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FREGE'S THEOREM AND THE PEANO POSTULATES

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Two thoughts about the concept of number are incompatible: that any zero or more things have a (cardinal) number, and that any zero or more things have a number (if and) only if they are the members of some one set. It is Russell's paradox that shows the thoughts incompatible: the sets that are not members of themselves cannot be the members of any one set. The thought that any (zero or more) things have a number is Frege's; the thought that things have a number only if they are the members of a set may be Cantor's and is in any case a commonplace of the usual contemporary presentations of the set theory that originated with Cantor and has become ZFC.

In recent years a number of authors have examined Frege's accounts of arithmetic with a view to extracting an interesting subtheory from Frege's formal system, whose inconsistency, as is well known, was demonstrated by Russell. These accounts are contained in Frege's formal treatise *Grundgesetze der Arithmetik* and his earlier exoteric book *Die Grundlagen der Arithmetik*. We may describe the two central results of the recent re-evaluation of his work in the following way: Let *Frege arithmetic* be the result of adjoining to full axiomatic second-order logic a suitable formalization of the statement that the Fs and the Gs have the same number if and only if the Fs and the Gs are equinumerous. Then (1) Frege himself succeeded in deriving arithmetic from Frege arithmetic and (2) Frege arithmetic is equi-consistent with full second-order arithmetic (and is thus consistent, with moral certainty). So Frege derived arithmetic from a single consistent and obvious-seeming principle, if not from logic alone, as he had hoped he had. A number of the articles responsible for the re-evaluation of Frege's formal work are collected in William Demopoulos' anthology, *Frege's Philosophy of Mathematics*.[1]

Frege, we now see, thus provided a consistent theory of the natural numbers altogether different from those of Dedekind, Russell and Whitehead, Zermelo, von Neumann, and the several founders of combinatory logic and the lambda-calculus. Particularly noteworthy is the difference between

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[1]William Demopoulos, ed., *Frege's Philosophy of Mathematics*, Harvard University Press, Cambridge, Massachusetts, 1995.

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Frege's account and that found in *Principia Mathematica*: for Frege, numbers are certain objects that lie at the bottom level of a type-theoretical hierarchy, while for Russell and Whitehead, they are situated two levels above the bottom of a similar hierarchy.

Here we investigate the Fregean conception of number; we are going to show that on Fregean definitions of *zero*, *(immediately) precedes*,[2] and *natural number*, but without use of the aforementioned principle, the Peano postulates contain a notable redundancy, whose discovery can, with only a small amount of exaggeration, be attributed to Frege, since the proof of redundancy is an adaptation of Frege's proof that the series of natural numbers is infinite. Thus the study of the details of Frege's work has yielded a result concerning the extremely familiar Peano axiomatization. We begin the investigation by recalling an old observation of Henkin's concerning a dependency among the Peano postulates.[3]

Suppose that the Dedekind-Peano postulates are formulated as usual in a second-order language containing a constant $\mathbf{0}$ (for zero) and a one-place function sign $\mathbf{s}$ (for successor):

(i) $\forall x\, \mathbf{0} \neq \mathbf{s}x$
(ii) $\forall x \forall y(\mathbf{s}x = \mathbf{s}y \rightarrow x = y)$
(iii) Mathematical induction: $\forall F(F\mathbf{0} \wedge \forall x(Fx \rightarrow F\mathbf{s}x) \rightarrow \forall x Fx)$.

Henkin observed that (iii) *implies the disjunction of* (i) *and* (ii).

We may see this as follows: by (iii), the range of the first order variables is the class of denotations of $\mathbf{0}$, $\mathbf{s0}$, $\mathbf{ss0}$, ... Either these denotations are all distinct, in which case (i) and (ii) both hold, or they are not and for some least m, $\mathbf{s}^m\mathbf{0}$ denotes the same object as $\mathbf{s}^n\mathbf{0}$, for some $n > m$. If $m = 0$, (i) fails but (ii) holds; if $m > 0$, (ii) fails but (i) holds.

It is easy to construct models in which each of the seven conjunctions $\pm$(i) $\pm$ (ii) $\pm$ (iii) other than $-$(i) $-$ (ii) $+$ (iii) holds; so no other dependencies among (i), (ii), and (iii) await discovery.

Our main interest is in the Fregean conception of number and we shall accordingly follow Frege in not *assuming* that the relation *precedes* is the graph of a total function on the natural numbers, or even a functional relation. Accordingly, we formulate the Peano postulates in the language $\{\mathbf{0}, P, N\}$, with P a binary predicate letter for 'precedes' and N a unary predicate letter for 'is a natural number', as the universal closures of the following seven formulas:

(a) $N\mathbf{0}$

[2] We use 'precedes' to mean 'immediately precedes' (rather than 'is less than').

[3] Leon Henkin, "On mathematical induction", *American Mathematical Monthly* 67, April 1960, 323-338.

(b) $Nm \wedge mPn \rightarrow Nn$
(c) $Nm \wedge mPn \wedge mPn' \rightarrow n = n'$
(d) $Nm \rightarrow \exists n \ \ mPn$
(e) $Nn \rightarrow \neg nP\mathbf{0}$
(f) $Nm \wedge Nm' \wedge mPn \wedge m'Pn \rightarrow m = m'$
(g) $F\mathbf{0} \wedge \forall m \forall n(Fm \wedge mPn \rightarrow Fn) \rightarrow Fn.$

The background logic of our investigation will be axiomatic second-order logic.

(g) is of course a formulation of mathematical induction. It is immediately implied by the stronger-looking

(h) $F\mathbf{0} \wedge \forall m \forall n(Nm \wedge Fm \wedge mPn \rightarrow Fn) \rightarrow Fn.$

But (h) is equivalent to (g) in the presence of (a) and (b): Let $Gn \equiv Nn \wedge Fn$. Then (a), (b) and the antecedent of (h) yield $G\mathbf{0}$ and $\forall m \forall n(Gm \wedge mPn \rightarrow Gn)$; so by (g), Gn, whence Fn.

The conjunction of (a), (b) and (g) is equivalent to the celebrated Dedekind-Frege-Russell definition of N from $\mathbf{0}$ and P:

Def N $\quad \forall x(Nx \leftrightarrow \forall F(F\mathbf{0} \wedge \forall m \forall n(Fm \wedge mPn \rightarrow Fn) \rightarrow Fx)).$

We first examine the dependencies among (c), (d), (e) and (f) under the assumption of Def N. It will help matters if we give (c), (d), (e), and (f) more suggestive designations. Thus we dub (c) **F**unctionality, (d) **T**otality, (e) **E**xiledom (the sequence of natural numbers never returns to its point of origin), and (f) **I**njectivity.

We shall use: $\vdash^*$ to denote derivability from Def N (in axiomatic second-order logic).

Formalizing Henkin's argument, we see that $\vdash^* \mathbf{F} \wedge \mathbf{T} \rightarrow \mathbf{E} \vee \mathbf{I}$.

In a similar vein, we may see that $\vdash^* \mathbf{F} \wedge \neg\mathbf{T} \rightarrow \mathbf{E} \wedge \mathbf{I}$, by formalizing the following argument: If **F** holds, every object satisfying Nx bears (the) P(-relation) to at most one object; if $\neg\mathbf{T}$ holds, some object satisfying Nx bears P to no object. By (a), (b) and (g), the objects satisfying Nx are just the denotation a_0 of $\mathbf{0}$, the unique object a_1, if it exists, to which a_0 bears P, the unique object a_2, if it exists, to which a_1 bears P, ... Now if $i < j$ and a_j exists, then a_i exists, but a_j does not bear P to a_i; otherwise we have a loop and so **T** holds; nor can every a_i exist, for then, since every a_i bears P to a_{i+1}, **T** again holds. So the a_is eventually give out without ever repeating. But then **E** and **I** both hold.

So we have

$$\vdash^* \neg(\mathbf{F} \wedge \mathbf{T} \wedge \neg\mathbf{E} \wedge \neg\mathbf{I}) \quad \text{(Henkin)},$$

as well as

$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \mathbf{E} \wedge \neg\mathbf{I}),$$
$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \neg\mathbf{E} \wedge \mathbf{I}), \quad \text{and}$$
$$\vdash^* \neg(\mathbf{F} \wedge \neg\mathbf{T} \wedge \neg\mathbf{E} \wedge \neg\mathbf{I}).$$

We shall discuss the twelve other possibilities below. One of them, however, deserves mention here: It is not the case that $\vdash^* \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \to \mathbf{T}$, for (a), (b), (g), **F**, **E** and **I** are all true and **T** false in the model consisting of a single element which does *not* bear P to itself; of course, that single element will be the denotation of **0**, and it will therefore satisfy Nx.

The use of the constant **0**, rather than a monadic predicate letter Z (for 'is *a* zero') in the formulation of **E** and Def N is important. In particular, the assumption that there are not two zeros has significant consequences. For suppose we reformulate the Peano Postulates in the language $\{Z, P, N\}$ as the universal closures of:

(a′)	$Zn \to Nn$
EZ	$\exists nZn$
UZ	$Zm \wedge Zn \to m = n$
(b)	$Nm \wedge mPn \to Nn$
F	$Nm \wedge mPn \wedge mPn' \to n = n'$
T	$Nm \to \exists n\, mPn$
E′	$Zm \wedge Nn \to \neg nPm$
I	$Nm \wedge Nm' \wedge mPn \wedge m'Pn \to m = m'$
(g′)	$\forall F(\forall x(Zx \to Fx) \wedge \forall d\forall a(Fd \wedge dPa \to Fa) \to \forall x(Nx \to Fx))$

and, correspondingly, redefine N:

Redef N $\quad \forall n(Nn \leftrightarrow \forall F(\forall x(Zx \to Fx) \wedge \forall d\forall a(Fd \wedge dPa \to Fa) \to Fn))$.

We can derive (a′) $\wedge$ (b) $\wedge$ (g′) from Redef N, and thus also derive (h′), i.e.,

$$\forall F(\forall x(Zx \to Fx) \wedge \forall d\forall a(Nd \wedge Fd \wedge dPa \to Fa) \to \forall x(Nx \to Fx)).$$

But not only can $\mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \to \mathbf{T}$ not be derived (in axiomatic second-order logic) from Redef $N \wedge$ EZ $\wedge$ UZ, as the model with a single element a such that Za, $\neg aPa$, and Na shows, neither can any of

$$\mathbf{F} \wedge \mathbf{T} \to \mathbf{E}' \vee \mathbf{I},$$
$$\mathbf{F} \wedge \neg\mathbf{T} \to \mathbf{E}' \vee \mathbf{I},$$
$$\mathbf{F} \wedge \neg\mathbf{T} \to \mathbf{E}' \vee \neg\mathbf{I}, \quad \text{and}$$

$$\mathbf{F} \wedge \neg\mathbf{T} \to \neg\mathbf{E}' \vee \mathbf{I}$$

be derived from Redef $N \wedge$ EZ, as the following easy countermodels respectively show:

$$\langle\{a, b, c\}, \{a, b\}, \{aa, bc, cc\}\rangle,$$
$$\langle\{a, b, c, d\}, \{a, b, c\}, \{aa, bd, cd\}\rangle,$$
$$\langle\{a, b, c\}, \{a, b\}, \{aa, bc\}\rangle, \quad \text{and}$$
$$\langle\{a, b, c\}, \{a, b\}, \{ac, bc\}\rangle.$$

$(a, b, c, d$ are any four objects. The members of the second set in each model are the objects satisfying Z there. The members of the third are the ordered pairs satisfying P. In each model all objects satisfy N.)

So, as will appear later, **F**, **T**, **E**, **I** are totally independent if Redef N and EZ but nothing else is assumed.

We now add to the treatment of number given so far the distinctly Fregean ingredient. It will be of interest to stick with the language $\{Z, P, N\}$ for the time being.

Frege held that numbers are objects, but that what he called "a statement of number", i.e., a statement saying how many things there are of a certain kind,[4] contains an assertion about a concept. Concepts, according to Frege, are denotations of predicates, or values of monadic second-order variables; objects are denoted by terms that may appear on either side of the sign of identity. To express this aspect of Frege's account of number, we introduce a sign $^{\#}$ for a totally defined function. $^{\#}$ is read "the number of things that ... " and is a function-sign of "mixed type": when attached to a monadic second-order variable F, $^{\#}$ produces a term $^{\#}F$ of the same type as a first-order variable n. So $n = {}^{\#}F$ is well-formed. We emphasize that $^{\#}$ is to denote a total function and thus that $\forall F \exists! n\ n = {}^{\#}F$ will be valid. We assume that the axioms and rules of second-order logic guarantee in some manner the derivability of $\forall F \exists! n\ n = {}^{\#}F$.

We now define Z and P:

Def Z $\quad \forall n(Zn \leftrightarrow \exists F(n = {}^{\#}F \wedge \forall x \neg Fx));$

Def P $\quad \forall m \forall n(mPn \leftrightarrow \exists F \exists G \exists y(Gy \wedge \forall x(Fx \leftrightarrow Gx \wedge x \neq y) \wedge m = {}^{\#}F \wedge n = {}^{\#}G).$

Def P is a slight reformulation of Frege's definition of 'precedes'.

We shall use: $\vdash'$ to denote derivability from Def Z, Def P, and Redef N.

According to Def Z, a zero is the number belonging to some empty concept.

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[4]One of Frege's examples is "Jupiter has four moons".

By Def Z alone, $\vdash'$ EZ, for $\exists F \forall x(Fx \leftrightarrow x \neq x)$ is one of the comprehension axioms of second-order logic, and therefore $\vdash' \exists F \forall x \neg Fx$, whence $\vdash' \exists n \exists F(n = {}^{\#}F \wedge \forall x \neg Fx)$, i.e., $\vdash' \exists n Zn$. Also by Def Z alone,

$$\vdash' \mathrm{UZ} \leftrightarrow \forall F \forall G(\forall x \neg Fx \wedge \forall x \neg Gx \rightarrow {}^{\#}F = {}^{\#}G),$$

whose right side may be read: Empty concepts have the same number.

"Empty concepts ... " is an immediate consequence of *Functional Extensionality (FE)*

$$\forall x(Fx \leftrightarrow Gx) \rightarrow {}^{\#}F = {}^{\#}G,$$

which could well be considered a truth of logic, as it is valid under any extensional semantics for languages containing the function sign $^{\#}$. For if $\forall x(Fx \leftrightarrow Gx)$ is true, then the same class of objects will be assigned to the variables F and G, and then ${}^{\#}F = {}^{\#}G$ will be true.[5] The converse of FE is inconsistent, as the argument of Russell's paradox shows. But FE is itself an immediate consequence of *Hume's principle*

$${}^{\#}F = {}^{\#}G \leftrightarrow F \approx G,$$

where $F \approx G$ is some standard second-order formula for equinumerosity. By now it is well known than Hume's principle is consistent. However, Hume's principle holds in no finite domain and cannot be considered a truth of logic.

Our principal result is that $\mathbf{T}$ can be derived from $\mathbf{F}$, $\mathbf{E}'$, and $\mathbf{I}$ in $\vdash'$, i.e., that $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}$, which is perhaps surprising in view of the earlier observation that $\nvdash^{*} \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}$.

To prove this theorem we define the ancestral R^* of an arbitrary relation R:

$$xR^*y \equiv \forall F(\forall a(xRa \rightarrow Fa) \wedge \forall d \forall a(Fd \wedge dRa \rightarrow Fa) \rightarrow Fy).$$

We write $xR^{*=}y$ to mean $xR^*y \vee y = x$.

Among the consequences of the definitions of R^* and $R^{*=}$ are:

(1) $\forall x \forall y(xRy \rightarrow xR^*y)$,
(2) $\forall x \forall y(xR^*y \wedge yR^*z \rightarrow xR^*z)$,
(3) $\forall x \forall y(xR^*y \rightarrow \exists z(xR^{*=}z \wedge zRy))$,
(4) $\forall x \forall y(Fx \wedge xRy \rightarrow Fy) \rightarrow \forall x \forall y(Fx \wedge xR^{*=}y \rightarrow Fy)$

and

(5) $\forall x \forall y(xR^{*=}y \leftrightarrow \forall F(Fx \wedge \forall d \forall a(Fd \wedge dRa \rightarrow Fa) \rightarrow Fy))$.

Frege's definition of 'natural number' was $\forall n(Nn \leftrightarrow \mathbf{0}P^{*=}n)$, rather than Def N, but by (5), the definitions are equivalent.

[5] FE is further discussed in my "On the proof of Frege's theorem", to appear in a *Festschrift* for Paul Benacerraf edited by Adam Morton and Stephen Stich.

We now prove that $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}$.

LEMMA 1. $\vdash' \mathbf{E}' \wedge \mathbf{I} \rightarrow (Nx \rightarrow \neg xP^*x)$.

PROOF. Suppose Zx and xP^*x. By (a′), Nx. By (3), for some z, $xP^{*=}z$ and zPx. By (b′) and (4), Nz, contra $\mathbf{E}'$. So $\forall x(Zx \rightarrow \neg xP^*x)$.

Suppose Nd, dPa and aP^*a. By (b′), Na. By (3), for some z, $aP^{*=}z$ and zPa. By (1) and (2), zP^*z. By (b′) and (4), Nz. By $\mathbf{I}$, $z = d$. Thus dP^*d. So $\forall d\forall a(Nd \wedge \neg dP^*d \wedge dPa \rightarrow \neg aP^*a)$.

By (h′) (and the comprehension scheme of second-order logic), the lemma follows. (The proof does not cite Def Z or Def P, by the way.) ⊣

LEMMA 2. $\vdash' \mathbf{E}' \rightarrow (Za \rightarrow \exists G(aP\,^{\#}G \wedge \forall x(Gx \leftrightarrow Nx \wedge xP^{*=}a)))$.

PROOF. Suppose Za. Then by Def Z, for some F, $\forall x\neg Fx$ and $a = {}^{\#}F$. By second-order logic, for some G, $\forall x(Gx \leftrightarrow Nx \wedge xP^{*=}a)$. To show $aP\,^{\#}G$ it is enough, by Def P, to show that for some y, Gy and $\forall x(Fx \leftrightarrow Gx \wedge x \neq y)$. Let $y = a$. Since Za, Na; and trivially, $aP^{*=}a$. So Ga.

Suppose Gx and $x \neq a$. Then Nx and xP^*a. By (3), for some z, $xP^{*=}z$ and zPa. Since Nx and $xP^{*=}z$, Nz, by (b′) and (4). So Nz, zPa, and Za, contra $\mathbf{E}'$. Since $\forall x\neg Fx$, $\forall x(Fx \leftrightarrow Gx \wedge x \neq y)$. ⊣

LEMMA 3. $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow (Nd \wedge dPa \wedge dP\,^{\#}F \wedge \forall x(Fx \leftrightarrow Nx \wedge xP^{*=}d) \rightarrow \exists G(aP\,^{\#}G \wedge \forall x(Gx \leftrightarrow Nx \wedge xP^{*=}a)))$.

PROOF. Suppose Nd, dPa, $dP\,^{\#}F$, and $\forall x(Fx \leftrightarrow Nx \wedge xP^{*=}d)$. By second-order logic, for some G, $\forall x(Gx \leftrightarrow Nx \wedge xP^{*=}a)$. We must show $aP\,^{\#}G$.

By $\mathbf{F}$, $a = {}^{\#}F$. So it is enough to show ${}^{\#}FP\,^{\#}G$. Since Nd and dPa, Na; and trivially $aP^{*=}a$. So by Def P it suffices to show

$$\forall x(Nx \wedge xP^{*=}a \wedge x \neq a \leftrightarrow Nx \wedge xP^{*=}d).$$

Assume Nx.

Suppose $xP^{*=}a$ and $x \neq a$. Then xP^*a and by (3), for some z, $xP^{*=}z$ and zPa. Since Nx and $xP^{*=}z$, Nz by (b′) and (4). By $\mathbf{I}$, $z = d$, whence $xP^{*=}d$.

Conversely, suppose $xP^{*=}d$. Since dPa, xP^*a, whence $xP^{*=}a$. If $x = a$, then xP^*x, contra Lemma 1. So $x \neq a$. ⊣

THEOREM 1. $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}$.

PROOF. By comprehension, (h′), and Lemmas 2 and 3,

$$\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow (Nm \rightarrow \exists G(mP\,^{\#}G \wedge \forall x(Gx \leftrightarrow Nx \wedge xP^{*=}m))).$$

Thus $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow (Nm \rightarrow \exists n\, mPn)$, i.e., $\vdash' \mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}$. ⊣

Let us now write $\vdash$ to mean derivability from "Empty concepts have the same number", Def $\mathbf{0}$, Def P and Def N, where Def $\mathbf{0}$ is $\forall n(n = \mathbf{0} \leftrightarrow \exists F(n = {}^{\#}F \wedge \forall x \neg Fx))$.

We then have

$$\vdash \forall n(Zn \leftrightarrow n = \mathbf{0}) \rightarrow \text{Def } Z \wedge \text{Redef } N.$$

Combining our results, we have

$$\begin{aligned} &\vdash \mathbf{F} \wedge \mathbf{T} \rightarrow \mathbf{E} \vee \mathbf{I}, \\ &\vdash \mathbf{F} \wedge \neg\mathbf{T} \rightarrow \mathbf{E} \wedge \mathbf{I}, \quad \text{and} \\ &\vdash \forall n(Zn \leftrightarrow n = \mathbf{0}) \rightarrow (\mathbf{F} \wedge \mathbf{E}' \wedge \mathbf{I} \rightarrow \mathbf{T}). \end{aligned}$$

Substituting $\cdot = \mathbf{0}$ for $Z\cdot$, we have

$$\vdash \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}.$$

It follows (by propositional logic) that

$$\vdash \mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}).$$

We now show that this result is best possible in the sense that no stronger truth-functional compound of $\mathbf{F}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{I}$ can be derived.

Let

$$\begin{aligned} [1] &= \{\mathbf{F}, \mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [2] &= \{\mathbf{F}, \mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [3] &= \{\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \\ [4] &= \{\neg\mathbf{F}, \mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [5] &= \{\neg\mathbf{F}, \mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [6] &= \{\neg\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \\ [7] &= \{\neg\mathbf{F}, \mathbf{T}, \neg\mathbf{E}, \neg\mathbf{I}\}, \\ [8] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \mathbf{I}\}, \\ [9] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \neg\mathbf{I}\}, \\ [10] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \neg\mathbf{E}, \mathbf{I}\}, \quad \text{and} \\ [11] &= \{\neg\mathbf{F}, \neg\mathbf{T}, \neg\mathbf{E}, \neg\mathbf{I}\}. \end{aligned}$$

Since $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$ is truth-functionally equivalent to $\bigwedge[1] \vee \cdots \vee \bigwedge[11]$, in order to show that $\vdash A$ for no truth-functional compound A of $\mathbf{F}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{I}$ stronger than $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$, it will suffice to provide a model $\langle D, f \rangle$ of each $[i]$, $1 \leq i \leq 11$. Henceforth we write: mPn to mean: P applies to m, n. In each of the eleven models, $f : \mathcal{P}D \rightarrow D$; and so "empty concepts" and indeed FE will hold; $\mathbf{0}$ will denote $f\emptyset$ and so Def $\mathbf{0}$ will hold; mPn if and only if there exist A, B and y in B such that $m = fA$, $n = fB$, $A \subseteq B$

and $A = B - \{y\}$, and so Def P will hold; and N will apply to $f\emptyset$ and those elements of D to which $f\emptyset$ bears P^* and so Def N will hold.

[1]: Let $D = N = \{\text{the naturals}\}$. Let $fA = n+1$ if $|A| = n$, $fA = 0$ if $|A|$ is infinite. Then if $m > 0$, mPn if and only if $n = m+1$ and $0Pn$ if and only if $n = 0$.

[2]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$ and $f\{0\} = f\{1\} = f\{0, 1\} = 1$. Then $0P1$, $1P1$, not: $0P0$, and not: $1P0$.

[3]: Let $D = \{0\}$. Let $f\emptyset = f\{0\} = 0$. Then $0P0$.

[4]: The model for [4] is the most complicated one. Let $D = N$, $E = \{\text{the evens}\}$, $O = \{\text{the odds}\}$. Let $f\emptyset = 1$, $fE = 1$, $fA = 2n+1$ if $|A| = n$, $fA = 2n$ if for some $B \subseteq O$, $0 < |B| = n$ and $A = E \cup B$, and $fA = 0$ otherwise. Then we have $1P3P5P7P\cdots$ and $1P2P4P6P\cdots$. Moreover, $0P0$ and $0P1$, and $0P2$, $0P4$, $0P6$, $\ldots$, and P holds between no other pairs except those indicated.

F fails because $1P3$ and $1P2$. **T** clearly holds. But **E** and **I** hold too, because the objects to which N applies in this model are the positive integers, and the only pairs of positive integers among which P holds are those indicated in: $1P3P57P\cdots$ and $1P2P46P\cdots$.

[5]: Let $D = \{0, 1, 2, 3\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 2$ and $fA = 3$ otherwise. Then $0P1$, $0P2$, $1P3$, $2P3$, $3P3$ and nothing else holds.

[6]: Let $D = N$. Let $fA = n$ if $|A| = n$, $fA = 0$ if A is infinite. Then $\mathbf{0}$ denotes 0; $0P0$, $0P1$, $1P2$, $2P3$, $\ldots$ and nothing else holds.

[7]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 0$ and $f\{0, 1\} = 0$. Then $0P0$, $0P1$, $1P0$, but not: $1P1$.

[8]: Let $D = N$. Let $f\emptyset = f(N - \{0\}) = 2$, $fN = 1$, $fA = n+2$ if $|A| = n$, and $fA = 0$ otherwise. Then $0P0$, $0P2$, $0P1$, $2P1$, $2P3P4P\cdots$ and nothing else holds. Here $\mathbf{0}$ denotes 2 and the objects to which N applies are 2, 3, 4, $\ldots$ and also 1. Since $2P1$ and $2P3$, **F** fails; since for all a, not: $1Pa$, **T** fails. But **E** and **I** hold, since although $0P2$ and $0P1$, N does not apply to 0.

[9]: Let $D = \{0, 1, 2, 3\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 2$, $f\{0, 1\} = 1$, $f\{0, 2\} = 2$, $f\{0, 1, 2\} = 1$, $f\{1, 2, 3\} = 2$, $f\{1, 2\} = 2$, $f\{1, 3\} = 1$, $f\{0, 1, 2, 3\} = 3$, and $fA = 1$ otherwise. Then $0P1$, $0P2$, $1P1$, $2P2$, $1P2$, $2P1$, $1P3$, $2P3$ and nothing else holds.

[10]: Let $D = \{0, 1\}$. Let $f\emptyset = 0$, $f\{0\} = f\{1\} = 0$, and $f\{0, 1\} = 1$. Then $0P0$ and $0P1$, but not: $1P0$ and not: $1P1$.

[11]: Let $D = \{0, 1, 2\}$. Let $f\emptyset = 0$, $f\{0\} = 1$, $f\{1\} = 0$, $f\{2\} = 0$, $f\{0, 1\} = 0$, $f\{0, 2\} = 0$, $f\{1, 2\} = 1$ and $f\{0, 1, 2\} = 2$. Then $0P0$, $0P1$, $1P0$, $0P2$, $1P2$, and nothing else holds.

So if $\vdash A$, A a truth-functional compound of **F**, **T**, **E**, **I**, then $\vdash (\mathbf{F} \to \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I})) \to A$.

In any model for the language $\{^{\#}\}$ of the form $\langle\{a\}, f\rangle$, it must be that aPa, since $f\emptyset = f\{a\} = a$. Thus the model described far above consisting of a single element not bearing P to itself, which shows that

$$\not\vdash^{*} \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T},$$

cannot be converted to a model for $\{^{\#}, \mathbf{0}, P, N\}$ of $\{\mathbf{F}, \neg\mathbf{T}, \mathbf{E}, \mathbf{I}\}$ (and Def $\mathbf{0}$, Def P, Def N and "Empty concepts ... ").

The central argument of Frege's *Foundations of Arithmetic* is a sketch of a proof that every natural number precedes some object (which will of course itself be a natural number). The argument sketched by Frege proceeds by showing that $\mathbf{0}P\,^{\#}[\,x : xP^{*=}\mathbf{0}\,]$ and that if d is a natural number[6], dPa and $dP\,^{\#}[\,x : xP^{*=}d\,]$, then $aP\,^{\#}[\,x : xP^{*=}a\,]$. Our proof that $\vdash' \mathbf{F} \wedge \mathbf{E} \wedge \mathbf{I} \rightarrow \mathbf{T}$ is a modification of Frege's argument: instead of showing that if n is a natural number, $nP\,^{\#}[\,x : xP^{*=}n\,]$, we in effect show from $\mathbf{F}$, $\mathbf{E}$ and $\mathbf{I}$ that any natural number n is the number of some concept G such that $\forall x(Gx \leftrightarrow 0P^{*=}xP^{*=}n)$. Frege's proof appealed to Hume's principle, which, as Frege showed, is also sufficient for $\mathbf{F}$, $\mathbf{E}$, and $\mathbf{I}$; our derivation of $(\mathbf{F} \rightarrow \mathbf{T} \wedge (\mathbf{E} \vee \mathbf{I}))$ appeals only to "empty concepts have the same number" and the definitions from $^{\#}$ of $\mathbf{0}$, P, and N.

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[6]Frege realized that he needed this condition on d only in *Grundsetze der Arithmetik* and not in *Die Grundlagen der Arithmetik*. For more details on his error, see "*Die Grundlagen der Arithmetik*, §§82–3, by George Boolos and Richard G. Heck, Jr., to appear in *Philosophy of Mathematics Today*, edited by Matthias Schirn.