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OPTIMAL PROOFS OF DETERMINACY

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Introduction. In this paper I shall present a method for proving determinacy from large cardinals which, in many cases, seems to yield optimal results. One of the main applications extends theorems of Martin, Steel and Woodin about determinacy within the projective hierarchy. The method can also be used to give a new proof of Woodin's theorem about determinacy in $L(\mathbb{R})$.

The reason we look for optimal determinacy proofs is not only vanity. Such proofs serve to tighten the connection between large cardinals and descriptive set theory, letting us bring our knowledge of one subject to bear on the other, and thus increasing our understanding of both. A classic example of this is the Harrington-Martin proof that Π^1_1-determinacy implies $(<\omega^2 - \Pi^1_1)$-determinacy.[1] This is an example of a *transfer theorem*, which assumes a certain determinacy hypothesis and proves a stronger one. While the statement of the theorem makes no mention of large cardinals, its proof goes through $0^{\#}$, first proving that

$$\Pi^1_1\text{-determinacy} \Longrightarrow 0^{\#} \text{ exists},$$

and then that

$$0^{\#} \text{ exists} \Longrightarrow (<\omega^2 - \Pi^1_1)\text{-determinacy}.$$

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[1]Recall that a set A of *reals* (points in the Baire space $\mathbb{R} = {}^{\mathbb{N}}\mathbb{N}$) is Π^1_n if it can be defined by a Π^1_n formula, i.e.,

$$(*) \qquad A = \{x \in \mathbb{R} \mid \forall y_1 \exists y_2 \dots \forall/\exists y_n R(x, y_1, \dots, y_n)\},$$

where the relation R is arithmetical; it is $\Pi^1_n(z)$ (Π^1_n relative to z) if $(*)$ holds with some R which is arithmetical in the real $z \in \mathbb{R}$; and it is $\boldsymbol{\Pi}^1_n$ (boldface Π^1_n) if it is $\Pi^1_n(z)$ for some z. A set A is $(\alpha - \Pi^1_1)$ for an ordinal $\alpha \leq \omega^2$ if there exists a recursive sequence $\langle B_\gamma \mid \gamma < \alpha \rangle$ of Π^1_1 sets such that

$$x \in A \iff \text{the least } \gamma \text{ such that } [\gamma = \alpha \text{ or } x \notin B_\gamma] \text{ is odd};$$

and A is $(<\alpha - \Pi^1_1)$ if it is $(\beta - \Pi^1_1)$ for some $\beta < \alpha$. The relativized classes $(<\alpha - \Pi^1_1)(z) = (<\alpha - \Pi^1_1(z))$ and the boldface classes $(<\alpha - \boldsymbol{\Pi}^1_1)$ are defined similarly.

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More recent examples of the connection between large cardinals and descriptive set theory include Steel's proof that

$$\mathrm{AD}^{L(\mathbb{R})} \Longrightarrow \mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH},$$

see [9], and several results of Woodin about models of AD^+, a strengthening of the axiom of determinacy AD which Woodin has introduced. These proofs not only use large cardinals, but also reveal a deep, structural connection between descriptive set theoretic notions and notions related to large cardinals. Steel proved $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}$ by showing that, in fact, $\mathrm{HOD}^{L(\mathbb{R})}$ is a *core model* below $\Theta^{L(\mathbb{R})}$. Many of Woodin's results about AD^+ rely on the relation between models of AD^+ and models of choice with infinitely many Woodin cardinals. From any ZFC model which satisfies "*there exist infinitely many Woodin cardinals*", it is possible to derive in a canonical way a model of AD^+, and conversely, every model of AD^+ is a derived model, see [10]. In [4] and [3] it is shown not just that $0^\#$ exists $\Longrightarrow (<\omega^2 - \Pi^1_1)$-determinacy, but also that $0^\#$ and $(<\omega^2 - \Pi^1_1)$ sets are very closely connected: $0^\#$ is recursively isomorphic to the "complete" $\Game(<\omega^2 - \Pi^1_1)$ real, where $\Game$ is the *game quantifier* of [7, Section 6D]. (We will recall the definition of the game quantifier at the end of this Introduction.)

Probably the most important result of the present paper is Lemma 1.7 which gives a general method for proving many determinacy results. This is formulated in Section 1, which also includes an intuitive survey of the notions needed for it. It is hoped that any interested logician should find Section 1 readable.

The main result of Section 2 is Theorem 2.5, an application of Lemma 1.7 which strengthens the following theorem in [5].

THEOREM (Martin-Steel). *If there exist n Woodin cardinals and a measurable cardinal above them, then $\mathbf{\Pi}^1_{n+1}$-determinacy holds.*

Theorem 2.5 is different from the Martin-Steel result in two ways. First, it proves more determinacy, showing that all $\Game^{(n)}(<\omega^2 - \Pi^1_1)$ sets are determined. Secondly, the hypothesis of Theorem 2.5 is weaker and is, in fact, optimal because of the following theorem of Woodin [10].

THEOREM (Woodin). *If $\mathbf{\Pi}^1_{n+1}$-determinacy holds, then for every real z, there exists a model M satisfying the assumptions of Theorem* 2.5 *with $z \in M$.*

Woodin's theorem and Theorem 2.5 combined yield the following transfer result

THEOREM. *$\mathbf{\Pi}^1_{n+1}$-determinacy $\Longrightarrow \Game^{(n)}(<\omega^2 - \mathbf{\Pi}^1_1)$-determinacy.*

For $n = 0$, this is the Harrington-Martin Theorem mentioned above; for odd n, it was established by Kechris and Woodin by a purely descriptive set

I	x_0		x_2		$\dots$	
II		x_1		x_3		$\dots$

FIGURE 1. The standard game $G(P)$ with payoff P.

theoretic argument, see [2]; and for even $n > 0$, there is no known proof which does not involve large cardinals.

The game quantifier. For each set $P \subseteq \mathbb{R}$, a *run* of the *standard* (perfect information, two-person) *game* $G(P)$ *with payoff* P is illustrated in Figure 1: the players I and II alternate choosing natural numbers $x_0, x_1, x_2, \dots$, forever, and in the end I wins the run if the jointly produced real $x = \langle x_0, x_1, x_2, \dots \rangle$ is in P, otherwise II wins. We say that I *wins* $G(P)$ (or P) if I has a winning strategy for $G(P)$—and similarly for II. Finally, for each set $A \subseteq \mathbb{R} \times \mathbb{R}$ and each real x, we set

$$(\Game y)A(x, y) \iff \text{I wins the game } G(\{y \mid \langle x, y\rangle \in A\}).$$

A set of reals is in $\Game^{(n)}(<\omega^2 - \Pi^1_1)$ if it can be defined from a $(<\omega^2 - \Pi^1_1)$ set using n successive applications of the game quantifier $\Game$. It is easy to see that every $\mathbf{\Pi}^1_{n+1}$ set is $\Game^{(n)}(<\omega^2 - \mathbf{\Pi}^1_1)$, but the converse is false.

§1. A few preliminaries and a lemma. To keep the level of technical details as low as possible, I will not give a precise definition of iteration trees. The reader who would like to see the actual definition should consult [6]. For our purposes, it will be enough to know that iteration trees are certain trees of models (of ZFC) and embeddings between them: we think of the nodes

$$\begin{tikzcd}
\vdots & & \vdots & \\
M_5 & M_4 & & M_3 \\
M_2 \arrow[u, "i_{2,5}"] & & M_1 \arrow[ul, "i_{1,4}"'] \arrow[ur, "i_{1,3}"] & \\
& M_0 \arrow[ul, "i_{0,2}"] \arrow[ur, "i_{0,1}"'] & &
\end{tikzcd}$$

of the tree as representing the models, while the branches represent the embeddings, e.g., $i_{0,2} : M_0 \to M_2$. We will only deal with trees which are finite or have height ω.

In real life, there are many restrictions on the structure of iteration trees. The embeddings must be the ultrapower embeddings by some *extenders*—more precisely, if M_ℓ and M_k are directly connected by a branch of the tree, then the embedding $i_{\ell,k}$ must be the ultrapower embedding $i_{\ell,k} : M_l \to \mathrm{Ult}(M_\ell, E_{k-1}) = M_k$, using some extender E_{k-1} in M_{k-1}.[2] Given a finite tree with a last model M_n, we can extend the tree by picking some extender $E_n \in M_n$ and applying it to the model M_k for some $k \leq n$, to get $i_{k,n+1} : M_k \to \mathrm{Ult}(M_k, E_n) = M_{n+1}$. There are still more details omitted, the most important one being that we require $V_{\kappa+1}^{M_k} = V_{\kappa+1}^{M_n}$, where $\kappa = \mathrm{crit}(E_n)$, i.e., M_k and M_n have the same subsets of the critical point κ of the extender. This agreement allows us to apply E_n to the model M_k.

Let T be some iteration tree of height ω, and let b be a branch of T. (Informally, $b = \langle M_{k_j} \mid j \in \omega\rangle$, and the tree has maps $i_{k_j,k_{j+1}} : M_{k_j} \to M_{k_{j+1}}$, for all $j < \omega$.) We let M_b be the direct limit of $\langle M_{k_j}, i_{k_j,k_{j'}}\rangle$, and $i_{k_j,b} : M_{k_j} \to M_b$ be the direct limit maps; and let $i_b = i_{0,b}$. It is a fact that all models along a finite tree are well founded. A natural question to ask about the tree T is whether any of the direct limit models is also well founded, i.e., *is there some infinite branch b of T such that M_b is well founded?* (One reason for considering this question and its importance will become clear in Section 2.) Let us call a model M 1-*iterable* if *for every iteration tree T with $M_0 = M$, there exists an infinite branch b of T such that M_b is well founded.* For simplicity, we will say that T is a *tree on M*, if T is an iteration tree with $M_0 = M$.

Note that we do not require the trees T to be in M or to be somehow definable over M. Each finite, initial segment of T will be definable over M from a finite sequence of extenders, but the *infinite* sequence of extenders which defines the whole of T need not be in M. If M_b is well founded, we can ask whether M_b itself is 1-iterable. In general, let us say that M is *n-iterable* (with $n \geq 1$), if *for every iteration tree T on M, there exists an infinite branch b of T such that M_b is $(n-1)$-iterable.*

As mentioned above, in this paper we will stick to trees of height ω, with ω-closed extenders. So in "real world" terms, our definition of n-iterability is the following:

Definition 1.1. A model M is 0-iterable if it is well founded; and M is *n-iterable* $(n \geq 1)$ if for every height-ω iteration tree T on M which uses only extenders whose ultrapowers are ω-closed in the models in which they appear, there exists an infinite branch b of T such that M_b (the direct limit along b) is $(n-1)$-iterable.

The following theorem of [6] shows that there are many models which are n-iterable:

Theorem 1.2 (Martin-Steel). *Let M be countable, let $\pi : M \to V_\theta$ be an elementary embedding of M into a rank of V, and let T be an iteration tree on*

[2] For technical reasons, let us use only extenders whose ultrapowers are ω-closed. This is not part of the standard definition, but it is useful for our purposes.

M; then there exists an infinite branch b of T and a map $\sigma : M_b \to V_\theta$, such that the diagram

$$\begin{tikzcd} V_\theta & \\ M \arrow[u, "\pi"] \arrow[r, "i_b"'] & M_b \arrow[ul, "\sigma"'] \end{tikzcd}$$

commutes.

PROOF. See [6]. ⊣

COROLLARY 1.3 (Martin-Steel). *Each model M as in Theorem* 1.2 *is n-iterable, for every $n \in \omega$.*

The models we will work with are class models and therefore not countable. However, every model M we will use will be of the form $L(V_\lambda^M)$, so that M satisfies the claim that "it is the constructible universe L, built over a rank-initial segment of the universe, V_λ". The following result gives one method of obtaining such models which are n-iterable.

THEOREM 1.4 (Martin-Steel). *Let $M = L(V_\lambda^M)$ with V_λ^M countable, suppose that $(V_\lambda^M)^\#$ exists and can be elementarily embedded into $V_\theta^\#$ for some θ, say $\pi : (V_\lambda^M)^\# \to V_\theta^\#$, and let T be an iteration tree on M; then there exists an infinite branch b of T and an embedding $\sigma : i_b\left((V_\lambda^M)^\#\right) \to V_\theta^\#$, such that the diagram*

$$\begin{tikzcd} V_\theta^\# & \\ (V_\lambda^M)^\# \arrow[u, "\pi"] \arrow[r, "i_b"'] & i_b\left((V_\lambda^M)^\#\right) \arrow[ul, "\sigma"'] \end{tikzcd}$$

commutes.

From this we can see that $i_b\left((V_\lambda^M)^\#\right) = (V_{i_b(\lambda)}^{M_b})^\#$, and by a proof similar to that of Corollary 1.3 we get:

COROLLARY 1.5 (Martin-Steel). *Every model M as in Theorem* 1.4 *is n-iterable, for every $n < \omega$.*

So, for example, if $V_\theta^\#$ exists and $L(V_\theta) \models$ "There exist n Woodin cardinals", then there exists an n-iterable class model with n Woodin cardinals: simply take $M = L(V_{\bar\theta}^M)$ to be the Skolem hull of the indiscernibles for V_θ. Thus the existence of an n-iterable model with n Woodin cardinals follows from, e.g., the existence of n Woodin cardinals and a measurable above them all. (The converse, of course, is false.)

Let us for now assume that M is a class model such that

I	x_0, T_0, h_0		x_2, T_1, h_1		x_4, T_2, h_2		$\dots$
II		x_1		x_3		$\dots$	

FIGURE 2. The game $\widehat{G}(\dot{A})$.

- $M \models$ "δ is Woodin", and
- (In V) there exists some g which is M-generic $/\mathrm{col}(\omega, \delta)$,

where $\mathrm{col}(\omega, \delta)$ is the usual forcing poset for making δ countable. So $g : \omega \twoheadrightarrow \delta$ is a surjection, and in particular, δ is countable in V. We begin by considering games with payoff in $M^{\mathrm{col}(\omega,\delta)}$. Let $\dot{A}$ be a name for a set of reals in $M^{\mathrm{col}(\omega,\delta)}$, so that $\dot{A}_g$ is a set of reals in $M[g]$, the model with the generic g adjoined. Let $G(\dot{A}_g)$ be the standard game illustrated in Figure 1, with payoff $\dot{A}_g$. This need not be determined, since $M[g]$ satisfies, after all, the Axiom of Choice, but, as it turns out, we can obtain by changing it slightly a game which is (roughly) determined and which is quite useful in proving many forms of determinacy.

DEFINITION 1.6. In the game $\widehat{G}(\dot{A})$, the players I and II alternate moves as in Figure 2, where each x_i must be an integer and the auxilliary objects played by Player I must satisfy the following:

- Each T_i is a finite iteration tree, and $i < j \Longrightarrow T_j$ extends T_i.
- Each h_i is a function with $\mathrm{dom}(h_i) = T_i$ (so, informally, for every model N in the tree T_i, $h_i(N)$ is defined), and $i < j \Longrightarrow h_i \subset h_j$.

At the end of a run, let

$$x = \langle x_0, x_1, \dots \rangle,$$
$$T = \bigcup_{i<\omega} T_i$$
$$h = \bigcup_{i<\omega} h_i,$$

so that T is then an iteration tree of height ω, and h is a map with $\mathrm{dom}(h) = T$. Now h induces a (continuous) map $\tilde{h}$ with $\mathrm{dom}(\tilde{h}) = \{\text{infinite branches of } T\}$, by

$$\tilde{h}(b) = \bigcup_{N \in b} h(N), \quad \text{for } b \text{ an infinite branch of } T.$$

I wins the game $\widehat{G}(\dot{A})$ if whenever b is an infinite branch of T such that M_b is well founded, then the following two conditions are satisfied:

(i) $\tilde{h}(b)$ is M_b-generic $/\mathrm{col}(\omega, i_b(\delta))$.
(ii) $x \in i_b(\dot{A})_{\tilde{h}(b)}$.

Compared with $G(\dot{A}_g)$, the game $\widehat{G}(\dot{A})$ is easier for I in two ways. First, I is allowed to move $\dot{A}$ by using an iteration tree—replacing the name $\dot{A}$ by $i_b(\dot{A})$. Secondly, I is allowed to choose the generic object $\tilde{h}(b)$ which is used to interpret the name $i_b(\dot{A})$.

LEMMA 1.7. *For M as above, $\dot{A}$ a name for a set of reals in $M^{\mathrm{col}(\omega,\delta)}$, and g M-generic $/\mathrm{col}(\omega,\delta)$:*

either (I) : I *wins $\widehat{G}(\dot{A})$ via a strategy $\widehat{\sigma} \in M[g]$, which wins against all of* II*'s plays in V* (not only plays in $M[g]$)*;*

or (II) : II *wins the game $G(\dot{A}_g)$ via a strategy $\sigma \in M[g]$, which need only work against* I*'s plays in $M[g]$.*

In Sections 2 and 3, I will use this Lemma to prove several determinacy results. But before that, let us notice one important corollary.

COROLLARY 1.8. *Suppose M is* 1*-iterable, $M \models$ "δ is Woodin" and $($in $V)$ there exists some g which is M-generic $/\mathrm{col}(\omega,\delta)$; then every real z $($in $V)$ can be made generic over an iterate of M, i.e., for any real z, there exists an iteration tree T $($of height $\omega)$ on M such that for any infinite branch b of T with a well founded direct limit model M_b, there is an h, M_b-generic $/\mathrm{col}(\omega, i_b(\delta))$ with $z \in M_b[h]$.*

To prove this we apply the Lemma with $\dot{A} = \{$all reals in $M^{\mathrm{col}(\omega,\delta)}\}$.

Compared with Woodin's iterations to make reals generic (see the forthcoming [3]), this method has the disadvantage of using a forcing which is not δ-c.c. $(\mathrm{col}(\omega,\delta))$, and the more serious disadvantage of requiring full information about the real z. For example, this method cannot be used to make a Turing degree generic without first picking a representative of the degree. The main advantage of this method is that it uses only iteration trees of height ω, and so it can be used to make reals generic over models which are just 1-iterable.

§**2. The projective hierarchy.** Let us now see how to use Lemma 1.7 to establish the determinacy of various classes of projective sets. As a simple first example:[3]

THEOREM 2.1. *If M is* 1*-iterable, $M \models$ "δ is Woodin" and $($in $V)$ there exists a g which is M-generic $/\mathrm{col}(\omega,\delta)$, then Π^1_2-determinacy holds $($in $V)$.*

REMARK. The existence of such M is a large cardinal assumption. As mentioned in Section 1, it is weaker than "there exists a Woodin cardinal and a measurable cardinal above it".

[3]A set A is Σ^1_n if its complement $\mathbb{R} \setminus A$ is Π^1_n; it is Δ^1_n if it is both Σ^1_n and Π^1_n; and similarly for the relativized and the boldface notions. A set is *projective* if it is $\boldsymbol{\Sigma}^1_n$ for some n,

PROOF. Let A be a Σ^1_2 set of reals, say

$$x \in A \iff V \models \phi[x],$$

where ϕ is some Σ^1_2 formula. To show that the standard game $G(A)$ with payoff set A is determined, we apply Lemma 1.7 to the canonical name $\dot{A}$ for $A \cap M[g]$, i.e.,

$$\dot{A} = \{\langle \dot{y}, p\rangle \mid p \Vdash^{M}_{\mathrm{col}(\omega,\delta)} \phi(\dot{y})\}.$$

Case I. I wins $\widehat{G}(\dot{A})$, say by $\widehat{\sigma} \in M[g]$. Let σ be the strategy for I obtained by ignoring the auxilliary moves T_i, h_i in $\widehat{G}(\dot{A})$.

CLAIM : *σ is a winning strategy for* I *in $G(A)$.*

PROOF. We have to show that if x is a play by σ, then $x \in A$. Now since x is by σ, there exists a corresponding play x, T, h by $\widehat{\sigma}$. As M is 1-iterable, the tree T has a well founded infinite branch b, and since $\widehat{\sigma}$ is winning for I in $\widehat{G}(\dot{A})$:

(i) $\tilde{h}(b)$ is M_b-generic /$\mathrm{col}(\omega, i_b(\delta))$, and

(ii) $x \in i_b(\dot{A}_{\tilde{h}(b)})$.

But

$$\begin{aligned} i_b(\dot{A}) = \{\langle \dot{y}, p\rangle \mid\ & p \text{ is a condition in } \mathrm{col}(\omega, i_b(\delta)), \\ & \dot{y} \text{ is a name for a real in } M_b^{\mathrm{col}(\omega, i_b(\delta))}, \text{ and} \\ & M_b \models \text{“}p \Vdash \phi(\dot{y})\text{”}\}, \end{aligned}$$

so by (ii), $M_b[\tilde{h}(b)] \models \phi[x]$. Since ϕ is Σ^1_2, it is absolute for $M_b[\tilde{h}(b)]$ and V, so $V \models \phi[x]$ and $x \in A$. ⊣ [Claim]

Case II. II wins $G(\dot{A}_g)$ via a strategy $\sigma \in M[g]$, which works against all plays in $M[g]$.

CLAIM : *σ is, in fact, a strategy for* II *in $G(A)$, which works against all plays (in V).*

PROOF. If not, then there is (in V) a play x by σ which is won by I, i.e., $x \in A$. So

$$V \models (\exists x)[\phi[x] \,\&\, x \text{ is by } \sigma].$$

Since ϕ is Σ^1_2, this entire statement is Σ^1_2, hence absolute. So

$$M[g] \models (\exists x)[\phi[x] \,\&\, x \text{ is by } \sigma];$$

but then for some x in $M[g]$, x is by σ, and $x \in A$—which is a contradiction, since σ is winning for II in $M[g]$. ⊣ [Claim]

Since by Lemma 1.7 either Case I or Case II must hold, the game $G(A)$ is determined. ⊣

So we can very easily use Lemma 1.7 if A is absolute enough between $M[g]$, V and between $M_b[h]$, V. Notice that we did not, in fact, use iterability in Case II. The following observation of John Steel shows how we can, to a certain extent, avoid iterability in Case I as well.

LEMMA 2.2. *Let M be as in Theorem 2.1, except that M need not be 1-iterable, let*

$$A = \{x \mid V \models \phi[x]\}$$

be a Σ^1_2 set of reals, let

$$\dot{A} = \{\langle \dot{y}, p \rangle \mid p \Vdash^M_{\mathrm{col}(\omega,\delta)} \phi(\dot{y})\},$$

and assume that I *wins $\widehat{G}(\dot{A})$; then there exists $\bar{\sigma} \in M$* (not $M[g]$)*, a strategy for player* I*, such that if $x \in M$* (again, not $M[g]$) *is a play by $\bar{\sigma}$, then $x \in A$.*

PROOF. We avoid using iterability by passing to a submodel of M. More precisely, *working in M*, let $\bar{M}$ be a countable, transitive model which can be embedded into V^M_θ for some large enough θ. Say $\pi : \bar{M} \to V^M_\theta$, and assume that $\delta, \dot{A} \in \mathrm{range}(\pi)$, $\delta = \pi(\bar{\delta})$, $\dot{A} = \pi(\dot{\bar{A}})$. Now, by Corollary 1.3,

$$M \models \text{“}\bar{M} \text{ is iterable”}.$$

Furthermore

$$M \models \text{“there exists } \bar{g} \text{ which is } \bar{M}\text{-generic } /\mathrm{col}(\omega, \bar{\delta})\text{”},$$

and by the elementarity of π,

$$\bar{M} \models \text{“I wins } \widehat{G}(\dot{\bar{A}})\text{”}.$$

As in the proof of Theorem 2.1, we can obtain $\bar{\sigma} \in \bar{M}[\bar{g}] \subset M$, such that if $x \in M$ is a play by $\bar{\sigma}$, then there exist T, $\tilde{h}$ with the properties that:

- T is an iteration tree on $\bar{M}$.
- If b is an infinite branch of T such that $\bar{M}_b$ is well founded, then
  - $\tilde{h}(b)$ is $\bar{M}_b$-generic $/\mathrm{col}(\omega, i_b(\bar{\delta}))$, and
  - $x \in \bar{M}_b[\tilde{h}(b)]$ and $\bar{M}_b[\tilde{h}(b)] \models \phi[x]$.

Since $\bar{M}$ is 1-iterable (in M), we can find a branch b such that $\bar{M}_b$ is well founded, and so $\bar{M}_b[\tilde{h}(b)] \models \phi[x]$. But ϕ is Σ^1_2, and so reflects upwards from set models, so $M \models \phi[x]$.

(It is essential here that ϕ is Σ^1_2. A Π^1_2 statement, for example, would not reflect from $\bar{M}_b[\tilde{h}(b)]$ to M, since $\bar{M}_b[\tilde{h}(b)]$ does not contain all the ordinals.) $\dashv$

Lemma 2.2 together with the absoluteness of Σ^1_2 statements for class models gives the following result:

Corollary 2.3. *If* $M \models$ "δ is Woodin" *and* (in V) *there exists* g *which is* M*-generic* $/\mathrm{col}(\omega, \delta)$*, then* Δ^1_2*-determinacy holds in* V*.* Notice that we do not assume here that M is iterable.

This was first proved by Woodin, see [10]. With some extra effort it seems possible to replace in this Corollary the requirement

"(in V) there exists g which is M-generic $/\mathrm{col}(\omega, \delta)$"

with the slightly weaker

"δ is countable (in V)".

It is not known if "$\delta \leq \omega_1^V$" could suffice, but it is known that nothing weaker than "$\delta \leq \omega_1^V$" is enough. This follows from the following result of Woodin:

Theorem (Woodin). *If* Δ^1_2*-determinacy holds, then there exists a class inner model* M *and an ordinal* $\delta \leq \omega_1^V$*, such that* $M \models$ "δ is Woodin".

It is simple to adapt the proof of Corollary 2.3 to give:

Theorem 2.4. *Suppose* $M \models$ "δ is Woodin"*, assume that* (in V) *there exists a* g *which is* M*-generic* $/\mathrm{col}(\omega, \delta)$*, and let* A *be a set of reals such that there exist trees* S *and* S^* *in* M *which project to* A *and* $\mathbb{R} \setminus A$ *respectively; then* A *is determined, and the player who wins* $G(A)$ *has a winning strategy in* M*.* Here S and S^* are usual trees, i.e., $S, S^* \subseteq (\omega \times \lambda)^{<\omega}$ for some ordinal λ, $A = \{x \in \mathbb{R} \mid (\exists f)(\forall n < \omega)[\langle x_0, f_0, x_1, f_1, \ldots, x_n, f_n\rangle \in S]\}$, and $\mathbb{R} \setminus A = \{x \in \mathbb{R} \mid (\exists f)(\forall n < \omega)[\langle x_0, f_0, x_1, f_1, \ldots, x_n, f_n\rangle \in S^*]\}$.

Theorem 2.4 is more general than Corollary 2.3. Indeed it is very simple to derive Corollary 2.3 from Theorem 2.4, by using the Shoenfield trees for A and $\mathbb{R} \setminus A$ when A is Δ^1_2.

Let us now move further up the projective hierarchy. We assume from now on that $(\forall x \in \mathbb{R})[x^\# \text{ exists}]$. By a result of Martin [4], a set $A \subseteq \mathbb{R}$ is $\Game(<\omega^2 - \Pi^1_1)(z)$ if and only if there is some k and some formula ϕ such that

$$x \in A \iff L[x] \models \phi[x, z, c_0, \ldots, c_{k-1}],$$
$$\text{whenever } c_0, \ldots, c_{k-1} \text{ are indiscernibles for } x \text{ and } z.$$

Theorem 2.5. *Suppose* M *is* n*-iterable* $(n < \omega)$*,* $z \in M \cap \mathbb{R}$*,* $M \models$ "$\delta_1 < \cdots < \delta_n$ are Woodin" *and assume that* (in V) *there exists a* g *which is* M*-generic* $/\mathrm{col}(\omega, \delta_n)$*; then* $\Game^{(n)}(<\omega^2 - \Pi^1_1)(z)$*-determinacy holds* (in V).

Theorem 2.6. *If* $B \subseteq \mathbb{R}$ *is in* $\Game^{(n+1)}(<\omega^2 - \Pi^1_1)(z)$*, then there exists a formula* ϕ *and some* $k \in \omega$*, such that for every real* x*,*

$$x \in B \iff M \models \phi[x, z, c_0, \ldots, c_{k-1}],$$

whenever M is an n-iterable model such that

- $M \models$ "$\delta_1 < \cdots < \delta_n$ are Woodin",
- $x, z \in M$,
- (in V) *there exists a g which is M-generic* $/\mathrm{col}(\omega, \delta_n)$.
- $c_0, \dots, c_{k-1}$ *are indiscernibles for* $V_{\delta_n}^M$, *and*

Theorems 2.5 and 2.6 are proved together by induction on n. Theorem 2.6 for the integer n supplies us with the absoluteness needed to apply Lemma 1.7 in proving Theorem 2.5 for the integer $n+1$. Note the similarity between Theorem 2.6 and Martin's result mentioned above; for $n = 0$, Theorem 2.6 *is* Martin's result.

§3. The full axiom of determinacy. Finally, using the ideas of this paper, we can also give a new and simpler proof of the following theorem of Woodin.

THEOREM 3.1 (Woodin). *Let δ be a limit of Woodin cardinals* (in V), *let H be V-generic* $/\mathrm{col}(\omega, <\delta)$, *and let $\mathbb{R}^*$ be the set of reals in the symmetric collapse, i.e.,*

$$\mathbb{R}^* = \bigcup_{\alpha<\delta} \mathbb{R}^{V[H \cap \mathrm{col}(\omega, <\alpha)]};$$

then $L(\mathbb{R}^*) \models \mathrm{AD}$.

We need some terminology. A *coarse premouse* is a structure $\mathcal{M} = \langle M, \in, \delta^{\mathcal{M}} \rangle$ satisfying the fragment of ZFC specified in [6, Definition 1.1]. We say that $\mathcal{M}$ is an *ω-Woodin mouse* just in case $\mathcal{M} \models$ "$\delta^{\mathcal{M}}$ is a limit of Woodin cardinals". We say $\mathcal{M}$ is *ω-iterable* just in case II wins the weak iteration game of length ω on $\mathcal{M}$ (cf. [6, §5], where the game is called $WG(\mathcal{M}, \omega)$). The proof of 1.3 easily shows that if $\langle V_\alpha, \in, \delta \rangle$ is a coarse premouse and $\mathcal{M}$ is a countable elementary submodel of $\langle V_\alpha, \in, \delta \rangle$, then $\mathcal{M}$ is ω-iterable. Finally, if $\mathcal{M}$ is a coarse premouse, we say that $\mathbb{R}^*$ is (the set of reals of) a symmetric collapse of $\mathcal{M}$, just in case for some H which is M-generic $/\mathrm{col}(\omega, <\delta^{\mathcal{M}})$,

$$\mathbb{R}^* = \bigcup_{\alpha<\delta^{\mathcal{M}}} \mathbb{R} \cap M[H \cap \mathrm{col}(\omega, <\alpha)].$$

We use the following absoluteness lemma which is a slight variant of a theorem due to Steel and Woodin, see [8, §4].

LEMMA 3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be countable, ω-iterable, ω-Woodin mice. Let $\mathbb{R}^*_{\mathcal{M}}$ and $\mathbb{R}^*_{\mathcal{N}}$ be the symmetric collapses of $\mathcal{M}$ and $\mathcal{N}$ respectively. Suppose θ is a sentence and*

$$\alpha_0 = \text{the least } \alpha \in OR^{\mathcal{M}} \text{ such that } L_\alpha(\mathbb{R}^*_{\mathcal{M}}) \models \theta,$$
$$\beta_0 = \text{the least } \beta \in OR^{\mathcal{M}} \text{ such that } L_\beta(\mathbb{R}^*_{\mathcal{N}}) \models \theta.$$

(We suppose there are such α and β.) *Let $\phi(v)$ be a formula and $x \in \mathbb{R}^*_{\mathcal{M}} \cap \mathbb{R}^*_{\mathcal{N}}$; then*

$$L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \phi[x] \iff L_{\beta_0}(\mathbb{R}^*_{\mathcal{N}}) \models \phi[x].$$

Proof (Sketch). The idea is to simultaneously build plays of the weak iteration games $WG(\mathcal{M}, \omega)$ and $WG(\mathcal{N}, \omega)$, in a "back-and-forth" fashion. These yield iteration maps $i : \mathcal{M} \to \mathcal{P}$ and $j : \mathcal{N} \to \mathcal{Q}$, and we can arrange that there are symmetric collapses $\mathbb{R}^*_{\mathcal{P}}$ of $\mathcal{P}$ and $\mathbb{R}^*_{\mathcal{Q}}$ of $\mathcal{Q}$ such that $\mathbb{R}^*_{\mathcal{P}} = \mathbb{R}^*_{\mathcal{Q}}$. Corollary 1.8 (or, alternatively, Woodin's genericity theorem) makes the successor step of the construction possible (indeed trivial). The homogeneity of the collapse poset and the elementarity of i and j now yield the desired conclusion. $\dashv$

Proof of 3.1 (Sketch). If 3.1 fails, then a simple Löwenheim-Skolem argument produces a countable, ω-iterable, ω-Woodin mouse $\mathcal{M}$, such that for some symmetric collapse $\mathbb{R}^*_{\mathcal{M}}$ of $\mathcal{M}$, and some $\alpha \in OR^{\mathcal{M}}$, $L_\alpha(\mathbb{R}^*_{\mathcal{M}}) \models \neg\mathrm{AD}$. Let α_0 be the least such α, and let θ be the sentence "$\neg$AD". Every set in $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ is ordinal definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ from some real; to simplify notation, let us assume the real is 0 in our case; that is, there is a $B \subseteq \mathbb{R}^*_{\mathcal{M}}$, ordinal definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ such that $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models$ "B is not determined". By minimizing the ordinal parameters, we may assume that B is, in fact, definable over $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}})$ from no parameters. Say $B = \{x \mid L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \phi[x]\}$.

Let $\delta = \delta^{\mathcal{M}}$, and let $\delta_0 < \delta$ be the least Woodin cardinal of M. Let $\dot{A}$ be the following name for a set of reals in $M^{\mathrm{col}(\omega,\delta_0)}$:

$$\begin{aligned}\dot{A} = \{\langle \dot{x}, p\rangle \mid\ & \dot{x} \text{ is a name for a real in } M^{\mathrm{col}(\omega,\delta_0)},\\ & p \in \mathrm{col}(\omega, \delta_0), \text{ and}\\ & p \Vdash_{\mathrm{col}(\omega,\delta_0)} \text{“} \Vdash_{\mathrm{col}(\omega,<\delta)} \text{“}L_{\alpha_0}(\mathbb{R}^*) \models \phi[\check{\dot{x}}],\\ & \qquad \text{where } \alpha_0 \text{ is the least } \alpha \text{ such that } L_{\alpha_0}(\mathbb{R}^*) \models \neg\mathrm{AD}\text{” ”}\}.\end{aligned}$$

We now use Lemma 1.7 with $\dot{A}$. We may assume Case I holds, as otherwise we can just replace $\dot{A}$ with its complement. (Notice that Case II cannot hold for both, as then we could just pit the relevant strategies against each other and get a contradiction.)

Let $g \in \mathbb{R}^*_{\mathcal{M}}$ be M-generic $/\mathrm{col}(\omega, \delta_0)$. Lemma 1.7 gives us a strategy $\hat{\sigma} \in M[g]$ which is winning for I in $\widehat{G}(\dot{A})$. From $\hat{\sigma}$ we can obtain a strategy σ by ignoring I's auxilliary moves. Since $\sigma \in \mathbb{R}^*_{\mathcal{M}}$, σ is not winning for I in $G(B)$, and so we can fix a play x by σ such that $x \in \mathbb{R}^*_{\mathcal{M}} \setminus B$. Let x, T, h be the corresponding play by $\hat{\sigma}$.

Since $\mathcal{M}$ is ω-iterable, we have a cofinal branch b of T such that $\mathcal{M}_b$ is ω-iterable. Since $\hat{\sigma}$ is winning, $x \in i_b(\dot{A})_{\hat{h}(b)}$, where $\hat{h}(b)$ is the M_b-generic on $\operatorname{col}(\omega, i_b(\delta_0))$ produced by h. Inspecting $\dot{A}$, we see that this means that for some symmetric collapse $\mathbb{R}^*_{\mathcal{M}_b}$ of $\mathcal{M}_b$, there is a β such that $L_\beta(\mathbb{R}^*_{\mathcal{M}_b}) \models \neg\mathrm{AD}$, and for β_0 the least such β, $L_{\beta_0}(\mathbb{R}^*_{\mathcal{M}_b}) \models \phi[x]$. But $x \notin B$, so $L_{\alpha_0}(\mathbb{R}^*_{\mathcal{M}}) \models \neg\phi[x]$, and this contradicts Lemma 3.2. $\dashv$

The proof of 3.2 also shows that if $\mathcal{M} = \langle M, \in, \delta\rangle$ is an ω-Woodin mouse with $OR \subseteq M$ and V^M_δ countable, and if $\mathcal{M}$ is (forced to be) ω-iterable in $V^{\operatorname{col}(\omega, 2^{\aleph_0})}$, then $L(\mathbb{R}) \equiv L(\mathbb{R}^*_{\mathcal{M}})$ for any symmetric collapse $\mathbb{R}^*_{\mathcal{M}}$ of M. So the existence of such an $\mathcal{M}$ implies $\mathrm{AD}^{L(\mathbb{R})}$. The existence of such an $\mathcal{M}$ follows, using the proof of 1.2, from the existence of ω Woodin cardinals with a measurable above them all.

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