\mathbb{R}$ could be reduced to a "few strokes of the pen" in his new arithmetic:

$$(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}\,.$$

Cantor went on to present his theory of order types. He characterized the order type η of the rationals as the countable dense linear order without endpoints, introducing the now familiar back-and-forth argument of model theory. He also characterized the order type θ of the reals as the perfect linear order with a countable dense set; whether a realist or not, Cantor the mathematician had been able to provide a characterization of the continuum.

The second *Beiträge* developed the *Grundlagen* ideas by focusing on well-orderings and construing their order types as the ordinal numbers. Here at last was the general proof via order comparison of well-ordered sets that ordinal numbers are comparable. Cantor went on to describe ordinal arithmetic as a special case of the arithmetic of order types and after giving the basic properties of the second number class defined $\aleph_1$ as its cardinal number. The last sections were given over to a later preoccupation, the study of ordinal exponentiation in the second number class. The operation was defined via a transfinite recursion and used to establish a normal form, and the pivotal ε-numbers satisfying $\varepsilon = \omega^\varepsilon$ were analyzed.

The two parts of the *Beiträge* are not only distinct by subject matter, cardinal number and the continuum vs. ordinal number and well-ordering, but between them there developed a wide, irreconcilable breach. In the first part nowhere is the [1891] result $\mathfrak{a} < 2^{\mathfrak{a}}$ stated even in a special case; rather, it is made clear [1895, 495] that the procession of transfinite cardinal numbers is to be secured through their construal as the alephs, defined as the powers of the set of predecessors of ordinal numbers. However, the second *Beiträge* does not mention any aleph beyond $\aleph_1$, nor does it mention CH, which could now have been stated as $2^{\aleph_0} = \aleph_1$.[20] Ordinal comparability was secured, but cardinal comparability was not reduced to it. During this period Cantor was again intensely at work on the Continuum Problem,[21] and the tentativeness of the second *Beiträge* was to become a memorial to his failure. Having ushered in arbitrary functions through cardinal exponentiation Cantor had introduced an irreconcilable tension into his view that all sets are well-ordered, and there was little point to developing the theory of the higher alephs without the assurance of their gauging all the cardinal numbers.

Thus, the Continuum Problem was embedded in the very interstices of the early development of set theory, and in fact the structures that Cantor built, while now of intrinsic interest, emerged out of efforts to articulate and establish CH. The tension uncovered by his diagonal argument between well-ordering and arbitrary functions (and hence power set), remains central to set theory as the main source of its vitality and fascination. David Hilbert [1900] when he presented his famous list of 23 problems at the 1900 International Congress of Mathematicians in Paris made establishment of CH the very first problem and pointed out Cantor's main difficulty by suggesting that a definable well-ordering of the reals should be found.

The next, 1904 International Congress of Mathematicians at Heidelberg was to be a turning point for set theory. Julius König announced a proof that $2^{\aleph_0}$ is not an aleph, that the continuum is not well-orderable. The argument combined the result that $\aleph_\alpha < \aleph_\alpha^{\aleph_0}$ when the cofinality of α is ω, a consequence of the now familiar König inequality, with a result that alas does not universally hold.[22] Cantor was understandably upset with the prospect that the continuum would simply escape the number context that he had built for its analysis. Much of his life had been devoted to the Continuum Problem, and in his preoccupation with the second number class he had never entertained the basic distinction between regular and singular alephs. By the next day Zermelo had found the flaw, and the torch had passed from Cantor to Zermelo.

§2. Mathematization.

2.1. Axiom of Choice and axiomatization. Ernst Zermelo[23] (1871–1953), born when Cantor was establishing his trigonometric series results, had begun to investigate Cantorian set theory at Göttingen under the influence of Hilbert. In just over a month after the Heidelberg congress, and presumably stimulated by the König incident, Zermelo [1904] formulated what he soon called the *Axiom of Choice* (AC) and with it, established his Well-Ordering Theorem:

Every set can be well-ordered.

Nowadays we would say that this theorem has essentially one proof: For a given set M a choice function on the collection of non-empty subsets of M is used to recursively define a well-ordering of M. But the function concept was just then pivoting toward extensionalization, and Zermelo himself argued in terms of associating to each (non-empty) subset M' of M a "distinguished" element of M', this resulting in a "covering" γ.[24] He then defined a *γ-set* to be a well-ordered subset N of M such that every $a \in N$ is the distinguished element of

$$\{b \in N \mid b \text{ does not precede } a \text{ in the well-ordering of } N\}.$$

This is a clever and economical way of specifying an initial segment of the desired well-ordering! Zermelo proceeded to show that the union of all the γ-sets is again a γ-set and that it must be M itself, so that M is well-orderable.

Zermelo [1904, 516] noted without much ado that his result implies that every infinite cardinal number is an aleph and satisfies $\mathfrak{m}^2 = \mathfrak{m}$, and that it secured Cardinal Comparability—so that the main issues raised by Cantor's *Beiträge* are at once resolved. Zermelo maintained that AC is a "logical principle" which "is applied without hesitation everywhere in mathematical deduction," and this is reflected in the Well-Ordering Theorem being regarded as a theorem in itself. The axiom is consistent with Cantor's view of the finite and transfinite as unitary, in that it posits for infinite sets an unproblematic feature of finite sets. On the other hand, the Well-Ordering Theorem shifted the weight from Cantor's well-orderings with their residually temporal aspect of numbering through *successive* choices to the use of a function for making *simultaneous* choices.[25] Cantor's work had served to accentuate a growing stress among mathematicians, who were already exercised by two related issues: whether infinite collections can be mathematically investigated at all, and how far the function concept is to be extended. The positive use of an arbitrary function having been made explicit, there was open controversy after the appearance of Zermelo's proof.[26] This can be viewed as a turning point for mathematics, with the subsequent tilting toward the acceptance of AC symptomatic of a basic shift in mathematics.

In response to his critics Zermelo published a second proof [1908] of his Well-Ordering Theorem, and with axiomatization assuming a general methodological role in mathematics he also published the first full-fledged axiomatization [1908a] of set theory. But as with Cantor's work this was no idle structure building but a response to pressure for a new mathematical context. In this case it was not for the formulation and solution of a *problem* like the Continuum Problem, but rather to clarify a specific *proof*. Zermelo's main motive for axiomatizing set theory was to buttress his Well-Ordering Theorem by making explicit its underlying set existence assumptions.[27]

The objections raised against Zermelo's first proof [1904] mainly played on the ambiguities of a γ-set's well-ordering being only implicit, as for Cantor's sets, and also on the definition of a γ-set having an impredicative (we would now say recursive) flavor. To avoid these features Zermelo in his second proof [1908] resorted to an approach due to Gerhard Hessenberg [1906, 674ff.] with roots in Dedekind [1888]. Instead of γ-sets, the initial segments of the desired well-ordering, Zermelo shifted to the final segments, and proceeded to define the maximal reverse inclusion chain:

Zermelo first defined a Θ-*chain* to be a collection Θ of subsets of M such that: (a) $M \in \Theta$; (b) if $A \in \Theta$, then $A - \{\varphi(A)\} \in \Theta$ where $\varphi(A)$ is the distinguished element of A; and (c) if $B \subseteq A$, then $\bigcap B \in \Theta$. He then took the intersection of all Θ-chains; showed that it is again a Θ-chain; and that it is well-ordered by reverse inclusion, the well-ordering corresponding to one for M. (Note that this proof is less parsimonious than the [1904] proof, relying on the power set of the power set of M.)

Zermelo's axiomatization encased his proof in just the necessary set existence principles, the salient ones being the generative Power Set[16] and Union Axioms, the Axiom of Choice of course, and the Separation Axiom. His setting allowed for urelements, objects without members yet distinct from each other. But Zermelo focused on sets, and his Axiom of Extensionality announced the espousal of an extensional viewpoint. In line with this AC, a "logical principle" in [1904] expressed in terms of an informal choice function, was framed less instrumentally: It posited for a set consisting of non-empty, pairwise disjoint sets the existence of a *set* that meets each one in a unique element.[28] However, the Separation Axiom retained an intensional aspect, with its "separating" out using a *definit* property [definite Eigenschaft] in Zermelo's language.

Fully two decades earlier Dedekind [1888] had provided an incisive analysis of the natural numbers and their arithmetic in terms of sets [Systeme], and several overlapping aspects serve as points of departure for Zermelo's axiomatization.[29] The first is how Dedekind's chain argument extends to Zermelo's [1908] proof of the Well-Ordering Theorem, which in the transfinite setting brings out the role of AC. Both Dedekind and Zermelo set down rules for sets in large part to articulate arguments involving simple set operations like "set of," union, and intersection. In particular, both had to argue for the equality of sets resulting after involved manipulations, and extensionality became operationally necessary. However vague the initial descriptions of sets, sets are to be determined solely by their elements, and the membership question is to be determinate.[30] The looseness of Dedekind's description of sets allowed him [1888, §66] the latitude to "prove" the existence of infinite sets, but Zermelo just stated the Axiom of Infinity as a set existence principle.

The main point of departure has to do with the larger issue of the role of proof for articulating sets. By Dedekind's time proof had become basic for mathematics, and indeed his work did a great deal to enshrine proof as the vehicle to algebraic abstraction and generalization.[31] Like the algebraic constructs sets were new to mathematics and would be incorporated by setting down the rules for their proofs. Just as calculations are part of the sense of numbers, so proofs would be part of the sense of sets, as their "calculations." Just as Euclid's axioms for geometry had set out the permissible geometric constructions, the axioms of set theory would set out the specific rules for set generation and manipulation. But unlike the emergence of mathematics from marketplace arithmetic and Greek geometry, sets and transfinite numbers were neither laden nor buttressed with substantial antecedence. Like strangers in a strange land stalwarts developed a familiarity with them guided hand in hand by their axiomatic scaffolding. For Dedekind [1888] it had sufficed to work with sets and functions by merely giving a few definitions and properties. For Zermelo [1908a] more rules had to be given in the form of various axioms to articulate the proof of the Well-Ordering Theorem. Mathematization does not provide explanation, but rather contextualization through rules and their use at various levels of organization.

Zermelo's axiomatization shifted the focus away from the transfinite numbers to an abstract view of sets structured solely by $\in$ and simple operations. For Cantor the transfinite numbers had become central to his investigation of definable sets of reals and the Continuum Problem, and sets had emerged not only equipped with orderings but only as the developing context dictated, with the "set of" operation never iterated more than three or four times. For Zermelo, his second reverse inclusion chain proof of the Well-Ordering Theorem served to eliminate any residual role that the transfinite numbers may have played in the first proof and highlighted the set-theoretic operations. This approach to (linear) ordering was to preoccupy his followers for some time, and through this period the elimination of the use of transfinite numbers where possible, like ideal numbers, was regarded as salutary.[32] Hence, Zermelo rather than Cantor should be regarded as the creator of abstract set theory.

Outgrowing Zermelo's pragmatic purposes axiomatic set theory could not long forestall the Cantorian initiative, as even $2^{\aleph_0} = \aleph_1$ could not be asserted directly, and in the 1920's John von Neumann was to fully incorporate the transfinite using Replacement (see 3.1). (Textbooks usually establish the Well-Ordering Theorem by first introducing Replacement, formalizing transfinite recursion, and only then defining the well-ordering using (von Neumann) ordinals; this amounts to another historical misrepresentation, but one that resonates with how acceptance of Zermelo's proof broke the ground for formal transfinite recursion.) Generally in terms of later developments, Zermelo's axioms had the advantages of schematic simplicity and open-endedness. The generative set formation axioms, especially Power Set and Union, were to lead to Zermelo's [1930] cumulative hierarchy picture of sets, and the vagueness of the *definit* property in the Separation Axiom was to invite Thoralf Skolem's [1923] proposal to base it on first-order logic, enforcing extensionalization (see 3.2).

2.2. Logic and paradox. At this point, the incursions of a looming tradition can no longer be ignored. Gottlob Frege is regarded as the greatest philosopher of logic since Aristotle for developing quantificational logic in his *Begriffsschrift* [1879], establishing a logical foundation for arithmetic in his *Grundlagen* [1884], and generally stimulating the *analytic tradition* in philosophy. The architect of that tradition was Bertrand Russell who in his earlier years, influenced by Frege and Guiseppe Peano, wanted to found all of mathematics on the certainty of logic. But from a logical point of view Russell [1903] became exercised with paradox. He had arrived at Russell's Paradox in late 1901 by analyzing Cantor's diagonal argument applied to the class of all classes,[33] a version of which is now known as Cantor's Paradox of the largest cardinal number. Russell [1903, §301] also refocused the Burali-Forti Paradox of the largest ordinal number, after reading Cesare Burali-Forti's [1897].[34] Russell's Paradox famously led to the tottering of Frege's mature formal system, the *Grundgesetze* [1893, 1903].[35] Yet Russell's vaulting ambition led to another foundational edifice, Alfred Whitehead and Russell's *Principia Mathematica* [1910, 1912, 1913], in which a large part of mathematics was derived from a modicum of logical concepts and axioms. But in traumatic reaction to the paradoxes, *Principia* was encased in a complex logical system of different types and intensional predications ultimately breaking under Russell's Axiom of Reducibility, a fearful symmetry imposed by an artful dodger.

The mathematicians did not imbue the paradoxes with such potency. Unlike Russell who wanted to get at everything but found that he could not, they started with what could be got at and peered beyond. And as with the invention of the irrational numbers, the outward push eventually led to the positive subsumption of the paradoxes.

Cantor in 1899 correspondence with Dedekind considered the collection Ω of all ordinal numbers as in the Burali-Forti Paradox, but he used it *positively* to give mathematical expression to his Absolute.[36] He defined an "absolutely infinite or inconsistent multiplicity" as one into which Ω is injectible, and proposed that these collections be exactly the ones that are not sets. He would thus probe the very limits of sethood using his positive concept of power! Cantor stated some set existence principles, most notably the Union Axiom and forms of Separation and Replacement, and proceeded to argue that every set can be well-ordered: If a collection cannot be well-ordered, then Ω can be injected into it, and hence it must be absolutely infinite. Presumably the injection is to be given by a recursion, one that could never exhaust the collection before running through Ω, and so this prefigured von Neumann's formalization in the 1920's with Replacement (see 3.1). Cantor never published his argument, perhaps sensing the lack of a structured context, and in retrospect what was needed was something akin to the Zermelian axiomatic framework. The 1899 correspondence only appeared in 1932 when Zermelo brought out Cantor's collected works (Cantor [1932]); it must have been astonishing to see so much anticipation of the later structuring of set theory.

Zermelo found Russell's Paradox independently and before 1903,[37] but like Cantor, he did not regard the emergence of the paradoxes so much as a crisis as an overall delimitation for sets. In the Zermelian generative view [1908, 118], " ... if in set theory we confine ourselves to a number of established principles such as those that constitute the basis of our proof—principles that enable us to form initial sets and to derive new sets from given ones—then all such contradictions can be avoided." For the first theorem of his axiomatic theory Zermelo [1908a] subsumed Russell's Paradox, putting it to use as is done nowadays to establish that for any set x there is a $y \subseteq x$ such that $y \notin x$, and hence that there is no universal set.[38]

The differing concerns of Frege-Russell logic and the emerging set theory are further brought out by the analysis of the function concept as discussed below in 2.4, and those issues are here rehearsed with respect to the existence of the null class, or empty set. The null class played a crucial role in the definition of zero in Frege's [1884, 74] reduction of arithmetic to logic. Frege [1897] in his criticism of Ernst Schröder's [1890] argued that Schröder cannot maintain both that a class is merely a collection of objects and that there is a null class, since it contains no objects. Logic enters in giving unity to a class as an extension of concept and thus making the null class viable. Adhering to an intensional (class-concept) vs. extensional (class) dichotomy, Russell [1903, §36] criticized Peano much as Frege had Schröder by pointing out the incompatibility of a class being merely a collection of terms yet the null class subsisting without any terms.

As for the set theorists, Cantor did not dwell on the empty set although he at one point [1880, 356] introduced a sign O and used $P \equiv O$ to assert that the set P has no members. Dedekind [1888, §2] deliberately excluded the empty set though he saw its possible usefulness in other contexts. Then Zermelo [1908a] with his emphasis on set-theoretic operations posited the existence of the empty set, though something of intension remains in his calling it "a (fictitious) set." Thus, the set theorists attributed little significance to the empty set beyond its usefulness. Although embracing both extensionality and the null class may engender philosophical difficulties for the logic of classes, the empty set became commonplace in mathematics simply through use, like its intimate, zero.

2.3. Measure, category, and Borel hierarchy. During this period Cantor's two main legacies, the investigation of definable sets of reals and the extension of number into the transfinite, were further incorporated into mathematics in direct initiatives. The axiomatic tradition would be complemented by another, one that would draw its life more directly from mathematics.

The French analysts Emile Borel, René Baire, and Henri Lebesgue took on the investigation of definable sets of reals in what was to be a paradigmatically constructive approach. Cantor [1884] had established the perfect set property for closed sets and formulated the concept of *content* for a set of reals, but he did not pursue these matters. With these as antecedents the French work would lay the basis for measure theory as well as *descriptive set theory,* the definability theory of the continuum.[39]

Soon after completing his thesis Borel [1898, 46ff.] considered for his theory of measure those sets of reals obtainable by starting with the intervals and closing off under complementation and countable union. The formulation was axiomatic and in effect impredicative, and seen in this light, bold and imaginative; the sets are now known as the *Borel sets* and quite well-understood.

Baire in his thesis [1899] took on a dictum of Lejeune Dirichlet's that a real function is any arbitrary assignment of reals, and diverging from the 19th Century preoccupation with pathological examples, sought a constructive approach via pointwise limits. He formulated the following classification of real functions: *Baire class 0* consists of the continuous real functions, and for countable ordinal numbers $\alpha > 0$, *Baire class* α consists of those functions f not in any previous class yet obtainable as pointwise limits of sequences $f_0, f_1, f_2, \ldots$ of functions in previous classes, i.e., $f(x) = \lim_{n\to\infty} f_n(x)$ for every real x. The functions in these classes are now known as the *Baire* functions, and this was the first stratification into a transfinite hierarchy after Cantor.[40]

Baire's thesis also introduced the now basic concept of *category.* A set of reals is *nowhere dense iff* its closure under limits includes no open set, and a set of reals is *meager* (or *of first category*) *iff* it is a countable union of nowhere dense sets—otherwise, it is *of second category*. Baire established the Baire Category Theorem: *Every non-empty open set of reals is of second category.* His work also suggested a basic property: A set of reals has the

Baire property iff it has a meager symmetric difference with some open set. Straightforward arguments show that every Borel set has the Baire property.

Lebesgue's thesis [1902] is fundamental for modern integration theory as the source of his concept of measurability. Inspired in part by Borel's ideas and notably non-constructive, Lebesgue's concept of measurable set through closure under countable unions subsumed the Borel sets, and his analytic definition of measurable function through closure under pointwise limits subsumed the Baire functions. Category and measure are quite different; there are co-meager (complement of a meager) sets of reals that are of Lebesgue measure zero (cf., end of 1.1).[41] Lebesgue's first major work in a distinctive direction was to be the seminal paper in descriptive set theory:

In the memoir [1905] Lebesgue investigated the Baire functions, stressing that they are exactly the functions definable via analytic expressions (in a sense made precise). He first established a correlation with the Borel sets by showing that they are exactly the pre-images of open intervals via Baire functions. With this he introduced the first hierarchy for the Borel sets, his *open sets of class* α being those sets not in any previous class that are the pre-images of some open interval via some Baire class α function. After verifying various closure properties and providing characterizations for these classes Lebesgue established two main results. The first demonstrated the necessity of exhausting the countable ordinal numbers: *The Baire hierarchy is proper, i.e., for every countable* α *there is a Baire function of class* α*; correspondingly the hierarchy for the Borel sets is analogously proper.* The second established transcendence beyond countable closure for his concept of measurability: *There is a Lebesgue measurable function which is not in any Baire class; correspondingly there is a Lebesgue measurable set which is not a Borel set.*

The first result was the first of all hierarchy results, and a precursor of fundamental work in mathematical logic in that it applied Cantor's enumeration and diagonalization argument to achieve a transcendence to a next level. Lebesgue's second result was also remarkable in that he actually provided an explicitly defined set, one that was later seen to be the first example of a non-Borel analytic set (see 2.5). For this purpose, the reals were for the first time regarded as encoding something else, namely countable well-orderings, and this not only further embedded the transfinite into the investigation of sets of reals, but foreshadowed the later coding results of mathematical logic.

Lebesgue's results, along with the later work in descriptive set theory, can be viewed as pushing the mathematical frontier of the actual infinite past $\aleph_0$, which arguably had achieved a mathematical domesticity through increasing use in the late 19th Century, through Cantor's second number class to $\aleph_1$. It is somewhat ironic but also revealing, then, that this grew out of work by analysts with a definite constructive bent. Baire [1899, 36] viewed the infinite ordinal numbers and hence his function hierarchy as merely *une façon de parler*, and continued to view infinite concepts only in potentiality. Borel [1898] took a pragmatic approach and seemed to accept the countable ordinal numbers. Lebesgue was more equivocal but still accepting; recalling Cantor's early attitude Lebesgue regarded the ordinal numbers as an indexing system, "symbols" for classes, but nonetheless he exposed their basic properties, even providing a formulation [1905, 149] of proof by transfinite induction. All three analysts expressed misgivings about AC and its use in Zermelo's proof.[42]

As descriptive set theory was to develop, a major concern became the extent of the *regularity properties*, those properties indicative of well-behaved sets of reals of which Lebesgue measurability, the Baire property, and the perfect set property are the prominent examples. These properties seemed to get at basic features of the extensional construal of the continuum, yet resisted inductive approaches. Early explicit uses of AC through a well-ordering of the reals showed how it allowed for new constructions: Guiseppe Vitali [1905] established that there is a non-Lebesgue measurable set, and Felix Bernstein [1908], that there is a set without the perfect set property, in fact an uncountable set of reals such that both it and its complement meet every perfect set. Soon it was seen that neither of these examples have the Baire property. Thus, Cantor's very contention that the reals are well-orderable permitted constructions that precluded the universality of the regularity properties, in particular his own approach to the Continuum Problem through the perfect set property. How these results would be taken was to turn on the issues of AC and set existence.

2.4. Hausdorff and functions. Felix Hausdorff was the first developer of the transfinite after Cantor, the one whose work first suggested the rich possibilities for a mathematical investigation of the higher transfinite. A mathematician *par excellence*, Hausdorff took that sort of mathematical approach to set theory and extensional, set-theoretic approach to mathematics that would dominate in the years to come. While the web of 19th Century intension in Cantor's work, especially his approach toward functions, now seems rather remote, Hausdorff's work seems familiar as part of the modern language of mathematics.

In [1908] Hausdorff brought together his extensive work on *uncountable* order types. Deploring all the fuss being made over foundations by his contemporaries (p. 436) and with Cantor having taken the Continuum Problem as far as seemed possible, Hausdorff proceeded to venture beyond the second number class with vigor. He provided an elegant analysis of scattered linear order types (those having no dense subtype) in a transfinite hierarchy, and constructed the η_α sets, prototypes for saturated model theory. He first stated the *Generalized Continuum Hypothesis* (GCH), clarifed the significance of cofinality, and first considered (p. 443) the possibility of an uncountable regular limit cardinal, the first *large cardinal.*

Large cardinal hypotheses posit cardinals with properties that entail their transcendence over smaller cardinals, and as it turned out, provide a superstructure of hypotheses for the analysis of strong propositions in terms of consistency. Hausdorff observed that uncountable regular limit cardinals, also known now as *weakly inaccessible cardinals*, are a natural closure point for cardinal limit processes. In penetrating work of only a few years later Paul Mahlo [1911, 1912, 1913] investigated hierarchies of such cardinals based on higher fixed-point phenomena, the *Mahlo cardinals*. The theory of large cardinals was to become a mainstream of set theory.[43]

Hausdorff's classic text, *Grundzüge der Mengenlehre* [1914], broke the ground for a generation of mathematicians in both set theory and topology. A compendium of a wealth of results, it emphasized mathematical approaches and procedures that would eventually take firm root.[44] Typical of small but significant matters he (p. 37) pointed out the now commonplace correlation of *sets* with their characteristic *functions*, a significant correlation at the time and one through which Cantor's [1891] diagonal argument would become regarded as being about power sets. After giving a clear account of Zermelo's first proof [1904] of the Well-Ordering Theorem, Hausdorff (p. 140ff.) emphasized its maximality aspect by giving synoptic versions of Zorn's Lemma two decades before Zorn [1935], one of them now known as Hausdorff's Maximality Principle.[45] Also, Hausdorff (p. 304) provided the now standard account of the Borel hierarchy of sets, with the still persistent F_σ and G_δ notation. Of particular interest, Hausdorff (p. 469ff., and also in [1914a]) used AC to provide what is now known as Hausdorff's Paradox, an implausible decomposition of the sphere and the source of the better known Banach-Tarski Paradox from Stefan Banach and Alfred Tarski's [1924].[46] Hausdorff's Paradox was the first, and a dramatic, synthesis of classical mathematics and the Zermelian abstract view.

Hausdorff's reduction of functions through a defined ordered pair highlights the differing concerns of the earlier Frege-Russell logic and the emerging set theory. Frege [1891] had two fundamental categories, function and object, with a function being "unsaturated" and supplemented by objects as arguments. A concept is a function with two possible values, the True and the False, and a relation is a concept that takes two arguments. The extension of a concept is its graph or course-of-values [Werthverlauf], which is an object, and Frege [1893, §36] devised an iterated or double course-of-values [Doppelwerthverlauf] for the extension of a relation. In these involved ways Frege connected functions to relations. As for the ordered pair Frege in his *Grundgesetze* [1893, §144] defined one derivatively as an extension of a concept and for a specific purpose, while Charles S. Peirce [1883], Schröder

[1895], and Peano [1897] essentially regarded a relation from the outset as just a collection of ordered pairs.[47] Whereas Frege was attempting an analysis of thought, Peano was mainly concerned about recasting ongoing mathematics in economical and flexible symbolism and made many reductions, e.g., construing a *sequence* in analysis as a *function* on the natural numbers.

Peano's symbolism was the inspiration, and Frege's work a bolstering, for Whitehead and Russell's *Principia Mathematica* [1910, 1912, 1913], in which relations distinguished in intension and in extension were derived from "propositional" functions taken as fundamental and other "descriptive" functions derived from relations. They [1910, $*55$] like Frege defined an ordered pair derivatively, in their case in terms of classes and relations, and also for a specific purpose.[48] Previously Russell [1903, §27] had criticized Peirce and Schröder for regarding a relation "essentially as a class of couples," although he overlooked this shortcoming in Peano.[49] Commenting obliviously on *Principia* Peano [1911, 1913] simply reaffirmed an ordered pair as basic, defined a relation as a class of ordered pairs, and a function extensionally as a kind of relation, referring to the final version of his *Formulario Mathematico* [1905–8, 73ff.] as the source. Capping this to and fro Norbert Wiener [1914] provided a definition of the ordered pair in terms of unordered pairs of classes only, thereby reducing relations to classes. Although Russell thought highly of Sheffer's stroke, the logical connective not-both, he was not impressed with Wiener's reduction, which could not have been considered a genuine one.[50] Unlike Russell, Willard V.O. Quine in his major philosophical work *Word and Object* [1960, §53] regarded the reduction of the ordered pair as a paradigm for philosophical analysis.

Making no intensional distinctions Hausdorff [1914, 32ff., 70ff.] defined an ordered pair in terms of unordered pairs, formulated functions in terms of ordered pairs, and ordering relations as collections of ordered pairs. (He did not so define an arbitrary relation, for which there was then no mathematical use, but he was first to consider general *partial* orderings, as in his maximality principle.[51]) Hausdorff thus made both the Peano [1911, 1913] and Wiener [1914] moves *in* mathematical practice, completing the reduction of functions to sets. This may have been congenial to Peano, but not to Frege nor Russell, they having emphasized the primacy of functions. Following the pioneering work of Dedekind and Cantor[52] Hausdorff was at the crest of a major shift in mathematics of which the transition from an intensional, rule-governed conception of function, to an extensional, arbitrary one, was a large part, and of which the eventual acceptance of the Power Set Axiom and the Axiom of Choice was symptomatic.

In his informal setting Hausdorff took the ordered pair of x and y to be $\{\{x, 1\}, \{y, 2\}\}$ where 1 and 2 are distinct objects alien to the situation. This causes a problem when all objects are to be sets, one precluded by the now-standard definition $\{\{x\}, \{x, y\}\}$ due to Kazimierz Kuratowski [1921, 171], notably a by-product of his analysis of Zermelo's [1908] proof of the Well-Ordering Theorem.[53]

2.5. Analytic and projective sets. A decade after Lebesgue's seminal paper [1905], descriptive set theory emerged as a distinct discipline through the efforts of the Russian mathematician Nikolai Luzin. Luzin had become acquainted with the work and views of the French analysts while he was in Paris as a student, and from the beginning a major topic of his seminar was the "descriptive theory of functions." Significantly, the young Wacław Sierpiński was an early participant while he was interned in Moscow in 1915. Not only did this lead to a decade-long collaboration between Luzin and Sierpiński, but it undoubtedly encouraged the latter in his founding of the Polish school of mathematics and laid the basis for its interest in descriptive set theory.

Of the three regularity properties (see 2.3), two were immediate for the Borel sets: Lebesgue measurability and the Baire property. But little had been known about the perfect set property beyond Cantor's own result that the closed sets have it. Luzin's student Pavel Aleksandrov established the groundbreaking result that *the Borel sets have the perfect set property*, so that "CH holds for the Borel sets."[54]

In the work that really began descriptive set theory another student of Luzin's, Mikhail Suslin, investigated the *analytic sets* following a mistake he had found in Lebesgue's paper. Suslin [1917] formulated these sets in terms of an explicit operation $\mathcal{A}$: A *defining system* is a family $\{X_s\}_s$ of sets indexed by finite sequences s of natural numbers. The result of the Operation $\mathcal{A}$ on such a system is that set $\mathcal{A}(\{X_s\}_s)$ defined by:

$$x \in \mathcal{A}(\{X_s\}_s) \text{ } \mathit{iff} \text{ } (\exists f : \omega \to \omega)(\forall n \in \omega)(x \in X_{f|n})$$

where $f|n$ denotes that sequence determined by the first n values of f. For X a set of reals, X is *analytic iff* $X = \mathcal{A}(\{X_s\}_s)$ for some defining system $\{X_s\}_s$ consisting of closed sets of reals.

Suslin [1917] announced two fundamental results: *a set of reals is Borel iff it is both analytic and co-analytic;* and *there is an analytic set which is not Borel.*[55] This was to be his sole publication, for he succumbed to typhus in a Moscow epidemic in 1919 at the age of 25. In an accompanying note Luzin [1917] announced the regularity properties: *Every analytic set is Lebesgue measurable, has the Baire property, and has the perfect set property*, the last result attributed to Suslin.

Luzin and Sierpiński in their [1918, 1923] provided proofs, and the latter paper was instrumental in shifting the emphasis toward the co-analytic sets: Suppose that Y a co-analytic set of reals, i.e., $Y = \mathbb{R} - X$ with $X = \mathcal{A}(\{X_s\}_s)$

for some closed sets X_s, so that for reals x,

$$x \in Y \textit{ iff } x \notin X \textit{ iff } (\forall f : \omega \to \omega)(\exists n \in \omega)(x \notin X_{f|n}).$$

For finite sequences s_1 and s_2 define: $s_1 \prec s_2$ *iff* s_2 is a proper initial segment of s_1. For a real x define: $T_x = \{s \mid x \in X_t$ for every initial segment t of $s\}$. Then:

$$x \in Y \textit{ iff } \prec \text{ on } T_x \text{ is a well-founded relation,}$$

i.e., there is no infinite descending sequence $\ldots \prec s_2 \prec s_1 \prec s_0$. Thus, by negating the formulation of the operation $\mathcal{A}$, Luzin and Sierpiński had arrived at the basic *tree representation* of co-analytic sets, one from which the main results of the period flowed, and it is here that well-founded relations entered mathematical practice (cf., 3.2).

After the first wave in descriptive set theory brought about by Suslin [1917] and Luzin [1917] had crested, Luzin [1925a] and Sierpiński [1925] extended the domain of study to the *projective sets*. For $Y \subseteq \mathbb{R}^{k+1}$, the *projection of* Y is

$$pY = \{\langle x_1, ..., x_k\rangle \mid \exists y(\langle x_1, ..., x_k, y\rangle \in Y)\}.$$

Suslin [1917] had essentially noted that a set of reals is analytic *iff* it is the projection of a Borel subset of $\mathbb{R}^2$. (Borel subsets of $\mathbb{R}^k$ are defined analogously to those of $\mathbb{R}$.) Luzin and Sierpiński took the geometric operation of projection to be basic and defined the projective sets as those sets obtainable from the Borel sets by the iterated applications of projection and complementation. The corresponding hierarchy of projective subsets of $\mathbb{R}^k$ is defined as follows: For $A \subseteq \mathbb{R}^k$,

$$A \text{ is } \mathbf{\Sigma}^1_1 \textit{ iff } A \text{ is analytic}$$

(defined as for $k = 1$ in terms of a defining system consisting of closed subsets of $\mathbb{R}^k$) and recursively for integers n,

$$A \text{ is } \mathbf{\Pi}^1_n \textit{ iff } \mathbb{R}^k - A \text{ is } \mathbf{\Sigma}^1_n, \text{ and}$$
$$A \text{ is } \mathbf{\Sigma}^1_{n+1} \textit{ iff } A = pY \text{ for some } \mathbf{\Pi}^1_n \text{ set } Y \subseteq \mathbb{R}^{k+1}.$$

Luzin [1925a] and Sierpiński [1925] recast Lebesgue's use of the Cantor diagonal argument to show that the projective hierarchy is proper, and soon its basic properties were established. However, this investigation encountered basic obstacles from the beginning. Luzin [1917, 1925a] emphasized that whether the $\mathbf{\Pi}^1_1$ sets, the co-analytic sets at the bottom of the hierarchy, have the perfect set property was not known.[56] Luzin [1925b] also noted that at the next level, whether the $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable was not known. Both these difficulties were also pointed out by Sierpiński [1925]. This basic impasse in descriptive set theory was to remain for over a decade, to be surprisingly resolved by penetrating work of Gödel involving metamathematical methods (see 3.4).

2.6. Equivalences and consequences. In this period AC and CH began to be explored no longer as underlying axiom and primordial hypothesis but as part of mathematics. Consequences were drawn and even equivalences established, and this mathematization, like the development of non-Euclidean geometry, led eventually to a deflating of metaphysical attitudes and attendant concerns about truth and existence.

Friedrich Hartogs [1915] established an equivalence result for AC, and this was the first substantial use of Zermelo's axiomatization after his own in the Well-Ordering Theorem. The axiomatization had initially drawn ambivalent response among commentators,[57] especially those exercised by the paradoxes. However, the eventual success of the combinatorial framework would be secured by its increasing mathematical use to structure and clarify arguments. As noted in 1.3, Cardinal Comparability had become a problem for Cantor by the time of his *Beiträge* [1895]; Hartogs showed in Zermelo's system *sans* AC that *Cardinal Comparability implies that every set can be well-ordered.* Thus, an evident consequence of every set being well-orderable also implied that well-ordering principle, and this first "reverse mathematics" result established the equivalence of the well-ordering principle, Cardinal Comparability, and AC over the base theory.

Hartogs actually established without AC what is now called *Hartogs's Theorem*: *For any set M, there is a well-orderable set E not injectible into M.* Cardinal Comparability would then imply that M is injectible into E and hence is well-orderable. For the proof Hartogs first worked out a theory of ordering relations in Zermelo's system in terms of reverse inclusion chains as in Zermelo's [1908] proof.[58] He then used Power Set and Separation to get the set M_W of well-orderable subsets of M and the set E of equivalence classes partitioning M_W according to order-isomorphism. Finally, he showed that E itself has an inherited well-ordering and is not injectible into M. (As with Zermelo's Well-Ordering Theorem, textbooks usually establish Hartogs's Theorem after first introducing Replacement and (von Neumann) ordinals, and this amounts to a historical misrepresentation.) Reminiscent of Zermelo's subsumption of Russell's Paradox in the denial of a universal set, Hartogs's Theorem can be viewed as a subsumption of the Burali-Forti Paradox into the Zermelian setting.

The first explicit uses of AC mostly amounted to appeals to a well-ordering of the reals, Cantor's preoccupation. Those of Vitali [1905] and Bernstein [1908] were mentioned in 2.3, and Hausdorff's Paradox [1914, 1914a], in 2.4. Georg Hamel [1905] constructed by transfinite recursion a basis for the reals as a vector space over the rationals; cited by Zermelo [1908, 114], this provided a useful basis for later work in analysis and algebra. These various results, jarring at first, broached how a well-ordering allows for a new kind of *arithmetical* approach to the continuum.

The full exercise of AC in ongoing mathematics first occurred in the pioneering work of Ernst Steinitz [1910] on abstract fields. This was the first instance of an emerging phenomenon in algebra and topology: the study of axiomatically given structures with the range of possibilities implicitly including the transfinite. Steinitz studied algebraic closures of fields and even had an explicit transfinite parameter in the transcendence degree, the number of indeterminates necessary for closure. Typical of the generality in the years to come was Hausdorff's [1932] result using well-orderings that *every vector space has a basis*. As algebra and topology developed, however, such results as these came to be based on the maximal principles that Hausdorff had first broached (see 2.4) and which began to dominate after the appearance of Zorn's Lemma [1935]. Explicit well-orderings seemed out of place at this level of organization, and Zorn's Lemma had the remarkable feature that its hypothesis was easily checked in most applications.

The Polish school of mathematics carried out a penetrating investigation of the role of AC in set theory and analysis. Sierpiński's earliest publications culminating in his survey [1918] not only dealt with specific constructions but showed how deeply embedded AC was in the informal development of cardinality, measure, and the Borel hierarchy (see 2.3), supporting Zermelo's contention [1904, 516] that the axiom is applied "everywhere in mathematical deduction." Tarski [1924] explicitly building his work on Zermelo's system provided several propositions of cardinal arithmetic equivalent to AC, most notably that $\mathfrak{m}^2 = \mathfrak{m}$ for every infinite cardinal $\mathfrak{m}$. Adolf Lindenbaum and Tarski in their [1926] gave further cardinal equivalents, some related to the Hartogs [1915] result, and announced that GCH, in the form that $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$ holds for no infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$, implies AC. This study of consequences led to other choice principles, further implications and sometimes converses in a continuing cottage industry.[59]

The early mathematical study of AC extended to the issue of its independence. Abraham Fraenkel's first investigations [1922] directly addressed Zermelo's axioms, pointing out the need for the Replacement Axiom and attempting an axiomatization of the *definit* property for the Separation Axiom (see 3.1). The latter was motivated in part by the need to better articulate independence proofs for the various axioms. Fraenkel [1922a] came to the fecund idea of using urelements, objects without members yet distinct from each other, and starting with those and some initial sets closing off under set-theoretic operations to get a model. For the independence of AC he started with urelements $a_n, \overline{a_n}$ for $n \in \omega$ and the set $A = \{\{a_n, \overline{a_n}\} \mid n \in \omega\}$ of unordered pairs and argued that for any set M in the resulting model there is a co-finite $A_M \subseteq A$ such that M is invariant if members of any $\{a_n, \overline{a_n}\} \in A_M$ are permuted. This immediately implies that there is no choice function for

A in the model. Finally, Fraenkel argued that the model satisfies the other Zermelo axioms, except Extensionality because of the urelements.

Fraenkel's early model building emphasized the Zermelian generative framework, anticipated well-founded recursion, and foreshadowed the later play with models of set theory. That Extensionality was not to be had precluded settling the matter, but just as for the early models of non-Euclidean or finite geometries Fraenkel's achievement lay in stimulating interest in mathematical constructions despite relaxing some basic tenet. Fraenkel tried to develop his approach from time to time, but it needed the articulation that would come with the formalization of the satisfaction relation. In the late 1930's Lindenbaum and Andrzej Mostowski so cast and extended Fraenkel's work. Mostowski [1939] forged a method according to post-Gödelian sensibilities, bringing out the importance of groups of permutations leaving various urelements fixed, and the resulting models as well as later versions are now known as the *Fraenkel-Mostowski models.*

Even more than AC, Sierpiński investigated CH, and summed up his researches in a monograph [1934]. He provided several notable equivalences to CH, e.g., (p. 11) the plane $\mathbb{R}^2$ is the union of countably many curves, where a *curve* is a set of form $\{\langle x, y\rangle \mid y = f(x)\}$ or $\{\langle x, y\rangle \mid x = f(y)\}$ with f an injective real function.

Moreover, Sierpiński presented numerous consequences of CH from the literature, one in particular implying a host of others: Mahlo [1913a] and Luzin [1914] showed that CH *implies the existence of a Luzin set*, where a *Luzin set* is an uncountable set of reals that has countable intersection with every meager set (2.3). To state one consequence, a set X of reals has *strong measure zero iff* for any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of positive reals there is a sequence of intervals $I_0, I_1, I_2, \ldots$ such that the length of I_n is less than ε_n for each n and $X \subseteq \bigcup_n I_n$. Borel [1919] conjectured that such sets are countable. However, Sierpiński [1928] showed that *a Luzin set has strong measure zero.* Analogous to a Luzin set, a *Sierpiński set* is an uncountable set of reals that has countable intersection with every Lebesgue measure zero set. Sierpiński [1924] showed that CH *implies the existence of a Sierpiński set*, and emphasized [1934] an emerging duality between measure and category. Later, Fritz Rothberger [1938] showed that if both Luzin and Sierpiński sets exist, then they have cardinality $\aleph_1$, so that the joint existence of such sets of power the continuum implies CH; Rothberger [1939, 1948] considered other sets and implications between cardinal properties independent of whether CH holds, and this approach was to blossom half a century later in the study of *cardinal invariants* of the continuum.[60]

These results cast CH in a new light, as a construction principle. Conclusions had been drawn from having a well-ordering of the reals, but one with order type ω_1 allowed for recursive constructions where at any stage only countably many conditions corresponding to as many reals had to be handled. The construction of a Luzin set was a simple recursive application of the Baire Category Theorem (see 2.3), and later constructions took advantage of the possibility of diagonalization at each stage. However, whereas the new constructions using AC though jarring at first were eventually subsumed as concomitant with the acceptance of the axiom and as expressions of the richness of possibility, constructions from CH clashed with that very sense of richness for the continuum. It was the *mathematical* investigation of CH that increasingly raised doubts about its truth and certainly its provability (cf., end of 3.4).

§3. Consolidation.

3.1. Ordinals and Replacement. In the 1920's fresh initiatives structured the loose Zermelian framework with new features and corresponding developments in the axiomatics: von Neumann's work with ordinals and Replacement; the focusing on well-founded sets and the cumulative hierarchy; and extensionalization in first-order logic. Von Neumann effected a counter-reformation of sorts: The transfinite numbers had been central for Cantor but peripheral to Zermelo; von Neumann reconstrued them as *bona fide* sets, nowadays called simply the *ordinals*, and established their efficacy by formalizing transfinite recursion.

Von Neumann [1923, 1928], and before him Dimitry Mirimanoff [1917, 1917a] and Zermelo in unpublished 1915 work, isolated the now familiar concept of ordinal, with the basic idea of taking precedence in a well-ordering simply to be membership.[61] Appealing to forms of Replacement Mirimanoff and Von Neumann then established the key instrumental property of Cantor's ordinal numbers for ordinals: *Every well-ordered set is order-isomorphic to exactly one ordinal with membership.* Von Neumann in his own axiomatic presentation took the further step of ascribing to the ordinals the role of Cantor's ordinal number. Thus, like Kepler's laws by Newton's, Cantor's several principles of generation for ordinal numbers would be subsumed by the Zermelian abstract approach. For this and already to define the arithmetic of ordinals von Neumann saw the need to formalize transfinite recursion. And Replacement was necessary even for the very formulation, let alone the proof. With the ordinals in place von Neumann [1928, 1928a] completed the restoration of the Cantorian transfinite by *defining* the cardinals as the initial ordinals, codifying a strategy that had in fact appeared in that Cantor 1899 correspondence with Dedekind (see 2.2).

Fraenkel [1921, 1922] and Skolem [1923] had independently proposed the addition of Replacement to Zermelo's axioms, the first somewhat vaguely in terms of a restricted notion of function and the second in the now familiar framework of first-order logic. Both pointed to the inadequacy of Zermelo's axioms to establish that

$$\{\omega, \mathcal{P}(\omega), \mathcal{P}(\mathcal{P}(\omega)), \dots\}$$

is a set (where $\mathcal{P}$ denotes power set). Motivating Replacement through the need for specific sets like this or the (von Neumann) ordinal $\omega + \omega$ is certainly significant, and renders it a generalization of the Axiom of Infinity in introducing new generators into the Zermelian setting. However, it was von Neumann's formal incorporation of a *method* into set theory, transfinite recursion, that necessitated the full exercise of Replacement.

That Replacement became central for von Neumann was intertwined with his taking of function, in its full extensional sense, instead of set as primitive and his establishing of a context for handling *classes*, collections not necessarily sets. He [1925, 1928a] formalized the idea that a class is *proper*, i.e., not a set, exactly when it is bijective with the entire universe, and this exactly when it is not an element of any class. This thus brought in another Cantorian move from that 1899 correspondence. However, von Neumann's axiomatization [1925, 1928] of function was complicated, and reverting to sets as primitive Paul Bernays [1937+] recast and simplified von Neumann's system. Still, the formal incorporation of proper classes introduced a superstructure of objects and results distant from mathematical practice. What was to be inherited was a predisposition to entertain proper classes in the mathematical development of set theory, a willingness that would have crucial ramifications (cf., 3.6).

3.2. Well-foundedness and cumulative hierarchy. With ordinals and Replacement, set theory continued its shift away from pretensions of a general foundation to a more specific theory of the transfinite, a process fueled by the incorporation of well-foundedness. Mirimanoff [1917] was the first to study the well-founded sets, and the later cumulative hierarchy is clearly anticipated in his work. But interestingly enough well-founded relations next occurred in the direct definability tradition from Cantor, descriptive set theory (see 2.5).

In the axiomatic tradition Skolem [1923] and von Neumann [1925] considered the salutary effects of restricting the universe of sets to the well-founded sets. Well-founded relations in general were first explicitly defined much later in Zermelo [1935], in a notable adumbration of infinitary proofs. This was ostensibly an attempt to provide a logical framework for Zermelo's [1930] axiomatization, itself partly a response to a new, logical initiative:

The prescient Skolem [1923] had also made the proposal of taking for Zermelo's *definit* properties for the Separation Axiom those expressible in first-order logic. After Leopold Löwenheim [1915] had broken the ground for model theory with his result about the satisfiability of a first-order sentence, Skolem [1920, 1923] had located the result solidly in first-order logic and generalized it to the Löwenheim-Skolem Theorem: *If a countable collection of first-order sentences is satisfiable, then it is satisfiable in a countable domain.* That Skolem intended for set theory to be a first-order system without a privileged interpretation for $\in$ becomes evident in the initial application of the Löwenheim-Skolem Theorem to get the Skolem Paradox: In first-order logic Zermelo's axioms are countable, Separation having become a *schema*; the Theorem then implies the existence of countable models of set theory although that theory entails the existence of uncountable sets. Skolem intended by this means to deflate the possibility of set theory becoming a foundation for mathematics. Exercised by this relativism and by the recent work of Fraenkel and von Neumann, Zermelo [1929] in his first publication in set theory in two decades proposed an axiomatization of his *definit* property in second-order terms. In direct response Skolem [1930] pointed out possible difficulties with this approach and reaffirmed his first-order formulation, completing the backdrop for a new axiomatic synthesis.

Zermelo in his remarkable response [1930] to Skolem offered his final axiomatizations of set theory as well as a striking view of a procession of natural models. The main axiomatization incorporated Replacement but also the Axiom of Foundation, i.e., that the membership relation is well-founded. On the other hand, it was given in staunchly second-order terms, allowed urelements, and eschewed the Axiom of Infinity. But shorn of these features the axiomatization is recognizable as ZF; indeed, the appellation "Zermelo-Fraenkel" first appears in and emanates from this paper. Nowadays ZF does not include AC, the specific inclusion indicated by ZFC; Zermelo also did not include AC as he had done in his first axiomatization [1908a], but he assumed it as part of his underlying logic. To emphasize, Zermelo's theory was cast in second-order terms, whereas current ZF is a first-order theory; however, it is clear from his tussle with Skolem that Zermelo himself was opposed to any such revisionism.

In the same paper [1930] Zermelo described a succession of models for set theory, asserting that Foundation ranks the sets in these models into a *cumulative hierarchy.* In current terms the axiom layers the formal universe V of sets into ranks V_α, where

$$V_0 = \emptyset;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ V_\delta = \bigcup_{\alpha<\delta} V_\alpha \text{ for limit ordinals } \delta\,;$$

and

$$V = \bigcup_\alpha V_\alpha\,.$$

Mirimanoff [1917, 51ff.] and von Neumann [1929, 236ff.] had also formulated the cumulative hierarchy, but more to specific purposes;[62] Zermelo substantially advanced the schematic generative picture with his adoption of Foundation in an axiomatization. In a notable inversion this *iterative conception* became a heuristic for motivating the axioms of set theory generally.[63]

It is nowadays almost banal that Foundation is the one axiom unnecessary for the recasting of mathematics in set-theoretic terms, but the axiom is also the salient feature that distinguishes investigations specific to set theory as an autonomous field of mathematics. Indeed, it can be fairly said that current set theory is at base the study of well-foundedness, the Cantorian well-ordering doctrines adapted to the Zermelian generative conception of sets.

Zermelo established a second-order categoricity of sorts for his axioms: He showed that his models are characterized up to isomorphism by two cardinals, the number of their urelements and the height of their ordinals. Moreover, he established that if two models have the same number of urelements yet different heights, then one is isomorphic to an initial segment of the other's cumulative hierarchy. Grappling with Power Set and Replacement he characterized the heights of his models ("Grenzzahlen") as $\aleph_0$ or the (*strongly*) *inaccessible cardinals*, those uncountable regular cardinals κ that are strong limit. i.e., if $\lambda < \kappa$, then $2^\lambda < \kappa$.

Zermelo posited an endless procession of models, each a set in a next, advocating a dynamic view of sets that was a marked departure from Cantor's (and later, Gödel's) realist presumption of a fixed universe of sets. In synthesizing the sense of progression inherent in the new cumulative hierarchy picture and the sense of completion in the limit numbers, the inaccessible cardinals, he promoted the crucial idea of internal models of set theory.[64] The open-endedness of Zermelo's original [1908a] axiomatization had been structured by Replacement and Foundation, but with his absolute view of Power Set and cardinal number he advanced a new open-endedness with an eternal return of models approaching Cantor's Absolute.

In the process, inaccessible cardinals became structurally relevant. Sierpiński-Tarski [1930] had formulated these cardinals arithmetically as those uncountable cardinals that are not the product of fewer cardinals each of smaller power and observed that they are weakly inaccessible, the first large cardinal concept, from Hausdorff [1908, 443] (2.4). Be that as it may, in the early model-theoretic investigations of set theory the inaccessible cardinals provided the natural models as envisioned by Zermelo. Moreover, strong large cardinal hypotheses emerging in the 1960's were to be formulated in terms of these initial segments of the cumulative hierarchy.[65]

The journal volume containing Zermelo's [1930] also contained Stanisław Ulam's seminal paper [1930] on *measurable cardinals*, the most important of all large cardinals. For a set S, U is a (non-principal) *ultrafilter over S iff* U is a collection of subsets of S containing no singletons, closed under the taking of supersets and finite intersections, and such that for any $X \subseteq S$, either $X \in U$ or $S - X \in U$. For a cardinal λ, an ultrafilter U is λ-*complete iff* for any $D \subseteq U$ of cardinality less than λ, $\bigcap D \in U$. Finally, an uncountable cardinal κ is *measurable iff* there is a κ-complete ultrafilter over κ. Thus, a measurable cardinal is a cardinal whose power set is structured with a two-valued measure having a strong closure property.

Measurability embodied the first large cardinal confluence of Cantor's two legacies, the investigation of definable sets of reals and the extension of number into the transfinite: Distilled from measure-theoretic considerations related to Lebesgue measure, the concept *also* entailed inaccessibility in the transfinite. Moreover, the initial airing generated an open problem that was to keep the spark of large cardinals alight for the next three decades: *Can the smallest inaccessible cardinal be measurable?* In the 1960's consequences of, and a structural characterization of, measurability were established that became fundamental in the setting structured by the new Zermelian emphasis on well-foundedness (see 3.6).

3.3. First-order logic and extensionalization. The final structuring of set theory before it was to sail forth on its independent course as a distinctive field of mathematics was its full extensionalization in first-order logic.[66] However influential Zermelo's [1930] and despite his subsequent advocacy [1931, 1935] of infinitary logic, his efforts to forestall Skolem were not to succeed, as stronger currents were at work in the direction of first-order formalization.

Hilbert effected a basic shift in the development of mathematical logic when he took Whitehead and Russell's *Principia Mathematica*, viewed it as an uninterpreted formalism, and made it an object of mathematical inquiry. The book [1928][67] by Hilbert and Wilhelm Ackermann reads remarkably like a recent text. In marked contrast to the formidable works of Frege and Russell with their forbidding notation and all-inclusive approach, it proceeded pragmatically and upward to probe the extent of structure, making those moves emphasizing forms and axiomatics typical of modern mathematics. After a complete analysis of sentential logic it distinguished and focused on first-order logic ("functional calculus," and later "(restricted) predicate calculus") as already the source of significant problems. Thus, while Frege and Russell never separated out first-order logic, Hilbert through his mathematical investigations established it as a subject in its own right.

In response to intuitionistic criticism by Brouwer and Weyl, Hilbert in the 1920's developed proof theory, i.e. metamathematics, and proposed his program of establishing the consistency of classical mathematics. The issues here gained currency because of Hilbert's preeminence, just as mathematics in the large had been expanded in the earlier years of the century by his reliance on non-constructive proofs and transcendental methods and his advocacy of new contexts.[68] Through this expansion the full exercise of AC had become a mathematical necessity (cf., 2.6) and arbitrary functions, and so Power Set, had become implicitly accepted in the extensive investigation of higher function spaces.

Hilbert-Ackermann [1928, 65ff., 72ff.] raised two crucial questions directed at the further possibilities for first-order logic: the completeness of its axioms and the Decision Problem [Entscheidungsproblem]. These as well as Hilbert's program for securing consistency were to be decisively informed by penetrating work that for set theory eventually led to its first sophisticated metamathematical result, the relative consistency of AC and GCH.

Kurt Gödel (1906–1978), born when Zermelo was devising his proofs of the Well-Ordering Theorem, completed the mathematization of logic by submerging metamathematical methods into mathematics. The main source was of course the direct coding in his celebrated Incompleteness Theorem [1931], which led to the undecidability of the Decision Problem and the development of recursion theory. But starting an undercurrent, the earlier Completeness Theorem [1930] from his thesis answered Hilbert and Ackermann's question about completeness, clarified the distinction between the syntax and semantics of first-order logic, and secured its key instrumental property with the Compactness Theorem.

Tarski [1933, 1935] then gave his definition of truth, exercising philosophers to a surprising extent ever since. Through Hilbert-Ackermann [1928] and Gödel [1930] the satisfaction relation had been informal, and in that sense completeness could be said to have remained inadequately articulated. Tarski simply extensionalized truth in formal languages and provided a formal, *recursive* definition of the satisfaction relation in set-theoretic terms. This new response to a growing need for a mathematical framework became the basis for model theory, but thus cast into mathematics truth would leave behind any semantics in the real meaning of the word. Thus, the meaninglessness of the signs, often used pejoratively against Hilbertian formalism, would become the main *raison d'être* of model theory.

Tarski's [1933] was written around the same time as his [1931], a seminal paper that highlights the thrust of his initiative. Like Hilbert-Ackermann [1928] and Gödel [1931], Tarski [1931] streamlined the system of *Principia Mathematica*; he then gave a precise mathematical (that is, set-theoretic) formulation of the informal concept of a first-order definable set of reals, thus infusing the intuitive (or semantic) notion of definability into ongoing mathematics. This mathematization of intuitive or logical notions was accentuated by Kuratowski-Tarski [1931], where second-order quantificatin over the reals was correlated with the geometric operation of projection (cf., 2.5), beginning the process of explicitly wedding descriptive set theory to mathematical logic. The eventual effect of Tarski's [1933] mathematical formulation of so-called semantics would be not only to make mathematics within model theory out of the informal notion of satisfiability, but also to enrich ongoing mathematics with a systematic method for forming mathematical analogues of several intuitive semantical notions. (Incidentally, Tarski [1931] stated a result whose proof led to Tarski's well-known theorem [1951] that the elementary theory of real closed fields is decidable via the elimination of quantifiers.)

In this process of extensionalization first-order logic came to be accepted as the canonical language because of its mathematical possibilities as epitomized by the Compactness Theorem, and higher-order logics became downgraded as the workings of the power set operation in disguise. Skolem's early suggestion for set theory was thus taken up generally, and again the ways of paradox were positively subsumed, as the negative intent of the Skolem Paradox gave way to the extensive, internal use of Skolem functions from the Löwenheim-Skolem Theorem in set-theoretic constructions.

3.4. Relative consistency. So enriched and fortified by axioms, results, and techniques, axiomatic set theory was launched on its independent course by Gödel's construction of L [1938, 1939] leading to the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. Synthesizing all that came before, Gödel built on the von Neumann ordinals as sustained by Replacement to formulate a relative Zermelian universe of sets based on logical definability, a universe imbued with a Cantorian sense of order.

To summarize, for any set x let $\mathrm{def}(x)$ denote the collection of subsets of x definable over $\langle x, \in\rangle$ via a first-order formula allowing parameters from x. Now define

$$L_0 = \emptyset;\ L_{\alpha+1} = \mathrm{def}(L_\alpha);\ L_\delta = \bigcup_{\alpha<\delta} \text{ for limit ordinals } \delta\,;$$

and the *constructible universe*

$$L = \bigcup_\alpha L_\alpha\,.$$

The *Axiom of Constructibility* is $V = L$, i.e., $\forall x(x \in L)$. Why AC would hold in L is already apparent: L can be well-ordered by transfinite recursion, well ordering $L_{\alpha+1} - L_\alpha$ according to definitions and the previous well-ordering of the parameters from L_α.

Gödel's verification of GCH in L represents a steady intellectual development from his Incompleteness Theorem [1931] that was to extend to later speculations about large cardinals.[69] He [1938] regarded his constructible hierarchy as a transfinite completion of Russell's type hierarchy, one that "satisfies the impredicative axioms of set theory, because an axiom of reducibility can be proved for sufficiently high orders." Indeed, Gödel essentially established that if λ is an infinite cardinal and $x \in L_\lambda$, then for any $y \subseteq x$ in L, $y \in L_\lambda$. This he did in [1939] with the now well-known argument using Skolem functions (his term) and the crucial idea of taking a transitive collapse (cf., 3.6); the impredicative power set axiom was thus tamed in L leading to the consistency of GCH. But not only can this be viewed as a rectification of Russell's ill-fated Axiom of Reducibility, one provided by the *a priori* extent of the ordinals as given by Replacement, but it can also be viewed as the first instance of reflection (cf., below), later used for motivating large cardinals.[70]

Gödel apparently viewed L as an outright construction using transfinite reasoning in metamathematics.[71] There is an affinity of sorts with Zermelo [1930], and significantly, the main statement of consistency in Gödel [1939] appealed to what Zermelo had called "Grenzzahlen": *If* κ *is inaccessible, then* $L_\kappa \models \mathrm{ZFC} + \mathrm{GCH}$. To be sure, Zermelo did not formalize his logic while Gödel proceeded to transfinite systems in his investigation of the possibilities for formal systems. Indeed, L might be viewed as a synthesis of Zermelo [1930] and Skolem [1920, 1923] with the transfinite, well-founded cumulation of types refined by first-order definability and new isomorphism arguments involving Skolem functions.

Gödel [1939] described first-order definability and Skolem functions *ab initio* for his specific situation, and it is evident in succeeding arguments that he is taking $\mathrm{def}(x)$ to be a set. Yet only at the end does he note that L "can be defined and its theory developed in the formal systems of set theory themselves." In the monograph [1940] Gödel gave such a formal presentation of L. This time he generated L set by set with a transfinite recursion in terms of eight elementary set generators, a sort of Gödel numbering into the transfinite. These generators were based on Bernays's axiomatization [1937] which itself was based on von Neumann's axiomatization [1925, 1928]. Providing a rigorous formalization of his metamathematical construction Gödel now emphasized how it yields a class model and a finitary *relative* consistency result: *if* ZF is consistent, *then* so is ZF + AC + GCH. The new presentation highlighted the stark constrast between the elementary set formation processes and the extent of the ordinals, but at the cost of obscuring the cumulative definability and Skolem function argument.

Gödel's formalization not only recalled von Neumann's [1925, II] analysis of "subsystems," but also shed light on von Neumann's main concern: the categoricity of his axiomatization, or "whether it uniquely determines the system it describes." Fraenkel [1922] had expressed the desirability of closing off the Zermelian generative scheme by requiring through an "axiom of restriction" that there should be no further sets than those required by the axioms. It was to pursue this that von Neumann investigated subsystems for his axiomatization, but he concluded that there was probably no way to formally achieve Fraenkel's idea of a minimizing, and hence categorical, axiomatization. Gödel's axiom $V = L$ can be viewed as achieving this sense of categoricity since, as he essentially showed, the axiom uniquely specifies a well-founded proper class model up to isomorphism. However, Gödel quickly came to regard L as primarily a contrivance for establishing relative consistency results, much as von Neumann [1929, 236ff.] had viewed his formulation of the cumulative hierarchy in a wider setting. Nonetheless, the sort of fine analysis of Gödel's [1940] was itself to become a central concern of the mathematics of set theory.[72]

The synthesis at L extended to the resolution of difficulties in descriptive set theory (cf., end of 2.5). Gödel [1938] announced: *If $V = L$, then there is a set of reals both $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$ and not Lebesgue measurable, and a $\mathbf{\Pi}^1_1$ set of reals without the perfect set property.* Thus, the descriptive set theorists were confronting an obstacle insurmountable in ZFC! Gödel [1938] listed each of these impossibility results on an equal footing with his AC and GCH results. Unexpected, they were the first instances of metamathematical methods resolving outstanding mathematical problems that exhibited no prior connection to such methods. When eventually confirmed and refined, the results were seen to turn on a well-ordering of the reals in L defined via reals coding well-founded structures and thus connected to the well-founded tree representation of a $\mathbf{\Pi}^1_1$ set. Set theory had progressed to the point of establishing, in addition to a consistent resolution of CH, a consistent possibility for a definable well-ordering of the reals as Cantor had wanted, one that synthesizes the two historical sources of well-foundedness.

In later years Gödel speculated about the possibility of deciding propositions like CH with large cardinal hypotheses based on the heuristics of *reflection* and *generalization*. In 1946 remarks he (see Gödel [1990, 151]) suggested the consideration of "stronger and stronger axioms of infinity," and reflection as follows: "Any proof for a set-theoretic theorem in the next higher system above set theory (i.e., any proof involving the concept of truth ...) is replaceable by a proof from such an axiom of infinity." This ties in with the class of all ordinal numbers cast as Cantor's Absolute: A largeness property ascribable to the class might be used to derive some set-theoretic proposition; but any such property confronts the antithetical contention that the class is mathematically incomprehendable, fostering the synthetic move to a large cardinal posited with the property. For Gödel's [1939] proof of GCH in L using Skolem functions (see above), the property of being an infinite cardinal sufficed for a "strong axiom of infinity." In those 1946 remarks Gödel also formulated the concept of *ordinal definable* set, the formalization presumably to be based on a reflection argument akin to his GCH proof for L (cf., 3.6).

In an expository article [1947] on the Continuum Problem Gödel assumed that CH would be shown independent from ZF and speculated more concretely about possibilities with large cardinals. He advocated the Zermelian abstract view as refined by the iterative concept of set, and argued on this basis for new axioms that "assert the existence of still further iterations of the operations of 'set of'," citing Zermelo [1930] and echoing its theme. In an unpublished footnote 20 toward a 1966 revision of [1947] Gödel was to acknowledge ([1990, 260ff.]) "strong axioms of infinity of an entirely new kind," generalizations of properties of ω "supported by strong arguments from analogy." This heuristic of *generalization* ties in with Cantor's view of the finite and transfinite as unitary, with properties like inaccessibility and measurability technically satisfied by ω being too accidental were they not also ascribable to higher cardinals through the uniformity of the set-theoretic universe.[73] Gödel [1947] concluded by arguing against CH with six "highly implausible" consequences, one of them being the existence of a Luzin set (see 2.6) of cardinality of the continuum, and three others following from the existence of such a set.

3.5. Combinatorics. Gödel's construction of L was a culmination in all major respects of the early period in set theory. And for quite some time it was to remain an isolated monument in the axiomatic tradition. No doubt the intervening years of war were a prominent factor, but there was a continuing difficulty in handling definability within set theory and a stultifying lack of means for constructing models of set theory to settle issues of independence. It would take a new generation versed in emerging model-theoretic methods to set the stage for the next major methodological advance.

In the meanwhile, the direct investigation of the transfinite as extension of number was advanced, gingerly at first, by a new initiative. The seminal results of *combinatorics* were established beginning in the 1930's, and of course with the emergence of computer science combinatorics is now an enormous field of mathematics. As for algebra and topology it was natural to extend the concepts over the transfinite, and significantly, the combinatorics that would have the most bearing there had their roots in the mathematization of logic.

Frank Ramsey [1930] established a special case of the Hilbert-Ackermann [1928] Decision Problem, the decidability of validity for the $\exists\forall$ formulas with identity.[74] For this purpose he established a basic generalization of the pigeonhole principle. In a move that transcended purpose and context, he also established an infinite version implicitly applying the now familiar König's Lemma for trees. Stated more generally for graphs in Dénes König [1927, 121] the lemma had also figured implicitly in Löwenheim [1915].

The following establishes terminology for both Ramsey's results and König's Lemma for the transfinite context: For ordinals α, β, and δ and $n \in \omega$ the *partition property*

$$\beta \longrightarrow (\alpha)^n_\delta$$

is the assertion that for any partition of the n-element subsets of β into δ

cells there is an $H \subseteq \beta$ of order type α *homogeneous* for the partition, i.e., all the n-element subsets of H lie in the same cell.

Ramsey showed that for any k, n, and r all in ω, there is a $m \in \omega$ such that $m \longrightarrow (k)^n_r$. Skolem [1933] sharpened Ramsey's argument and thereby lowered the possibilities for the m's, but to this day the least such m's, the *Ramsey numbers*, have not been determined except in the simplest cases. Ramsey's infinite version is: $\omega \longrightarrow (\omega)^n_r$ for every $n, r \in \omega$. This partition property has been adapted to a variety of situations, and today *Ramsey theory* is a thriving field of combinatorics.[75]

A *tree* is a partially ordered set T such that the predecessors of any element are well-ordered. The *αth level* of T consists of those elements whose predecessors have order type α, and the *height* of T is the least α such that the αth level of T is empty. A *chain* of T is a linearly ordered subset, and an *antichain* is a subset consisting of pairwise incomparable elements. A *branch* of T is a maximal chain, and a *cofinal branch* of T is a branch with elements at every non-empty level of T. Finally, for a cardinal κ, a *κ-tree* is a tree of height κ each of whose levels has cardinality less than κ, and κ has the *tree property iff* every κ-tree has a cofinal branch.

Finite trees of course are quite basic to current graph theory and computer science. With infinite trees the concerns are rather different, typically involving cofinal branches. König's Lemma asserts that *ω has the tree property.* The first systematic study of infinite trees was carried out in Djuro Kurepa's thesis [1935]. Focusing on ω_1-trees Kurepa studied three kinds that together with their generalizations to higher cardinals became pivotal for relative consistency results and large cardinal hypotheses. An *Aronszajn tree* is an ω_1-tree without a cofinal branch, i.e., a counterexample to the tree property for ω_1. Kurepa (p. 96) gave Nachman Aronszajn's result that *there is an Aronszajn tree*. A *Suslin tree* is an ω_1-tree with the stronger property that there are no uncountable chains or antichains. Kurepa (p. 127ff.) reduced a hypothesis growing out of a problem of Suslin [1920] about the characterizability of the order type of the reals to a combinatorial property of ω_1 as follows: *Suslin's Hypothesis holds iff there are no Suslin trees.* Finally, a *Kurepa tree* is an ω_1-tree with at least ω_2 cofinal branches, and *Kurepa's Hypothesis* deriving from Kurepa [1942, 143] is the assertion that such trees exist. Much of this would be rediscovered, and both Suslin's Hypothesis and Kurepa's Hypothesis would be resolved three decades later with the advent of forcing.[76]

Paul Erdös, although an itinerant mathematician for most of his life, has been the prominent figure of a strong Hungarian tradition in combinatorics, and through some seminal results he introduced major initiatives into the detailed combinatorial study of the transfinite. Erdös and his collaborators simply viewed the transfinite numbers as a combinatorially rich source of intrinsically interesting problems, the concrete questions about graphs and mappings having a natural appeal through their immediacy. One of the earliest advances was Erdös-Tarski [1943] which concluded enticingly with an intriguing list of six combinatorial problems, the positive solution to any, as it turns out, amounting to the existence of a large cardinal. In contrast to Kurepa [1935] who had uncovered distinctive tree properties of ω_1, Erdös-Tarski [1943] was evidently motivated by strong properties of ω to formulate direct combinatorial generalizations to inaccessible cardinals by analogy. In a footnote various implications were noted, one of them being essentially that *for inaccessible κ, the tree property for κ implies $\kappa \longrightarrow (\kappa)^2_2$,* generalizing Ramsey's $\omega \longrightarrow (\omega)^2_2$ and making explicit the König Lemma property needed. The situation would be considerably clarified, but only two decades later.[77]

The detailed investigation of partition properties began in the 1950's, with Erdös and Richard Rado's [1956] being representative.[78] For a cardinal κ set $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. What became known as *the* Erdös-Rado Theorem asserts: *For any infinite cardinal κ and $n \in \omega$,*

$$\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa .$$

This was established using the basic tree argument underlying Ramsey's results, whereby a homogeneous set is not constructed recursively, but a tree is constructed such that its branches provide homogeneous sets, and a counting argument ensures that there must be a homogeneous set of sufficient cardinality. The $\beth_n(\kappa)^+$ was shown to be the least possible, and so unlike in the finite case an exact analysis was quickly achieved in the transfinite. This was to be a recurring phenomenon, that the gross features of transfinite cardinality make its combinatorics actually easier than in the analogous finite situation. And notably, iterated cardinal exponentiation figured prominently, so that shedding deeper concerns the power set operation became further domesticated in the arithmetic of combinatorics. In fact, assuming GCH simplifed results and formulations, and this was often done, as in Erdös, András Hajnal, and Rado's [1965], representative of the 1960's. Increasingly, a myriad of versions have been investigated in the larger terrain without GCH.[79]

Still among the Hungarians, Géza Fodor [1956] established the *regressive function lemma* for stationary sets.[80] It is a basic fact and a simple exercise now, but then it was the culmination of a progression of results beginning with a special case established by Aleksandrov and Paul Urysohn in their [1929]. The lemma is now seen as a synoptic formulation of the normality of the closed unbounded filter, and its use pervades set theory. The contrast with how the lemma's earlier precursors were considered difficult and even paradoxical is striking, indicative of both the novelty of uncountable cofinality and the great leap forward that set theory has made.

3.6. Model-theoretic methods. Model theory began in earnest with the method of diagrams[81] of Abraham Robinson's thesis [1951] and the related method of constants from Leon Henkin's thesis which gave a new proof [1949] of the Gödel Completeness Theorem. Tarski had set the stage with his definition of truth and more generally his casting of formal languages and structures in set-theoretic terms, and with him established at the University of California at Berkeley a large part of the development in the 1950's and 1960's would take place there. The construction of models freely used transfinite methods and soon led to new questions in set theory, but also set theory was to be decisively advanced by the infusion of model-theoretic methods.

The first relevant result was the generalization to well-founded relations of the Mirimanoff-von Neumann result, that every well-ordered set is order-isomorphic to exactly one ordinal with membership: A relation R is *extensional on X iff* for any $x \neq y$ both in X there is a $z \in X$ such that $\langle z, x\rangle \in R$ *iff* $\langle z, y\rangle \notin R$. A set x is *transitive iff* $\bigcup x \subseteq x$. *If X is a set and R is a well-founded relation extensional on X, there is a unique isomorphism of $\langle X, R\rangle$ onto a transitive set T with membership, i.e., a bijection $\pi\colon X \to T$ such that for any $x, y \in X$, $\langle x, y\rangle \in R$ iff $\pi(x) \in \pi(y)$.* T is the *transitive collapse* of X, and π the *collapsing isomorphism.* Thus, the linearity of well-orderings has been relaxed to an analogue of Extensionality, and with Foundation assumed the transitive sets become canonical representatives as ordinals are for well-orderings. Gödel [1939, 222] established this result *ab initio* for his application; Mostowski [1949, 147] in general terms much later; and John Shepherdson [1951, 171] in a structured setting that brought out a further necessary hypothesis for classes X: R is *set-like,* i.e., for any $x \in X$, $\{y \mid \langle y, x\rangle \in R\}$ is a set. The initial applications in Mostowski [1949] and Shepherdson [1953] were to establish the independence of the assertion that there is a transitive set M which with $\in$ restricted to it is a model of set theory. While the Mirimanoff-von Neumann result was basic to the analysis of number in the transfinite, the transitive collapse result grew in significance from specific applications and came to epitomize how well-foundedness made possible a coherent theory of models of set theory.

Shepherdson [1951, 1952, 1953] studied "inner" models of set theory, with [1952] giving a rigorous first-order account of the results of Zermelo [1930]. The term is nowadays reserved for a special case: A class M is an *inner model iff* it is transitive, contains all the ordinals, and with $\in$ restricted to it is a model of ZF. The archetypical inner model is Gödel's L, and $L \subseteq M$ for any inner model M since $L^M = L$, i.e., the construction of L carried out in M is again L. Because of this Shepherdson [1953] noted that the relative consistency of hypotheses like the negation of CH cannot be established via inner models.

However, Hajnal [1956, 1961] and Azriel Levy [1957, 1960] developed generalizations of L that were to become basic in a richer setting. For a set A, Hajnal formulated the *constructible closure $L(A)$ of A*, i.e., the smallest inner model M such that $A \in M$, and Levy formulated the *class $L[A]$ of sets constructible relative to A*, i.e., the smallest inner model M such that for every $x \in M$, $A \cap x \in M$.[82] $L(A)$ realizes the algebraic idea of building up a model starting from a set of generators, first foreshadowed by Fraenkel [1922a], and $L[A]$ realizes the idea of building up a model using A construed as a predicate. $L(A)$ may not satisfy AC since e.g., it may not have a well-ordering of A, yet $L[A]$ always satisfies that axiom. This distinction was only to surface later, as both Hajnal and Levy took A to be a set of ordinals, and then $L(A) = L[A]$. Hajnal and Levy (as well as Joseph Shoenfield [1959] who formulated a special version of Levy's construction) used these models to establish conditional independence results of the sort: if the failure of CH is consistent, then so is that failure together with $2^\lambda = \lambda^+$ for sufficiently large cardinals λ.

After Richard Montague [1956, 1961] applied reflection phenomena to investigate finite axiomatizability for set theory Levy [l960, 1960a, 1960b] also formulated reflection principles and established their broader significance for set theory. The *Reflection Principle* for ZF from Levy [1960], also figuring in Montague [1961], asserts: *For any formula $\varphi(v_1, \dots, v_n)$ and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \dots, x_n \in V_\alpha$,*

$$\varphi[x_1, \dots, x_n] \text{ iff } \varphi^{V_\alpha}[x_1, \dots, x_n],$$

i.e., the formula holds exactly when it holds with all the quantifiers restricted to V_α. Levy showed that this principle is equivalent to the conjunction of Replacement and Infinity. Moreover, he established results that in local form characterized cardinals in the Mahlo hierarchy (cf., 2.4), conceptually the least large cardinals after the inaccessible cardinals. The model-theoretic reflection idea thus provided a coherent scheme for viewing the bottom of the hierarchy of large cardinals as a generalization of Replacement and Infinity, one that resonates with the procession of models in Zermelo [1930] (see 3.2). The heuristic of reflection had been broached in 1946 remarks by Gödel (see 3.4); another point of contact is the formulation of the concept of *ordinal definable* sets in those remarks. The hereditarily ordinal definable sets form an inner model in which AC holds. The basic results about this inner model were to be rediscovered several times, and the formal definition turns on some form of the Reflection Principle for ZF (see Myhill-Scott [1971, 278]). In these several ways reflection phenomena both as heuristic and as principle became incorporated into set theory, bringing to the forefront what was to become a basic feature of the study of well-foundedness.

The set-theoretic generalization of first-order logic allowing transfinitely indexed logical operations was to lead to the solution of the problem of whether the smallest inaccessible cardinal can be measurable (cf., end of 3.2). Extending familiarity by abstracting to a new domain Tarski [1962] defined the *strongly compact* and *weakly compact* cardinals by ascribing natural generalizations of the key compactness property of first-order logic to the corresponding infinitary languages. These cardinals had figured in Erdös-Tarski [1943] (cf., 3.5) in combinatorial formulations that was later seen to imply that *a strongly compact cardinal is measurable,* and *a measurable cardinal is weakly compact.* Tarski's student William Hanf [1964] then established, using the satisfaction relation for infinitary languages, that *there are many inaccessible cardinals (and Mahlo cardinals) below a weakly compact cardinal. A fortiori,* (Tarski [1962]) *the smallest inaccessible cardinal is not measurable.* This breakthrough was the first result about the size of measurable cardinals since Ulam's original paper [1930] and was greeted as a spectacular success for metamathematical methods. Hanf's work radically altered size intuitions about problems coming to be understood in terms of large cardinals and ushered in model-theoretic methods into the study of large cardinals beyond the Mahlo cardinals.

Weak compactness was soon seen to have a variety of combinatorial (cf., 3.5) characterizations, most notably κ is weakly compact *iff* $\kappa \rightarrow (\kappa)^2_2$ *iff* $\kappa \rightarrow (\kappa)^n_\lambda$ for every $n \in \omega$ and $\lambda < \kappa$ *iff* κ is inaccessible and has the tree property. Erdös-Hajnal [1962] noted that the study of stronger partition properties had progressed to the point where a combinatorial proof that the smallest inaccessible cardinal is not measurable could have been given before Hanf. However, model-theoretic methods quickly led to far stronger conclusions, particularly through the connection that had been made in Ehrenfeucht-Mostowski [1956] between partition properties and *sets of indiscernibles*.[83]

The concurrent emergence of the *ultraproduct construction* in model theory set the stage for the development of the modern theory of large cardinals. With a precursor in Skolem's [1933a, 1934] construction of a non-standard model of arithmetic the ultraproduct construction was brought to the forefront by Tarski and his students after Jerzy Łoś's [1955] adumbration of its fundamental theorem. This new method of constructing concrete models brought set theory and model theory even closer together in a surge of results and a lasting interest in ultrafilters. Measurable cardinals had been formulated (see 3.2) in terms of ultrafilters construed as two-valued measures; Jerome Keisler [1962] struck on the idea of taking the ultrapower of a measurable cardinal κ by a κ-complete ultrafilter over κ to give a new proof of Hanf's result, seeing the crucial point that the completeness property led to a well-founded, and so in his case well-ordered, structure.

Then Dana Scott [1961] made the further, crucial move of taking the ultrapower of the universe V itself by such an ultrafilter. The full exercise of the transitive collapse as a generalization of the correlation of ordinals to well-ordered sets now led to an inner model M and an elementary embedding $j: V \to M$. With this Scott established[84]: *If there is a measurable cardinal, then* $V \neq L$. Large cardinal hypotheses thus assumed a new significance as a means for maximizing possibilities away from Gödel's delimitative construction. Also, the Cantor-Gödel realist view of a fixed set-theoretic universe notwithstanding, Scott's construction fostered the manipulative use of inner models in set theory. The construction provided one direction and Keisler [1962a] the other of a new characterization that established a central structural role for measurable cardinals: *There is an non-identical elementary embedding* $j: V \to M$ *for some inner model* M *iff there is a measurable cardinal.* This result is not formalizable in ZFC because of the use of the satisfaction relation and the existential assertion of a proper class, but technical versions are. Despite the lack of formalizability such existential assertions have been widely entertained since, and with this set theory in practice could be said to have overleaped the bounds of ZFC. On the other hand, that the existence of a class elementary embedding is equivalent to the existence of a certain set, the witnessing ultrafilter for a measurable cardinal, can be considered a means of formalization in ZFC, one that would be paradigmatic for such reductions. Through model-theoretic methods set theory was brought to the point of entertaining elementary embeddings into well-founded models,[85] soon to be transfigured by a new method for getting well-founded *extensions* of well-founded models.

§4. Independence.

4.1. Forcing. Paul Cohen (1934–), born just before Gödel established his relative consistency results, established the independence of AC from ZF and the independence of CH from ZFC [1963, 1964]. These were, of course, the inaugural examples of *forcing*, soon to become a remarkably general and flexible method with strong intuitive underpinnings for extending models of set theory.[86] If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's technique of forcing began its transformation into a modern one. According to Scott (Bell [1985, ix]): "Set theory could never be the same after Cohen, and there is simply no comparison whatsoever in the sophistication of our knowledge about models of set theory today as contrasted to the pre-Cohen era." In retrospect one can point to various precursors of forcing, but Cohen was unaware of these and started with simple and basic intuitions.

Cohen's approach was to start with a model M of ZF and adjoin a set G, one that would exhibit some desired new property. He realized that this had to be done in a minimal fashion in order that the resulting structure also model ZF, and so imposed restrictive conditions on both M and G.

He took M to be a countable standard model, i.e., a countable transitive set that together with the membership relation restricted to it is a model of ZF. (If ZF is consistent, then the assumption that such a model exists cannot be established in ZF by Gödel's (Second) Incompleteness Theorem. However, it is a convenient assumption for the presentation of forcing, and can be avoided in formal relative consistency proofs via the method.). The ordinals of M would then coincide with the predecessors of some ordinal ρ, and M would be the cumulative hierarchy $M = \bigcup_{\alpha<\rho} V_\alpha \cap M$. Cohen then established a system of terms to denote members of the new model, finding it convenient to use a ramified language: For each $x \in M$ let $\dot{x}$ be a corresponding constant; let $\dot{G}$ be a new constant; and for each $\alpha < \rho$ introduce quantifiers $\forall_\alpha$ and $\exists_\alpha$. Then develop a hierarchy of terms as follows: $\dot{M}_0 = \{\dot{G}\}$, and for limit ordinals $\delta < \rho$, $\dot{M}_\delta = \bigcup_{\alpha<\delta} \dot{M}_\alpha$. At the successor stage, let $\dot{M}_{\alpha+1}$ be the collection of terms $\dot{x}$ for $x \in V_\alpha \cap M$ and "abstraction" terms corresponding to formulas allowing parameters from $\dot{M}_\alpha$ and quantifiers $\forall_\alpha$ and $\exists_\alpha$. It is crucial that this ramified language with abstraction terms is entirely formalizable in M, through a systematic coding of symbols. Once a set G is provided from the outside, a model $M[G] = \bigcup_{\alpha<\rho} M_\alpha[G]$ would be determined by the terms, where each $\dot{x}$ is to be interpreted by x for $x \in M$ and $\dot{G}$ is to be interpreted by G, so that: $M_0[G] = \{G\}$; for limit ordinals $\delta < \rho$, $M_\delta[G] = \bigcup_{\alpha<\delta} M_\alpha[G]$; and $M_{\alpha+1}[G]$ consists of the sets in $V_\alpha \cap M$ together with sets interpreting the abtraction terms as the corresponding definable subsets of $M_\alpha[G]$ with $\forall_\alpha$ and $\exists_\alpha$ ranging over this domain.

But what properties can be imposed on G to ensure that $M[G]$ is a model of ZF? Cohen's key idea was to tie G closely to M through a system of sets in M called *conditions* that would approximate G. While G may not be a member of M, G is to be a subset of some $Y \in M$ (with $Y = \omega$ a basic case), and these conditions would "force" some assertions about the eventual $M[G]$ e.g., by deciding some of the membership questions, whether $x \in G$ or not, for $x \in Y$. The assertions are to be just those expressible in the ramified language, and Cohen developed a corresponding *forcing relation* $p \Vdash \varphi$, "p forces φ", between conditions p and formulas φ, a relation with properties reflecting his approximation idea. For example, if $p \Vdash \varphi$ and $p \Vdash \psi$, then $p \Vdash \varphi \;\&\; \psi$. The conditions are ordered according to the constraints they impose on the eventual G, so that if $p \Vdash \varphi$, and q is a stronger condition, then $q \Vdash \varphi$. Scott actually provided the now common forcing symbol $\Vdash$, and he also made an important simplification by suggesting the definition for negation: $p \Vdash \neg\varphi$ *iff* for no stronger condition q does $q \Vdash \varphi$. It was crucial to Cohen's approach that the forcing relation, like the ramified language, be definable in M.

The final ingredient is that the whole scaffolding is given life by incorporating a certain kind of set G. Stepping out of M and making the only use of its countability, Cohen enumerated the formulas of the ramified language in a countable sequence and required that G be completely determined by a countable sequence of stronger and stronger conditions $p_0, p_1, p_2, \ldots$ such that for every formula φ of the ramified language exactly one of φ or $\neg\varphi$ is forced by some p_n. Such a G is called a *generic* set. Cohen was able to show that the resulting $M[G]$ does indeed satisfy the axioms of ZF: Every assertion about $M[G]$ is already forced by some condition; the forcing relation is definable in M; and so the ZF axioms, holding in M, can be applied to derive corresponding forcing assertions about ZF axioms holding in $M[G]$.

The appeal to a countable model is a notable positive subsumption of the Skolem Paradox (see 3.2) into a new method. Remarkably, Skolem [1923, 229] had entertained the possibility of adjoining a new subset of the natural numbers to a countable model of Zermelo's system, and getting a new model, adding in a footnote that "it is quite probable" that the Continuum Hypothesis is not decided by Zermelo's axioms. Just as starting with a countable standard model is not formally necessary for relative consistency results, other features of Cohen's argument would soon be reformulated, reorganized, and generalized, but the main thrust of his constructive approach through definability and genericity would remain. His particular achievement lies in devising a concrete procedure for extending well-founded models of set theory in a minimal fashion to well-founded models of set theory with new properties but without altering the ordinals.[87] Set theory had undergone a sea-change, and beyond how the subject was enriched, it is difficult to convey the strangeness of it.

The creation of forcing is a singular phenomenon in the development of set theory not only since it raised the level of the subject dramatically but also since it could well have occurred decades earlier. But however epochal Cohen's advance there was a line of development for which it did provide at least a semblance of continuity: Interest in independence results for weak versions of AC had been on the rise from the mid-1950's, with more and more sophisticated Fraenkel-Mostowski models being constructed.[88] Solomon Feferman, who had associated with Cohen for several years, was the first after him to establish results by forcing; Levy soon followed; and among their first results were new independences from ZF for weak versions of AC (Feferman-Levy [1963], Feferman [1965]). Cohen [1965, 40] moreover acknowledged the similarities between his AC independence result and the previous Fraenkel-Mostowski models. In fact, consistencies first established via Fraenkel-Mostowski models were soon converted to those from ZF via forcing by correlating urelements with generic sets.[89]

After an initial result by Feferman [1963], Levy [1963, 1965, 1970] also probed the limits of ZFC definability, establishing consistency results about definable sets of reals and well-orderings and in descriptive set theory. Intriguingly, inaccessible cardinals were brought in to overcome a technical hurdle in this study; Levy [1963, IV] applied the defining properties of such a cardinal to devise its "collapse" to $\aleph_1$ by making every smaller ordinal countable, and this forcing is now known as the *Levy collapse*. Unlike earlier appeals to inaccessible cardinals (e.g., in Gödel [1939]), their use in the Levy collapse turned out to be essential in several cases (see below).

Forcing was quickly generalized and applied to achieve wide-ranging results, particularly by Robert Solovay. He above all epitomized this period of great expansion in set theory with his mathematical sophistication and fundamental results about and with forcing, and in the areas of large cardinals and descriptive set theory. Just weeks after Cohen's breakthrough Solovay [1963, 1965] elaborated the independence of CH by characterizing the possibilities for the size of 2^κ for regular κ and made the first exploration of a spectrum of cardinals. Then William Easton [1964, 1970] established the definitive result for powers of regular cardinals: *Suppose that* GCH *holds and F is a class function from the class of regular cardinals to cardinals such that for $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$ and the cofinality of $F(\kappa)$ is greater than κ. Then there is a forcing extension preserving cofinalities in which $2^\kappa = F(\kappa)$ for every regular κ.* Thus, the *only* restriction beyond monotonicity on the power function for regular cardinals is that given by König's inequality, as Solovay had seen locally. Easton's result vitally infused forcing not only with the introduction of proper classes of forcing conditions but the now basic idea of a product analysis and the now familiar concept of *Easton support*. Through its reduction Easton's result focused interest on the possibilities for powers of *singular* cardinals, and this *Singular Cardinals Problem* together with the *Singular Cardinals Hypothesis* would stimulate the further development of set theory much as the Continuum Problem and the Continuum Hypothesis had stimulated its early development.[90]

Just over a year after Cohen's breakthrough Solovay [1965b, 1970] established a result remarkable for its mathematical depth and revelatory of what standard of argument was possible with forcing: *If there is an inaccessible cardinal, then in an inner model of a forcing extension every set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.* (AC necessarily fails in this inner model, but the weaker Principle of Dependent Choices holds in it, sufficient to bolster it as a *bona fide* one for measure and category.) Solovay applied the Levy collapse and built on its definability properties as first exploited by Levy [1963, IV]; for the Lebesgue measurability he introduced a new kind of forcing beyond Cohen's direct ways of adjoining new sets of ordinals and collapsing cardinals, that of adding a *random real*. Solovay's work not only opened the door to a wealth of different forcing arguments, but to this day his original definability arguments remain vital in descriptive set theory.

The perfect set property, central to Cantor's direct approach to the Continuum Problem through definability (1.2, 2.3, 2.5), led to the first acknowledged instance of a new phenomenon in set theory: the derivation of *equiconsistency* results based on the complementary methods of forcing and inner models. A large cardinal hypothesis is typically transformed into a proposition about sets of reals by forcing that "collapses" that cardinal to $\aleph_1$ or "enlarges" the power of the continuum to that cardinal. Conversely, the proposition entails the same large cardinal hypothesis in the clarity of an inner model. Solovay's result provided the forcing direction from an inaccessible cardinal to the proposition that every set of reals has the perfect set property (and $\aleph_1$ is regular). But Ernst Specker [1957, 210] had in effect established that if this obtains, then $\aleph_1$ (of V) is inaccessible in L. Thus, Solovay's use of an inaccessible cardinal was actually necessary, and its collapse to $\aleph_1$ complemented Specker's observation. Other propositions figuring in the initial applications of inaccessibility to forcing turned out to require inaccessibility, further integrating it into the interstices of set theory.[91]

The emergence of such equiconsistency results is a subtle transformation of earlier hopes of Gödel (see 3.4): Propositions can be positively subsumed if there are enough ordinals, how many being specified by positing a large cardinal.[92] Forcing quickly led to the conclusion that there could be no direct implication for CH: Levy and Solovay (Levy [1964], Solovay [1965a], Levy-Solovay [1967]) established that measurable cardinals neither imply nor refute CH, with an argument generalizable to most inaccessible large cardinals. Rather, the subsumption for many other propositions would be in terms of the Hilbertian concept of consistency, the methods of forcing and inner models being the operative modes of argument. In a new synthesis of the two Cantorian legacies, hypotheses of length concerning the extent of the transfinite are correlated with hypotheses of width concerning sets of reals.

It was the incisive work of Scott and Solovay through this early period that turned Cohen's breakthrough into a general method of wide applicability. Scott simplified Cohen's original formulation as noted above; Solovay made the important move to general partial orders and generic filters; and they together developed, with vicissitudes, the formulation in terms of Boolean-valued models.[93] These models forcibly showed how to avoid Cohen's ramified language as well as his dependence on a countable model. With their elegant algebraic trappings and seemingly more complete information they held the promise of being the right approach to independence results. But Shoenfield [1971] showed that forcing with partial orders can get at the gist of the Boolean approach in a straightforward manner. Moreover, Boolean-valued models were soon found to be too abstract and unintuitive for establishing *new* consistency results, so that within a few years set theorists were generally working with partial orders. It is a testament to Cohen's concrete approach that in this return from abstraction even the use of ramified languages has played an essential role in careful forcing arguments at the interface of recursion theory and set theory.

4.2. Envoi. Building on his Lebesgue measurability result Solovay soon reactivated the classical descriptive set theory program (see 2.5) of investigating the extent of the regularity properties by providing characterizations for the $\mathbf{\Sigma}^1_2$ sets, the level at which Gödel established from $V = L$ the failure of the properties (see 3.4), and showed in particular that the regularity properties for these sets follow from the existence of a measurable cardinal. Thus, although measurable cardinals do not decide CH, they do establish the perfect set property for $\mathbf{\Sigma}^1_2$ sets (Solovay [1969]) so that "CH holds for the $\mathbf{\Sigma}^1_2$ sets"—a vindication of Gödel's hopes for large cardinals through a direct implication. Donald Martin and Solovay in their [1969] then applied large cardinal hypotheses weaker than measurability to push forward the old tree representation ideas of the classical descriptive set theorists (see 2.5), with the hypotheses cast in the new role of securing well-foundedness in this context.[94]

The method of forcing as part of the axiomatic tradition together with the transmutations of Cantor's two legacies, large cardinals furthering the extension of number into the transfinite and descriptive set theory investigating definable sets of reals, established set theory as a sophisticated field of mathematics, a study of well-foundedness expanded into one of consistency strength. With the further development of forcing through increasingly sophisticated iteration techniques questions raised in combinatorics and over a broad landscape would be resolved in terms of consistency, sometimes with equiconsistencies in terms of large cardinals. The theory of large cardinals would itself be much advanced with the heuristics of reflection and generalization and sustained through increasing use in the study of consistency strength. In the most distinctive and intriguing development of comtemporary set theory, the investigation of the determinacy of games, large cardinals would be further integrated into descriptive set theory. They were not only used to literally incorporate the well-foundedness of inner models into the study of tree representations, the historically first context involving well-foundedness, but also to provide the exact hypotheses, with Woodin cardinals, for gauging consistency strength.[95]

The thrust of mathematical research would gradually deflate the early metaphysical views and initiatives with an onrush of new models, hypotheses, and propositions, and shedding much of its foundational burden set theory would become an intriguing field of mathematics where formalized versions of truth and consistency became matters for manipulation as in algebra. From Skolem relativism to Cohen relativism the role of set theory for mathematics would become even more evidently one of an open-ended framework rather than an elucidating foundation. It is as a field *of* mathematics that both proceeds with its own internal questions and is capable of contextualizing over a broad range that set theory would become an intriguing and highly distinctive subject.

From this point of view, that the ZFC axioms do not determine the cardinality $2^{\aleph_0}$ of the reals seems an entirely satisfactory state of affairs.[96] With the richness of possibility for arbitrary reals and mappings, no axioms that do not directly impose structure from above should constrain a set as open-ended as the reals or its various possibilities for well-ordering. Decades after Georg Kreisel's [1971, 195ff.] railings against the analogy the situation today with CH is actually quite like the situation with the parallel axiom and non-Euclidean geometries.[97] The historical progression may detract somewhat from the analogy, but the cornucopia of models of set theory and of hypotheses and propositions has achieved both a stability of intrinsic mathematical interest as well as a sense of steady mathematical progress.

Notes

1. The history may be continued to the present in a subsequent article. Van Dalen in van-Dalen-Monna [1972] also gives a history of set theory from Cantor to Cohen. See the texts Jech [1978] or Kunen [1980] for basic set-theoretic terminology or unelaborated results.

2. Dauben [1979], Meschkowski [1983], and Purkert-Ilgauds [1987] are mathematical biographies of Cantor.

3. See Kechris-Louveau [1987] for recent developments in the Cantorian spirit about uniqueness for trigonometric series converging on definable sets of reals.

4. Correspondence from 1872 through 1882 has been published in Noether-Cavaillès [1937], and that of 1899 in Cantor [1932]. Both sets have been translated into French in Cavaillès [1962], and the main 1899 letter into English in van Heijenoort [1967, 113ff.]. Grattan-Guinness [1974] describes the rediscovery of this correspondence and raises textual issues about the 1899 letters. The most complete edition of Cantor's correspondence is Meschkowski-Nilson [1991].

5. Dedekind [1872] dated his conception of cuts to 1858, and antecedents to ideals in his work also occurred then. For Dedekind and the foundation

of mathematics see Dugac [1976], who accords him a crucial role in the development of the set-theoretic framework.

6. The exact date of birth can be ascertained as December 7, “a day of infamy” of course for other reasons. Cantor first gave a proof of the uncountability of the reals in a letter to Dedekind dated December 7, 1873, professing that “ ... it is only today that I have finished, it seems to me, with this business ... ,” and Dedekind’s notes reinforce this date. (See Noether-Cavaillès [1937, 14] or Cavaillès [1962, 189].)

7. Dauben [1979, 68ff.] suggests that the title and presentation of Cantor [1874] were deliberately chosen to avoid censure by Kronecker, one of the journal editors.

8. Kac-Ulam [1968, 13] wrote: “The contrast between the methods of Liouville and Cantor is striking, and these methods provide excellent illustrations of two vastly different approaches toward proving the *existence* of mathematical objects. Liouville’s is purely *constructive*; Cantor’s is purely *existential*.” See also Moore [1982, 39]. One exception to the misleading trend is Fraenkel [1930, 237][1953, 75], who from the beginning emphasized the constructive aspect of diagonalization.

9. Gray [1994] shows that Cantor’s original [1874] argument can be implemented by an algorithm that generates n digits of a transcendental number with time complexity $O(2^{n^{1/3}})$, and his later diagonal argument, with a tractable algorithm of complexity $O(n^2 \log^2 n \log\log n)$.

10. See Oxtoby [1971, §2]. On the other hand, Gray [1994] shows that every transcendental real is the result of diagonalization applied to *some* enumeration of the algebraic reals.

11. Cantor developed a bijective correspondence between $\mathbb{R}^2$ and $\mathbb{R}$ by essentially interweaving the decimal expansions of a pair of reals to define the associated real, taking care of the countably many exceptional points like $.100\ldots = .099\ldots$ by an *ad hoc* shuffling procedure. Such an argument now seems straightforward, but to have bijectively identified the plane with the line was a stunning accomplishment at the time. Cantor wrote to Dedekind (June 29, 1877; see Noether-Cavaillès [1937, 34] or Cavaillès [1962, 211]), in French in the text, “I see it, but I don’t believe it.”

Cantor’s work inspired a push to establish the “invariance of dimension,” that there can be no *continuous* bijection of any $\mathbb{R}^n$ onto $\mathbb{R}^m$ for $m < n$, with Cantor [1879] himself providing an argument. As topology developed, the stress brought on by the lack of firm ground led Brouwer [1911] to definitively establish the invariance of dimension in a seminal paper for algebraic topology.

12. This is emphasized by Hallett [1984] as Cantor’s “finitism.”

13. After describing the similarity between ω and $\sqrt{2}$ as limits of sequences, Cantor [1887, 99] interestingly correlated the creation of the transfinite numbers to the creation of the irrational numbers, beyond merely breaking new ground in different number contexts: "The transfinite numbers are in a certain sense *new irrationalities*, and in my opinion the best method of defining the *finite* irrational numbers [via Cantor's fundamental sequences] is wholly similar to, and I might even say in principle the same as, my method of introducing transfinite numbers. One can say unconditionally: the transfinite numbers *stand or fall* with the finite irrational numbers: they are like each other in their innermost being [Wesen]; for the former like the latter are definite delimited forms or modifications of the actual infinite."

14. Ferreirós [1995] suggests how the formulation of the second number class as a completed totality with a succeeding ordinal number emerged directly from Cantor's work on the operation P', drawing Cantor's ordinal numbers even closer to his earlier work on trigonometric series.

15. Moreover, diagonalization as such had already occurred in Paul du Bois-Reymond's theory of growth as early as in his [1869]. An argument is manifest in his [1875, 365ff.] for showing that for any sequence of real functions $f_0, f_1, f_2, \dots$ there is a real function g such that for each n, $f_n(x) < g(x)$ for all sufficiently large reals x.

16. The "power" in "power set" is from "Potenz" in the German for cardinal exponentiation, while Cantor's "power" is from "Mächtigkeit".

17. Remarkably, Cantor had already conjectured in the *Grundlagen* [1883, 590] that the collection of continuous real functions has the same power as the second number class (II), and that the collection of all real functions has the same power as the third number class (III). These are consequences of GCH and are indicative of the sweep of Cantor's conception.

18. This is emphasized by Lavine [1994]. A suggestion of the emerging tension is an assertion at the end of Cantor [1891]: "I have already shown by entirely other means in my *Grundlagen* ... that the powers have no maximum." He had not; even if powers were to be identified with his number classes, he had only established the special case that the second number class is of a higher power than the first number class.

19. Cantor wrote [1895, 486]: " ... by a '*covering* [Belegung] of N with M,' we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application. The element of M bound up with n is, in a way, a one-valued function of n, and may be denoted by $f(n)$; it is called a 'covering function [Belegungsfunktion] of n.' The corresponding covering of N will be called $f(N)$." A convoluted description! Arbitrary functions on arbitrary domains are now of course commonplace in mathematics, but

several authors at the time referred specifically to Cantor's concept of covering, most notably Zermelo [1904]. Jourdain in his introduction to his English translation [1915, 82] of the *Beiträge* wrote: "The introduction of the concept of 'covering' is the most striking advance in the principles of the theory of transfinite numbers from 1885 to 1895 . . . ."

With Cantor initially focusing on bijective correspondence [Beziehung] and these not quite construed as functions, Dedekind was the first to entertain an arbitrary function on an arbitrary domain. He [1888, §§21, 36] formulated $\phi\colon S \to Z$, "a mapping [Abbildung] of a system S in Z", in less convoluted terms, but did not consider the totality of such. He quickly moved to the case $Z = S$ for his theory of chains; see note 29.

20. Cantor did state $2^{\aleph_0} = \aleph_1$ in an 1895 letter; see Moore [1989, 99]. Also, Cantor did define the first several alephs and wrote "$\aleph_0, \aleph_1, \ldots, \aleph_{\omega_0}, \aleph_{\omega_0+1}, \ldots, \aleph_{\omega_1}, \ldots$" in 1899 correspondence with Dedekind, discussed in 2.2; see Cantor [1932, 446] or van Heijenoort [1967, 116].

21. See Dauben [1979, 194].

22. The *cofinality* of an ordinal number α is the least ordinal number β such that there is a set of the form $\{\gamma_\xi \mid \xi < \beta\} \subseteq \alpha$ unbounded in α, i.e, for any $\eta < \alpha$, there is a $\xi < \beta$ such that $\eta \leq \gamma_\xi$. An ordinal α is *regular* if its cofinality is itself, and otherwise α is *singular*. These concepts were not clarified until the work of Hausdorff, brought together in his [1908], discussed in 2.6.

König applied an equality, $\aleph_\alpha^{\aleph_0} = \aleph_\alpha \cdot 2^{\aleph_0}$, from Bernstein's 1901 Halle thesis as follows: If $2^{\aleph_0}$ were an aleph, say $\aleph_\beta$, then by Bernstein's equality $\aleph_{\beta+\omega}^{\aleph_0} = \aleph_{\beta+\omega} \cdot 2^{\aleph_0} = \aleph_{\beta+\omega}$, contradicting König's inequality. However, Bernstein's equality fails just in such cases, when α has cofinality ω and $2^{\aleph_0} < \aleph_\alpha$. König's published account [1905] acknowledged the gap. See Dauben [1979, 247ff.] or Moore [1982, 86ff.] for descriptions of this episode.

23. Pinl [1969, 221ff.] contains a biographical note on Zermelo. Peckhaus [1990] provides a biographical account of Zermelo's years 1897–1910 at Göttingen.

24. Cf., Note 19.

25. Zermelo himself stressed the importance of simultaneous choices over successive choices in criticism of an argument of Cantor's for the Well-Ordering Theorem in 1899 correspondence with Dedekind, discussed in 2.2. See Cantor [1932, 451] or van Heijenoort [1967, 117].

26. See Moore [1982, Chapter 2].

27. Moore [1982, 155ff.] supports this contention using items from Zermelo's *Nachlass*.

28. Russell [1906] had previously arrived at this form, his Multiplicative Axiom. The elimination of the "pairwise disjoint" by going to a choice

function formulation can be established with the Union Axiom, and this is the only use of that axiom in the Well-Ordering Theorem proof.

29. In current terminology, Dedekind [1888] considered arbitrary sets S and mappings $\phi\colon S \to S$ and defined a *chain* [Kette] to be a $K \subseteq S$ such that $\phi``K \subseteq K$. For $A \subseteq S$, the *chain of* $A$ is the intersection of all chains $K \supseteq A$. A set $N$ is *simply infinite iff* there is an injective $\phi\colon N \to N$ such that $N - \phi``N \neq \emptyset$. Letting 1 be a distinguished element of $N - \phi``N \neq \emptyset$ Dedekind considered the chain of $\{1\}$, the chain of $\{\phi(1)\}$, and so forth. Having stated an inherent induction principle, he proceeded to show that these sets have all the ordering and arithmetical properties of the natural numbers (that are established nowadays in texts for the (von Neumann) finite ordinals).

30. Dedekind [1888, §2] begins a footnote to his statement about extensional determination with: "In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances."

31. Cf., the first sentence of the preface to Dedekind [1888]: "In science nothing capable of proof ought to be accepted without proof."

32. Some notable examples: Ernst Lindelöf [1905] proved the Cantor-Bendixson result, that every uncountable closed set is the union of a perfect set and a countable set, without using transfinite numbers. Mikhail Suslin's [1917], discussed in 2.5, had the unassuming title, "On a definition of the Borel sets without transfinite numbers," hardly indicative of its results, so fundamental for descriptive set theory. And Kazimierz Kuratowski [1922] showed, pursuing the approach of Zermelo [1908], that reverse inclusion chains defined via transfinite recursion with intersections taken at limits can also be defined without transfinite numbers. Kuratowski [1922] essentially formulated Zorn's Lemma, and this was the main success of the push away from explicit well-orderings. Especially after the appearance of Zorn [1935] this recasting of AC came to dominate in algebra and topology.

33. Grattan-Guinness [1978], Coffa [1979], Moore [1988], and Garciadiego [1990] describe the evolution of Russell's Paradox.

34. Moore-Garciadiego [1981] and Garciadiego [1990] describe the evolution of the Burali-Forti Paradox.

35. See the exchange of letters between Russell and Frege in van Heijenoort [1967, 124ff.].

36. The 1899 correspondnece appeared in Cantor [1932] and, translated into French, in Cavaillès [1962]. The main letter is translated into English in van Heijenoort [1967, 113ff.]. Purkert [1989, 57ff.] argues that Cantor had already arrived at the Burali-Forti Paradox around the time of the *Grundlagen* [1883]. On the interpretations supported in the text *all* of the logical

paradoxes grew out of Cantor's work—with Russell shifting the weight to paradox.

37. See Zermelo [1908, Footnote 9]. Rang-Thomas [1981] describes Zermelo's discovery of Russell's Paradox.

38. In 2.6 Hartogs's Theorem is construed as a positive subsumption of the Burali-Forti Paradox.

39. See Kanamori [1995] for more on the emergence of descriptive set theory. See Moschovakis [1980] or Kanamori [1994] for the mathematical development.

40. Baire mainly studied the finite levels, particularly the classes 1 and 2. He [1898] pointed out that Dirichlet's function that assigns 1 to rationals and 0 to irrationals is in class 2 and also observed with a non-constructive appeal to Cantor's cardinality argument that there are real functions that are not Baire.

41. See Hawkins [1975] for more on the development of Lebesgue measurability. See Oxtoby [1971] for an account of category and measure in juxtaposition.

42. See Moore [1982, 2.3].

43. See Kanamori [1994, ∞] for more on large cardinals.

44. Hausdorff's mathematical attitude is reflected in a remark following his explanation of cardinal number in a revised edition [1937, §5] of [1914]: "This formal explanation says what the cardinal numbers are supposed to do, not what they are. More precise definitions have been attempted, but they are unsatisfactory and unnecessary. Relations between cardinal numbers are merely a more convenient way of expressing relations between sets; we must leave the determination of the 'essence' of the cardinal number to philosophy."

45. Hausdorff's Maximality Principle states that if A is a partially ordered set and B is a linearly ordered subset, then there is a $\subseteq$-maximal linearly ordered subset of A including B.

46. Hausdorff's Paradox states that a sphere can be decomposed into four pieces Q, A, B, C with Q countable and A, B, C, and $B \cup C$ all pairwise congruent. Even more implausibly, the Banach-Tarski Paradox states that a ball can be decomposed into finitely many pieces that can be rearranged by rigid motions to form two balls of the same size as the original ball. Raphael Robinson [1947] later showed that there is such a decomposition into just five pieces with one of them containing a single point, and moreover that five is the minimal number. See Wagon [1985] for more on these and similar results; they stimulated interesting developments in measure theory that, rather than casting doubt on AC, embedded it further into mathematical practice (cf., 2.6).

47. Frege [1893, §144] used his ordered pair $x; y$ to establish correspondence [Beziehung] between countable sequences toward characterizing their cardinal number; unfortunately, he defined his ordered pair using his famously inconsistent Basic Law V. Peirce [1883] used i and j schematically to denote the components, and Schröder [1895, 24] adopted this and also introduced $i : j$. Peano [1897, 579] introduced (x, y) at the outset and regarded it as fundamental, switching to $x; y$ in later writings.

48. Whitehead and Russell had first defined a cartesian product by other means, and only then defined their ordered pair $x \downarrow y$ as $\{x\} \times \{y\}$, a remarkable inversion from the current point of view. They [1910, $*56$] used their ordered pair initially to define the ordinal number 2.

49. Russell [1903, §98] argued that the ordered pair cannot be basic and would itself have to be given sense, which would be a circular or an inadequate exercise, and "It seems therefore more correct to take an intensional view of relations ... ."

50. Wiener defined the ordered pair $\langle x, y\rangle$ as $\{\{\{x\}, \Lambda\}, \{\{y\}\}\}$ when x and y are of the same type and Λ is the null class (of the next type). He used this to eliminate from the system of *Principia Mathematica* the Axiom of Reducibility for propositional functions of two variables. Wiener had written a doctoral thesis comparing the logics of Schröder and Russell.

Years later Russell [1959, 67] wrote: "I thought of relations, in those days, almost exclusively as *intensions*. ... It seemed to me—as indeed, it still seems—that, although from the point of view of a formal calculus one can regard a relation as a set of ordered couples, it is the intension alone which gives unity to the set."

51. Before Hausdorff and going beyond Cantor, Dedekind was first to consider non-linear orderings, e.g., in his remarkably early, axiomatic study [1900] of lattices.

52. See Note 19.

53. The general adoption of the Kuratowski pair proceeded through the later axiomatization discussed in 3.1. Von Neumann initially took the ordered pair as primitive but later noted [1928, 338][1929, 227] the reduction via the Kuratowski definition. In his recasting of von Neumann's system Bernays [1937, 68] explicitly acknowledged Kuratowski [1921] and began with its definition for the ordered pair.

54. After getting a partial result [1914, 465ff.] Hausdorff [1916] also showed that CH holds for the Borel sets, i.e., every Borel set is either countable or has the power of the continuum.

55. Luzin [1925] traced the term "analytic" back to Lebesgue [1905] and pointed out how the original example of a non-Borel Lebesgue measurable set there is in fact the first example of a non-Borel analytic set.

56. In a confident and prophetic passage Luzin [1925a] declared that his efforts towards the resolution of this problem led him to a conclusion "totally unexpected," that "one does not know *and one will never know*" of the family of projective sets, although it has cardinality $2^{\aleph_0}$ and consists of "effective sets," whether every member has cardinality $2^{\aleph_0}$ if uncountable, has the Baire property, or is even Lebesgue measurable.

57. See Moore [1982, 3.3].

58. This is better done in Kuratowski [1921]. The Hausdorff [1914] approach with an ordered pair could have been taken, but that only became standard later when more general relations were considered.

59. Rubin-Rubin [1985] provides numerous equivalents for AC. See Moore [1982], especially its 5.1, for other choice principles. This cottage industry has survived to the present, with e.g., Andreas Blass [1984] showing that Hausdorff's proposition that every vector space has a basis is actually equivalent to AC (over ZF *sans* the axiom, with Foundation needed in the proof).

60. See Miller [1984] for more on special sets of reals and van Douwen [1984] and Vaughan [1990] for cardinal invariants of the continuum. Much work has been done on the latter in terms of relative consistency; see Bartoszyński-Judah [1995] for the recent work on those cardinal invariants having mainly to do with measure and category.

61. See Hallett [1984, 8.1].

62. Von Neumann [1929, 236ff.] formally defined the cumulative hierarchy by transfinite recursion, and his argument established the relative consistency of Foundation, though this was not his main purpose. In its formality and purpose this work was a precursor of Gödel's construction of L (see 3.4).

63. Shoenfield [1967, 238ff.][1977], Wang [1974a], Boolos [1971], and Scott [1974] motivate the axioms of set theory in terms of an iterative concept of set based on stages of construction. Parsons [1977] raises issues about this approach. Potter [1990] is a historically informed textbook in set theory that develops the subject based on stages of construction.

64. In a polemical concluding paragraph Zermelo [1930, 47] wrote: "The two diametrically opposite tendencies of the thinking spirit, the idea of creative *progress* and of comprehensive *completion* [Abschluss], which also lie at the root of the Kantian 'antinomies', find their symbolic representation and symbolic reconciliation in the transfinite series of numbers based on the concept of well-ordering. This series in its boundless progression does not have a true conclusion, only relative stopping points, namely those 'limit numbers [Grenzzahlen, i.e., the inaccessible cardinals]' that separate the higher from the lower model types. And thus also, the set-theoretic 'antinomies' lead, if properly understood, not to a restriction or mutilation but rather to a presently unsurveyable unfolding and enrichment, of mathematical science."

65. See Kanamori [1994, Chapter 5].

66. See Goldfarb [1979] and Moore [1988b] for more on the emergence of first-order logic.

67. The historical development is clarified by the fact that while this book was published in light of the developments of the 1920's, it has a large overlap with unpublished lecture notes for a 1917-8 course given by Hilbert and Göttingen (see Moore [1988b, 114ff.]).

68. Cf., the dictum from Hilbert [1926, 170]: "From the paradise that Cantor has created for us no one will cast us out."

69. In a prescient footnote, 48a, to his incompleteness paper [1931] Gödel wrote: " ... the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite ... while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decided whenever appropriate higher types are added (for example, the type ω to the system P [Peano Arithmetic]). An analogous situation prevails for the axiom system of set theory."

70. In a resonating inversion textbooks often establish first the Reflection Principle for ZF, equivalent to Replacement and Infinity (see 3.6), and then apply it to establish GCH in L.

71. The quotation in note 69 and letters from Gödel in Wang [1974, 8ff.] support this contention.

72. In the mid-1960's Ronald Jensen [1972] developed a "fine structure" theory for L of intrinsic interest, and *inner model theory*, the study of constructibility and its generalizations, has become one of the mainstreams of current set theory.

73. See Wang [1974, §§1, 4] for more on Gödel's view on heuristics as well as the criteria of *intrinsic necessity* and *pragmatic success* for accepting new axioms.

74. See Dreben-Goldfarb [1979] for a general framework for the decidability of quantificational formulas.

75. See the text Graham-Rothschild-Spencer [1990] and the compendium Nešetřil-Rödl [1990] for the recent work on Ramsey Theory.

76. See Todorčević [1984] for a wide-ranging account of infinite trees.

77. The details of implications asserted at the end of Erdös-Tarski [1943] were worked out in an influential seminar conducted by Tarski and Mostowski at Berkeley in 1958–9, and appeared in Erdös-Tarski [1961]. In 3.6 the equivalent formulations of strong compactness and weak compactness and Hanf's pivotal result are discussed.

78. Several results of Erdös-Rado [1956] were derived independently by Kurepa [1959].

79. The results of Erdös-Hajnal-Rado [1965] were extended in Byzantine detail to the general situation without GCH by the book Erdös-Hajnal-Máté-Rado [1984].

80. For a cardinal λ, C is *closed unbounded in* λ *iff* C is an unbounded subset of λ containing all of its limits points less than λ. S is *stationary in* λ *iff* S is a subset of λ that has non-empty intersection with every set closed unbounded in λ. The regressive function lemma of Fodor [1956] asserts: *For* λ *regular and uncountable, if* S *is stationary in* λ *and* $f: S \to \lambda$ *is regressive* (*i.e.,* $f(\xi) < \xi$ *for* $\xi \in S$), *there is an* $\alpha < \lambda$ *such that* $\{\xi \in S \mid f(\xi) = \alpha\}$ *is stationary in* λ.

81. Genesis 2:20: "And Adam gave names to all cattle, and to the fowl of the air, and to every beast of the field ... " See Chang-Keisler [1990] and Hodges [1993] for model theory and its development.

82. To define $L(A)$, first recall def defined at the beginning of 3.4 and let $\mathrm{tc}(x)$ denote the smallest transitive set $\supseteq x$. Then define: $L_0(A) = \mathrm{tc}(\{A\})$ (to ensure that the resulting class is transitive); $L_{\alpha+1} = \mathrm{def}(L_\alpha(A))$; $L_\delta = \bigcup_{\alpha<\delta} L_\alpha(A)$ for limit $\delta > 0$; and finally $L(A) = \bigcup_\alpha L_\alpha(A)$. To define $L[A]$, first let $\mathrm{def}^A(x)$ denote the collection of subsets of x definable over $\langle x, \in, A \cap x\rangle$ via a first-order formula allowing parameters from x. Then define: $L_0[A] = \emptyset$; $L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A])$; $L_\delta[A] = \bigcup_{\alpha<\delta} L_\alpha[A]$ for limit $\delta > 0$; and finally $L[A] = \bigcup_\alpha L_\alpha[A]$.

83. See Kanamori [1994, §§7, 8, 9] for more on partition relations and sets of indiscernibles, particularly their role in the formulation of the set of natural numbers $0^{\#}$ and its role of transcendence over L.

84. Petr Vopěnka [1962] independently established Scott's result.

85. See Keisler-Tarski [1964] for a comprehensive account of the theory of large cardinals through the use of ultrapowers in the early 1960's.

86. See Moore [1988a] for more on the origins of forcing.

87. Scott continued (Bell [1985, ix]): "I knew almost all the set-theoreticians of the day, and I think I can say that no one could have guessed that the proof would have gone in just this way. Model-theoretic methods had shown us how many *non-standard* models there were; but Cohen, starting from very primitive first principles, found the way to keep the models *standard* (that is, with a well-ordered collection of ordinals)."

88. See Moore [1982, 5.1].

89. As one example, James Halpern [1961, 1964] first established the consistency of the Boolean Prime Ideal Theorem together with the failure of AC in a Fraenkel-Mostowski model. After the advent of forcing Levy saw how to effect this result relative to ZF, if a certain partition property were established. Halpern and Hans Läuchli in their [1966] duly established it, and so Halpern-Levy [1971] secured the consistency of the Boolean Prime Ideal Theorem together with the failure of AC relative to ZF.

Soon Thomas Jech and Antonin Sochor in their [1966] made the conversion of consistencies established via Fraenkel-Mostowski models into those via forcing relative to ZF systematic by establishing a general transfer theorem. They then proceeded to convert many of the Fraenkel-Mostowski consistencies in algebra established by Läuchli [1962]. David Pincus [1972] significantly extended the Jech-Sochor transfer theorem. See Felgner [1971] and Jech [1973] for more on independence of weak versions of AC and transfers.

90. See Gitik-Magidor [1992] and Jech [1995] for the recent work on the Singular Cardinals Problem.

91. The original application of the Levy collapse in Levy [1963, IV] also turned out to require an inaccessible cardinal (Levy [1970, 131ff.])—a remarkable turn of events for an apparently technical artifact at the beginning of forcing.

Many years later, Saharon Shelah [1980, 1984] was able to establish the necessity of Solovay's inaccessible for the proposition that every set of reals is Lebesgue measurable; on the other hand, Shelah also showed that the inaccessible is not necessary for the proposition that every set of reals has the Baire property. Jean Raisonnier [1984] provided a simpler proof of Shelah's result on Lebesgue measurability.

92. There is a telling antecedent in the result of Gerhard Gentzen [1936, 1943] that the consistency strength of arithmetic can be exactly gauged by an ordinal ε_0, i.e., transfinite induction up to that ordinal in a formal system of notations. Although Hilbert's program of establishing consistency by finitary means could not be realized, Gentzen provided an exact analysis in terms of ordinal length. Proof theory blossomed in the 1960's with the analysis of other theories in terms of such lengths, the proof theoretic ordinals.

93. Vopěnka had developed a similar concept in a reworking [1964] of the independence of CH. The concept was generalized and simplified in a series of papers on the so-called ∇-models from the active Prague seminar founded by Vopěnka (see Hájek [1971, 78]), culminating in the exposition Vopěnka [1967]. However, the earlier papers did not have much impact, partly because of an involved formalism in which formulas were valued in a complete lattice rather than Boolean algebra.

The Scott-Solovay method with Boolean algebras is completely general, since any partial order can be "completed" in a natural way to get a complete Boolean algebra. Scott popularized this approach in his own reworking [1967] of the independence of CH and in 1967 lecture notes eventually making their way to Bell [1985].

94. Richard Mansfield [1971] made the tree representations more explicit with a measurable cardinal.

95. See Kanamori [1994, ∞] for these recent developments.

96. Decades after his creation of forcing Cohen wrote (Albers-Alexanderson-Reid [1990, 54]): " ... I regard the present solution of the [Continuum] [P]roblem as very satisfactory. I think that it is the only possible solution. It gives one a feeling for what's possible and what's impossible, and in that sense I feel that one should be very satisfied .... There will be philosophical papers, but I don't think any mathematical paper will say that there is any answer other than the answer that it's undecidable."

97. Kreisel [1971] emphasized that CH is (provably) not independent of the full second-order version of Zermelo's axioms although which way it is decided is not known. Although he anticipated the obvious retort that this is no solution but a vicious circle, that retort is nonetheless there, particularly pointing to the conflation of some prior sense for the continuum that is then alleged to be captured by a formalization—another pitfall of second-order logic.

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