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For the Program Committee
WILFRID HODGES, PROGRAM CHAIR

Abstract of contributed talk

▶ CHRISTIAN EVEN AND RAMÓN PINO PÉREZ, *Extension of Scott-Koymans theorem to partial framework*.
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The Scott-Koymans theorem (see [1]) is the following one:

THEOREM 1.

(i) *Every reflexive object U of a cartesian closed category* (CCC *for short*) $\mathbf{C}$ *induces a λ-algebra $\mathcal{B}(U, \mathbf{C})$ (i.e., a combinatory algebra such that $\lambda \vdash t_\lambda = u_\lambda \Longrightarrow \mathcal{B}(U, \mathbf{C}) \models t = u$).*

(ii) *Every λ-algebra $\mathcal{A}$ induces a cartesian closed category $\mathbf{C}_\mathcal{A}$ and a reflexive object $U_\mathcal{A}$ such that $\mathcal{A} \simeq \mathcal{B}(U_\mathcal{A}, \mathbf{C}_\mathcal{A})$.*

This theorem shows the deep relationships between λ-calculus, combinatory logic and category theory.

In the partial framework, very useful for analyzing syntax and semantics of computations, we can find the notions of partial cartesian closed category (PCCC for short) introduced by Curien and Obtulowicz [3], partial λ-calculus introduced by Moggi [5] and partial combinatory logic (see [2]). So, it is natural to ask whether these notions are intimately linked. More precisely, has the Theorem 1 an analogous one in the partial framework?

First of all, remark that the partial notions generalize the "total" notions. But there is a mismatch between the partiality and the concept of partial λ-algebra as Klop pointed out [4]. Indeed every partial combinatory algebra satisfying the equation $S(KK)K = S(KS)(S(KK)K)$ is total.

The previous remark leads us to define *combinatory algebras with partial elements*. Very roughly speaking, these are combinatory algebras in which there is a partition of the universe into two sets: the total elements and the partial elements (possibly the empty set). The set

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of partial elements is a left and right ideal whereas the set of total elements can even be not closed by application. More precisely we define a first order theory called combinatory logic with partial elements (CLPE) on the λ-language of combinatory logic enriched with an unary predicate symbol, $\downarrow(\cdot)$. We write $t\downarrow$ instead of $\downarrow(t)$ and I instead of SKK. The axioms of CLPE are: $xy\downarrow \Longrightarrow x\downarrow \wedge y\downarrow$, $I\downarrow$, $x\downarrow \wedge y\downarrow \Longrightarrow Kx\downarrow \wedge Sxy\downarrow$, $x\downarrow \Longrightarrow y\lceil x = y$, $Ix = x$, $Sxyz = xz(yz)$, $x\lceil x = x$, $(x\lceil y)\lceil z = (x\lceil z)\lceil y$, $x\lceil(y\lceil z) = (x\lceil y)\lceil z$, $x(y\lceil z) = xy\lceil z$ and $(x\lceil z)y = xy\lceil z$.

The models (*in the sense of first order logic*) of CLPE are the combinatory algebras with partial elements.

Now to define λ_{pe}-algebras, we need the λ_{pe}-calculus. This calculus is derived from the λ_{p}-calculus (see [5] or [6]) by the following modifications: the axiom $M\lceil x = M$ is substituted by $\lambda x.(M\lceil x) = \lambda x.M$ and the rule of formation of terms "M is a term $\Longrightarrow \lambda x.M$ is a term" is substituted by "M is a term $\Longrightarrow (\lambda x.M)\lceil FV(M) \cup C(M)$ is a term", where $FV(M)$ and $C(M)$ are the free variables and the constants of M respectively.

A combinatory algebra with partial elements $\mathcal{A}$ is said to be a λ_{pe}-algebra if and only if $\lambda_{\mathrm{pe}} \vdash t_\lambda = u_\lambda \Longrightarrow \mathcal{A} \models t = u$.

MAIN RESULT. *The Theorem* 1 *in which we replace λ-algebras by λ_{pe}-algebras and cartesian closed category by partial cartesian closed category holds.*

The proof of this result is done by following the ideas of Scott and Koymans. The originality of our approach (and the main difficulty) was in finding the right notions, namely that of λ_{pe}-algebra.

REMARK 1. Our main result generalizes the Theorem 1 because every CCC is a PCCC in which all the morphisms are total and every λ-algebra is a λ_{pe}-algebra without partial elements. However our result is a strict generalization because there are λ_{pe}-algebras which are not λ-algebras.

REMARK 2. The λ_{pe}-calculus can be considered as a tool for reasoning about equivalence of programs in which the evaluation mechanism is call-by-value. One can ask why it is not an arbitrary tool. Our main result seems to indicate that this calculus is not arbitrary.

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