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MEASURES: BACK AND FORTH BETWEEN POINT SETS AND LARGE SETS

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It was questions about points on the real line that initiated the study of set theory. Points paved the way to point sets and these to ever more abstract sets. And there was more: Reflection on structural properties of point sets not only *initiated* the study of ordinary sets; it also supplied blueprints for defining extra-ordinary, "large" sets, transcending those provided by standard set theory. In return, the existence of such large sets turned out critical to settling open conjectures about point sets.

How to explain such action at a distance between the very large and the rather small? Rather than having an air of magic, could these results rest on deep structural similarities between the two superficially distant species of sets?

In this essay I dissect one group of such two-way results. Their linchpin is the notion of *measure*.[1]

§1. Vitali's impossibility result. Our starting point is a problem in measure theory regarding the notion of "Lebesgue measure." Before presenting the problem, I would like to review the notion of Lebesgue measure. Rather than listing its main properties, I would like to show how Lebesgue measure is born out of an attempt to generalize the notion of the length of an interval to arbitrary sets of reals. One tries to approximate arbitrary sets of reals by intervals, in the hope that the lengths of the intervals will induce a measure on these sets. The simplest case of such an approximation is that

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[1]Throughout this paper I avoid the question of whether large sets, usually referred to as "large cardinals," "really exist." I substitute the "whether" with a "where": Bracketing away the "philosophical" question—*whether* large cardinals exist—I rather focus on the mathematical question: *Where* exactly, in mathematical practice, does one encounter such sets? It is the main point of what follows that one does not have to be a specialist of abstract set theory to get involved with such "exotica": Led by mundane questions of measure theory and analysis, the most hum-drum mathematician is destined to encounter such "large" cardinals. I also believe that viewing large cardinals axioms as formulating *mathematical* (rather than distinctly set-theoretical) hypotheses takes away some of the mystery related to their "action at a distance" mentioned above. I relate to the more "philosophical" questions in a twin paper, "Set theory as a branch of mathematics."

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of approximating a point (on the real line) by intervals; since a point can be covered by intervals of arbitrary small length, its measure (or, to be precise, the measure of the singleton containing this point) ought to be zero. Thus the purported measure will be one that *vanishes on points*. If we let m denote our measure, then for every real number r, $m(\{r\}) = 0$.

We should note here that in the above example we were relying on our intuition that our notion of measure ought to be *monotone*: If one set is contained in another, then the "smaller" set should have measure less than or equal to that of the "larger" one.

Some sets cannot be so readily approximated by (a sequence of) *single* intervals. The simplest example is the disjoint union of two intervals. Here we need a principle that will help us combine the measures of the smaller sets to gauge the measure of the larger set. With the above example, it seems clear that we should opt for the principle that says that the measure of the union of the intervals is the sum of their measures. In fact, that should hold for the disjoint union of a pair of *arbitrary* sets. Using the induction principle, we can show that this requirement is equivalent to that of *finite additivity*: For any finite sequence of pairwise disjoint sets $\langle A_i : i < n\rangle$, we would like to have that $m(\bigcup_{i<n} A_i) = \sum_{i<n} m(A_i)$.

Should our measure abide by stronger principles of additivity? A principle which says that the measure of *any* disjoint union of sets is the sum of the measures of the individual sets is plainly false: *Any* set can be written as the disjoint union of the singletons of its members. Thus the interval $(0, 1)$ is the union of singleton r's, for reals r in $(0, 1)$; the entire real line is the disjoint union of all singletons of real numbers. Clearly, even though the measure of each singleton is zero, the unit interval or the real line will have positive measure. The problem lies with the *size* of the union: A *large* union of small sets may be large. In the above examples the union was definitely "large," being the size of the continuum. On the other hand, the measure on intervals is *countably additive*: If an interval is decomposed into countably many disjoint intervals, then the length of the original interval is the (infinite) sum of the lengths of the "component" intervals. This suggests that we may consider countable unions as "small." Thus we get the requirement of *countable additivity*: If $\langle A_i : i \in \omega\rangle$ is a countable sequence of disjoint sets, then $m(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} m(A_i)$.

Finally, the property of *translation invariance* suggests itself. Informally put: The measure of a given set is independent of its location on the real line. We can "move it around" without affecting its measure. Formally, we define the notion of a "shift" of a set as follows: A set Y is a shift of the set X (by the real number d) if Y is the "result" of adding d to every number in X (in symbols: $Y = X + d$, where $X + d$ is the set of all y's such that for some x in X, $y = x + d$.) *Translation invariance* says that if Y is a shift of X then X and Y have the same measure. This property is true for intervals measured by their lengths; and it ought to be true for the notion of measure on arbitrary sets of reals. It ought to be true for two reasons: First, our project is to approximate arbitrary sets by intervals, and it seems that translation invariance should be "passed on" from the measure on the intervals to the measure on arbitrary sets. But also, this requirement seems a precondition on any reasonable notion of measure on subsets of the *real line*: The notion of "moving a set around" the real line seems better understood than the notion of measure for an arbitrary set. How could the *size* of a subset of the real line change if all we did was move each point in it by some fixed number of units to the right (or to the left)?

We now know what principles we would like our measure to satisfy and (roughly) how we should set about finding the measure of an arbitrary set of reals. The question is: Is there such a measure on the real line, measuring *every* subset of it? Lebesgue measure is a measure on sets of reals which is obtained precisely by following the guidelines mentioned above.[2] Is Lebesgue measure total?

The turn of the century mathematician, Vitali, was the first to realize it was not. He relied on the axiom of choice in constructing a non-measurable set. In fact, his proof shows not only that Lebesgue measure is not total, but that *no* measure satisfying the above mentioned properties can be total. Let me first list these properties, and then review Vitali's proof.

Denote by m the Lebesgue measure on the real line (R).

We say that a subset X of R is *measurable* iff it belongs to the domain of m. Then m has the following properties:

(1) The measure of any finite interval is its length. E.g., if $J = (a, b)$, then $m(J) = b - a$. Infinite intervals have infinite measure.
(2) Non-triviality: For any real number r, $m(\{r\}) = 0$ and the measure of the entire real line is non-zero: $m(R) \neq 0$.
(3) Monotonicity: If X and Y are measurable subsets of R such that X is a subset of Y then $m(X) \leq m(Y)$.
(4) Countable additivity: If $\langle X_n : n \in \omega \rangle$ is a pairwise disjoint family of measurable subsets of the real line then $\bigcup_{n\in\omega} X_n$ is measurable and $m(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} m(X_n)$.
(5) Translation invariance: If X and Y are subsets of the real line and d is a real number such that $Y = X + d$, then Y is measurable iff X is and (if they are both measurable) $m(Y) = m(X)$.[3]

[2]Perhaps this can best be seen if one defines the Lebesgue measure of a set to be its *outer* measure if it (the outer measure) agrees with the inner measure. See [2] for a definition of inner and outer measures.

[3]For a definition of Lebesgue measure see [11, chapter 3]. In general, by a *measure* on a set X I will mean a function h from (all or some) subsets of X to the real line plus an

I will now review Vitali's proof: Consider the equivalence relation on the reals in the (closed) unit interval induced by: $x * y$ iff $x - y$ is rational $(0 \leq x, y \leq 1)$. Using the axiom of choice, pick a member from each equivalence class. The resulting set S is non-measurable: The unit interval can be covered by countably many sets, each a "shift" by a rational number (between -1 and 1) of the original set. By translation invariance all these "shifts" have the same measure (if they are measurable at all); thus either all have measure zero, which is impossible, since their countable union covers the unit interval,or else all have a fixed, positive, measure, equally impossible, for their union, which is contained in the interval $[-1, 2]$, would have infinite measure. Thus S is not measurable.[4]

The proof above does not rely on the specific definition of Lebesgue measure. It follows from properties 1.– 5. of Lebesgue measure listed above (and, in fact, does not depend on 1.: As long as finite intervals get finite, positive measures, the proof would still go through). Thus Vitali's construction of the non-measurable set not only shows that Lebesgue measure is not total; it establishes that there is no way of extending it to (or replacing it by) a total measure, if the extension (or replacement) is to assign finite positive measures to finite intervals and to be countably additive and translation invariant (recall that any measure satisfies, by stipulation, requirements 2.–3.). For any such measure admits the construction of a non-measurable set in the same manner described above. We thus get Vitali's impossibility result, (VI):

> (VI) There is no non-trivial, translation invariant, countably additive *total* measure on the real line (assigning finite measure to finite intervals).

I would like to emphasize that Vitali's result really stops us in our tracks. For it not only shows that some technical construction is impossible; rather it shows that the natural notion of measure on the real line cannot measure *all* subsets of the real line.

1.1. In the wake of Vitali's result. We now face the task of picking up the pieces. One path open to us is to turn our backs on the iceberg and sail back to the safe waters ruled by Lebesgue measure.

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additional point ("infinity"), satisfying (the analogs of) 2–3 above and *finite* additivity, i.e., if $\langle X_i : i < n\rangle$ is a finite sequence of subsets of X belonging to the domain of h, then $\bigcup_{i<n} X_i$ belongs to the domain of h and $h(\bigcup_{i<n} X_i) = \sum_{i<n} h(X_i)$.

Lebesgue measure is an example where the range of the function h is maximal, i.e., every point on the real line (plus "infinity") is the measure of some subset of the real line. Later we will discuss measures at the other extreme, where the range of the function h will consist only of two points.

[4]For details of this construction see [11], pp. 52–54.

The non-measurable set constructed is far from being a "natural" set of reals. It is a set specifically constructed to serve as a counter-example. It is built from "above," by using the "logical" operation of choosing representatives from equivalence classes. It contrasts with e.g., the *Borel* sets. These are built from "below": They are built from topologically simple sets (open intervals) by closing under simple "mathematical" operations (complementations, countable unions and countable intersections). The Borel sets of reals are Lebesgue measurable. Can one chart the navigable waters up to the non-measurable sets?

I mention two results that seek such cartography. Back in 1917, Lusin (in [6]) showed that all *analytic* sets of reals were Lebesgue measurable. Since the class of analytic sets extends the class of Borel sets, Lusin drew a new boundary for the class of Lebesgue measurable sets.[5]

In 1938, Gödel approached the boundary from the other side: Rather than displaying more complicated measurable sets, he came up with a simpler, *definable*, non-measurable set. But Gödel did not *prove* that there had to be such a set. Rather, he displayed a *model* of set theory where such a set existed.

Thus Gödel departed from the path taken by Vitali and Lusin in the *methods* he used to tackle the problem. He introduced *meta*-mathematical

[5]One can appreciate Lusin's result and the discussion that follows by introducing a scale measuring the complexity of the definitions (using formulas of first order set theory) of various sets of reals. On the two opposite ends, we have the Borel sets and Vitali's non-measurable set. *Borel* sets are all quite simply defined: These are the sets obtained from open intervals (themselves easily definable) by the operations of complementation and countable intersection (iterated countably many times). On the other hand, Vitali's set S described above is not definable at all: It contains one member from each equivalence class (of the relation $*$ defined above) but there is no way to define (in ZFC) which member was picked from each equivalence class. Thus we cannot explicitly define the members of this set. (The situation may change if we have a definable well ordering of the reals; see Footnote 6.) In between these extremes are definable sets of reals whose definition becomes more and more complicated. The collection of *projective* sets is a natural extension of the Borel sets: The projective sets are those definable in a first-order language, where the quantifiers range over the real numbers and the primitive relations are Borel relations. In fact, one can draw a further distinction *within* the class of projective sets: For each projective set, we count the number of quantifiers (ranging over the real numbers) needed for its definition. The more quantifiers needed to define a given set, the higher its rank in the hierarchy will be. This produces the projective *hierarchy*: The Borel sets form its basis and the other sets are divided along its ranks. The *analytic* sets mentioned above are close to the bottom of this hierarchy: Only one existential quantifier ranging over the real numbers is needed to define them. The set constructed by Gödel, to be mentioned in the main text below, is "almost" one rank higher: It requires two "real" quantifiers for its definition, but it has two definitions; one with an existential quantifier preceding a general quantifier, and one with a general quantifier preceding an existential quantifier. Such sets are standardly referred to as Δ^1_2 sets. I note that the set constructed by Gödel does not require any real parameters for its definition, even if we write out explicitly the formula for the Borel relation. See [3] and [18] for Gödel's result.

techniques in the hunt. Whereas Vitali and Lusin were directly analyzing the real numbers, Gödel was also analyzing the "proof system" employed by them. By displaying a model where all the assumptions used by Vitali and Lusin were verified (as well as preserving truth under their rules of inference) where a definable non-measurable set existed, Gödel was able to show that you cannot *prove*, using the same hypotheses as Vitali and Lusin, that all definable sets were Lebesgue measurable.

Gödel's model was a model of first order set theory, Zermelo-Fraenkel set theory (with choice), henceforth abbreviated ZFC. His result then showed that one cannot prove in ZFC that all definable sets are Lebesgue measurable. The principles used by Vitali and Lusin can be cast within ZFC but by no means do their principles exhaust ZFC. ZFC is a theory which describes many sets, of which sets of reals are only a very small fragment. Yet Gödel showed that even this theory could not prove that all definable sets of reals were Lebesgue measurable.

In fact, the non-measurable set in Gödel's model was a rather *simply defined* set of reals (not much more complicated than the analytic sets). Thus we get an even stronger result: One cannot prove, in ZFC, (much) more than what Lusin proved; viz., that all analytic sets were Lebesgue measurable. Gödel showed in effect that Lusin had hit a barrier, henceforth, "Lusin's Barrier," as to what was provable, in ZFC, about the extent of the Lebesgue measurable sets.[6,7]

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[6] Gödel constructed his non-measurable set in L, the model of constructible sets. In this model, there is a *definable* well ordering of the reals. It is the definability of the well ordering which enables one to define non-measurable sets. It is not clear what specific example (of a non-measurable set) Gödel had in mind (see [18]). But one such example (of the complexity asserted by Gödel) can be obtained by carrying out Vitali's own construction inside L, replacing his arbitrary choices of members from the equivalence classes by definable choices (namely, the least element from each equivalence class with respect to the well ordering mentioned above, which is itself definable). See also the previous footnote.

Gödel's result is considered again in Section 1.2. That section also contains a more detailed discussion of obtaining meta-mathematical results by using models. For more on ZFC and the sets it describes see Section 1.3. Gödel's model L is discussed in Section 3.

[7] I say in the text that Gödel's result shows that Lusin had hit a "barrier" as to the extent of the sets that could be shown to be measurable in ZFC. To be precise, there are classes of sets of reals which do extend the analytic sets (which were shown by Lusin to be Lebesgue measurable) which can be shown, in ZFC, to contain only Lebesgue measurable sets. In fact, Lusin himself, in a joint result with Sierpinski, ([7]), showed the Lebesgue measurable sets were closed under the Suslin operation $\mathcal{A}$ (See [9, p. 72], for the definition of the Suslin operation.) The closure of the analytic sets under this operation yields an example of a class of sets of reals extending the class of analytic sets in which all sets are Lebesgue measurable. The strongest result in this vein is an unpublished result of Solovay, proving (in ZFC) that if a set can be proved in ZFC to have a Δ^1_2 definition then it is Lebesgue measurable. The sets considered by Lusin and Solovay are all Δ^1_2; by Gödel's result, one *cannot* show (in ZFC) that *all* Δ^1_2 sets are Lebesgue measurable. Thus in terms of levels of the *projective hierarchy*, Gödel's result shows that Lusin's result is the best possible. Henceforth, whenever I talk about

We are now confronted by what I shall call the Vitali-Gödel Straits. Vitali presented us with a non-measurable set. This set was carefully constructed to serve as a counter-example and was not built up from below from simpler sets. On the other hand, Lusin presented us with a (relatively small) class of measurable sets, the analytic sets, all rather simply defined, and Gödel showed that this was roughly the best one could do (while working within ZFC set theory). But "most" sets of reals do not fall in either category: They are neither "fabricated" sets of reals nor are they as simple as the analytic sets. For any such set A (which is neither "fabricated" nor analytic), we naturally wonder:

Is A measurable or not?

The question seems simple enough. However, the above discussion shows that its solution cannot be so simple. Many possible routes are blocked even before any attempt to follow them has been made. We may be very ambitious and try to show *all* sets are Lebesgue measurable. Vitali's result puts an end to that quest. Or we might be less ambitious and just try to show that more sets (more than the analytic sets) are Lebesgue measurable. This time it is Gödel's result that stops us in our tracks, at least as long as we are operating within ZFC set theory.

Perhaps what the Vitali-Gödel results tell us is that ZFC does not exhaust the basic truths about sets. Might the true situation regarding the extent of Lebesgue measure be better understood if we use stronger (true) set theoretical principles than those described by the axioms of ZFC? In particular, can we hope to show, using a "strengthened" theory, that *more* (i.e., more than the analytic) sets are Lebesgue measurable?

Strengthening our hypotheses is one way of trying to get more information about our original problem. Another is to weaken our goals, by looking at a revised notion of measure. The "real" measure for sets of reals is Lebesgue measure, but it cannot be total. What if we look, e.g., at measures which are not translation invariant? They are not as intuitive as Lebesgue measure, but they pose an interesting question: Can such measures be total? Answering this question might not provide us directly with new information about the original problem; but indirectly, we might gain new insight as to the source of the malaise.

Caught between the Scylla and Charybdis of the Gödel-Vitali results, we may confront the Gödel end by strengthening our theory and hoping to prove more sets (more than the analytic sets) are Lebesgue measurable. Alternatively, we may bypass Vitali's trap by pursuing a revised notion of measure.

"more sets (more than the analytic) being Lebesgue measurable," I mean "higher levels of the projective hierarchy containing only Lebesgue measurable sets."

But we must take notice of the different kind of resolution involved. Whereas confronting Gödel's end leaves us with a *truth of the matter* oriented search conducted at the (real) universe of sets, Vitali's obstacle leads us to investigate *consistency* results and *models* of set theory.[8]

Our search for escape routes will thus bifurcate into truth of the matter results (involving Lebesgue measure) and consistency results (involving revised measure). I see the distinction between *truth of the matter* results and *consistency* results as a fundamental one and so will pause to expound it before detailing the two escape routes from the Vitali-Gödel straits.

1.2. Truth vs. consistency. Truth oriented results *describe* the reality of sets. They record, depict if you will, the way things really are in the one and only universe of sets.

Lusin's result is an example of the kind. Lusin proved, using the axioms of ZFC, that all analytic sets were Lebesgue measurable. Thus his result will hold in any model of ZFC. In particular, it must be true in the real universe of sets. It is this last which is the real force of Lusin's result: Regarding the *real* sets of reals, all analytic sets are Lebesgue measurable.

In contrast, the main goal of consistency results is to craft a model of set theory with some desired combinatorial properties in order to witness the consistency of certain propositions. The model is not the universe of sets but rather a consistency verifying artifact. We have just encountered one such result, Gödel's, about the existence, in a model, of a definable non-measurable set. Gödel's model is the class of constructible sets (mentioned if Footnote 6, see also Section 3). Gödel's result does not show that there exists a simple, definable, non-measurable set, only that the existence of such sets is consistent with ZFC, i.e., that one cannot prove, in ZFC, that such sets do not exist.

One final observation. Whether our end-results pertain to truth or consistency, we confront the question: what *means* are needed to reach those ends? When truth is our focus, we are not constructing models but investigating the universe of sets directly. We ask: What information about this universe is indispensable for ascertaining a *specific* result? E.g., to go back to Lusin's result, all that was needed to prove that the analytic sets were Lebesgue measurable were the axioms of ZFC. Thus looking at the universe of sets itself, we know that one need not exhaust all its riches to ascertain the truth of Lusin's result: Those structural features of the universe, which are

[8]The claim about the need to examine consistency results in order to bypass Vitali's trap is not obvious: Might we not be investigating the real universe of sets in order to determine whether, in it, there exists e.g., a non-translation invariant measure which is total? Indeed, there is a way to weaken the notion of measure which does allow for the construction, in the real universe of sets, of a total measure. I will mention this below (in Footnote 11). But the weakened notion of measure I will pursue will turn out to involve, and essentially so, discussion of consistency results. See Footnote 18.

revealed by the axioms of ZFC, suffice to ground his result. However, if we are to show that larger fragments of sets (i.e., more than the analytic sets) are measurable in the real universe of sets, stronger set theoretic principles will have to be used. This much is made clear by Gödel's result, which set limitations on what can be done when we are confined to the axioms of ZFC. In the results described below, the additional set theoretic principles strengthening our theory will all assert the existence of some *large cardinals*. By exploiting the fact that such large cardinals exist in the universe of sets, we will be able to show more sets of reals are Lebesgue measurable (i.e., more than the analytic sets).

The question of the means being used also arises in the context of consistency results. Here the question usually takes the form: What assumptions are needed in order to construct the model we are after? In Gödel's example of a non-measurable set mentioned above nothing more than ZFC was needed to construct the model. As we shall see, in order to construct models which will help us steer away from the Vitali-Gödel straits, principles stronger than those of ZFC, principles securing the existence of more sets, will be called for. Again it will be necessary to assume the existence of large cardinals. We must bear in mind, however, that the question thus posed is not whether there really exists this or that large cardinal in the universe of sets; it is rather what price must be paid to prove the consistency of ("provide a model for") a certain claim.

The rest of this paper is devoted to both the consistency and truth oriented escape routes from the Vitali-Gödel straits.[9]

1.3. Intermezzo: Large cardinals. I have mentioned the notion of a large cardinal and will use it extensively below. What are large cardinals? Rather than attempt to define them let me try to explain why they are called *large*. The "universe of sets" can be thought of in the following way: We start with the empty set and construct more and more sets. If we are given the ordinals, we can describe the construction using two operations: Power set and union. At successor stages we take the power set of the previous stage, at limit stages the union of the preceding stages.

Large cardinals are those (uncountable) cardinals which cannot be reached from "below" in this manner. The smallest large cardinal, the *inaccessible* cardinal, sets a good example. The inaccessible cardinals are defined as the

[9]Formally speaking, in consistency results we do not need to assume the existence of large cardinals, only that their existence is consistent with ZFC. As for the existence of large cardinals in the universe: Most set theorists believe, or feign to believe, that large cardinals exist. As noted above, I do not wish to take a stand on this issue. I would, however, like to draw the distinction between the use of large cardinals in truth of the matter results and in consistency results. In the former, the presence of the large cardinal sheds light on the very universe of sets; in the latter, the large cardinals merely serve as tools for constructing consistency-verifying models.

regular strong limit cardinals: If κ is an inaccessible cardinal, κ is not the union of less than κ sets each of size less than κ. Also, the power set of any cardinal less than κ must always itself be less than κ. Any cardinal satisfying these last two requirements is an inaccessible cardinal.

The inaccessible cardinals also exemplify another property which is common to all large cardinals and witnesses their size: If there is an inaccessible cardinal then the hierarchy described above, taken up to the first inaccessible cardinal, forms a model of ZFC set theory. In particular, the consistency strength of "ZFC + 'there exists an inaccessible cardinal' " is greater than that of the theory ZFC alone. This is typical of all large cardinals. As they cannot be reached using the operations of ZFC from "below," assuming their existence provides a way of forming a model of the theory of ZFC together with whichever cardinals (large or small) exist below them; and the assumption that such a cardinal exists always adds consistency strength to the theory.[10]

§2. The consistency path—revising the notion of measure. We first try to regain total measure by generalizing our notion of measure. From the various properties of Lebesgue measure, it is natural to ponder the subtraction of either countable additivity or translation invariance. Dispensing with translation invariance turns out to be more relevant to the theme of this paper, so I follow this path.[11] In fact, it will be useful at this point to dispense with both translation invariance and the requirement that the measure of finite intervals is their length. With both these requirements gone, we leave behind us the original, "concrete," geometrical form of measure. The only distinguishing mark of the measure space now is its cardinality; the remaining properties of measures do not tell apart different measure spaces of the same cardinality. Since a cardinal κ is, itself, a set of cardinality κ, we might as well think of the measures as being on the cardinals themselves. The original question about measures on the real line is thus transformed into: Is there a total, countably additive measure on the cardinal of the real numbers, $2^{\aleph_0}$? Before answering this question, we might ask: Are there *any* cardinals carrying total, countably additive measures? Ulam (in [19]) proved that the first cardinal κ with such a (total) measure has the property that all (total) measures on it are κ additive.[12] Hence in order to see whether there is any cardinal κ with a total, countably additive measure on it, we might as

[10] For more on large cardinals, see [5].

[11] It is possible to regain total measure if one drops the requirement of countable additivity (while holding on to the translation-invariance requirement). In fact, one can prove, in ZFC, that such a measure exists on the real line. The first to realize this was Banach. See [20, p. 156], for a discussion of this result.

[12] A measure on κ is κ additive, or κ complete, if whenever $\langle A_\alpha | \alpha < \beta \rangle$ is a sequence of pairwise disjoint subsets of κ of length *less than* κ (i.e., $\beta < \kappa$) the measure of the union of

well see whether there is any cardinal κ with a κ additive measure on it. This search bifurcates into a search for *atomic* and *non-atomic* measures. I will consider both in this order. We shall see that large cardinals are intrinsic to both.[13]

2.1. Atomic measures. An atom is a set in the measure space with positive measure which is, as its name suggests, unsplittable. Any attempt to "break" an atom into two will result in one of the two "pieces" having measure zero. In other words, any subset of an atom either has measure zero or has measure equal to that of the original set. We can restrict ourselves to one such atom, i.e., consider the atom, say A, as our new measure space, where the only sets to be measured are the subsets of A.

Since A was an atom, every subset of A has either measure zero or the same measure as A itself. Further, if A was a subset of κ, and the original measure on κ was κ additive, then A has cardinality κ. We thus face the notion of a *two valued* measure, i.e., a measure which takes only two values. Putting all this together, we arrive at a definition of a measurable cardinal:

(MC) A cardinal κ is *measurable* iff there exists a non-trivial, two valued, κ additive (or: κ complete) total measure on κ.

It is customary to assume that the measure assigned to κ by a two valued measure on κ is one. So if κ is a measurable cardinal, every subset of it gets assigned either zero or one by the measure. Note that of the five characteristic properties of Lebesgue measure mentioned in Section 2, it is the second, third and fourth that appear in the definition of this new notion: 2–3 are left intact (replacing R by κ) and 4 is strengthened—instead of demanding countable additivity we now require κ additivity. The new notion of measure is indeed a generalization of the original notion of Lebesgue measure.

How does the discovery of this new notion bear on our original quest for a total measure on the real line? At first glimpse, it seems not to. For measurable cardinals turn out to be *large* cardinals. They are inaccessible cardinals, and this already suffices to rule out the possibility that the cardinal of the real numbers, $2^{\aleph_0}$, will be measurable (it can be reached by the power set operation from the set of natural numbers), or that the existence of *any* measurable cardinal could be proved in ZFC. In fact, measurable cardinals are much larger than inaccessible cardinals: Below the first measurable cardinal κ there are κ inaccessible cardinals. Thus inasmuch as there emerged a concept of total measure, it is one that only applies to large cardinals. In

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the A_α's is the sum of their measures. According to this terminology, countably additive (or countably complete) measures are $\aleph_1$ additive ($\aleph_1$ complete).

[13]When opting to consider non-translation-invariant measures, we also decided to part with the requirement that finite intervals be assigned their lengths by the measure. The decision about the intervals is independent from the one about translation invariance, as can be seen from Solovay's work discussed later in this paper. See Section 2.3 and Footnote 15.

contrast, the cardinal of the set of real numbers, $2^{\aleph_0}$, is "small" and cannot carry such a measure. The measurable cardinals offer no immediate end to our quest for a measure on the reals; however, they do supply a critical clue for the construction of a suitable measure on the set of real numbers itself.

2.2. Non-atomic measures. The two valued measures were generated out of atomic measures. By focusing on non-atomic measures, we encounter a concept of measure which *can* sometimes apply to $2^{\aleph_0}$, that of *real valued* measure.

If a total, κ-complete measure on a cardinal κ has no atoms, then we can split any positive measured set into two smaller positive measured sets. This results in the measure having a multitude of values, in fact, as many values as the real numbers, as opposed to the two valued measures encountered before. Such measures are called "real valued measures." Cardinals carrying such measures are called "real valued measurable cardinals":

(RVMC) A cardinal κ is a *real valued measurable cardinal* iff there exists a non-trivial, κ additive (or: κ complete) measure on κ.

Although $2^{\aleph_0}$ cannot be a (two-valued) measurable cardinal, it can sometimes be a *real valued* measurable cardinal. At the end of the sixties, R. Solovay constructed a model for:

$2^{\aleph_0}$ is a real valued measurable cardinal.[14]

From measures on large cardinals we are now down to measures on $2^{\aleph_0}$, which is the cardinality of the set of reals.

This is much closer to our original question, which concerned measures on the real line. Can we get even closer to our original target? Yes: Let $2^{\aleph_0}$ be a real valued measurable cardinal. One can now construct a *total* countably additive measure, m^*, on the real line which extends the *Lebesgue* measure: Any set which is Lebesgue measurable, will be assigned by m^* its Lebesgue measure. But, unlike Lebesgue measure, m^* will assign *any* subset of the real line *some* value.

Relying on m^*, Solovay constructed a model of

ZFC + "Lebesgue measure has a countably additive (non-translation invariant) extension to all subsets of the unit interval"

Solovay thus found one escape route out of the Vitali-Gödel straits: It *is* possible to generalize the notion of Lebesgue measure in a way that makes total measure on the reals compatible with the axioms of ZFC. It is possible, but what are the "costs?" What assumptions were made by Solovay in constructing the above described model?[15]

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[14]See [17]. The assumptions needed to construct this model will be discussed in the next section (2.3).

[15]Solovay's model demonstrates that one cannot, in general, deduce that a countably additive measure on the real line is translation invariant just because finite intervals are

2.3. Measurable cardinals and measuring point sets. Solovay started out with a model of set theory, alias *the initial model*. He transformed it into *the final model* where $2^{\aleph_0}$ was a real valued measurable cardinal. Solovay assumed that the initial model, besides being a model of ZFC, contained a *measurable cardinal*.

In fact, it is the measurable cardinal itself, κ, which becomes at once $2^{\aleph_0}$ and the real valued measurable cardinal in Solovay's *final* model. The two valued measure on κ is no longer total in the final model, but Solovay showed that it could be extended to a total *real valued* measure on κ, i.e., on $2^{\aleph_0}$. This real valued measure, in turn, enabled Solovay to construct a countably additive extension of *Lebesgue measure* to all subsets of the real line.

The presence of a measurable cardinal is thus *sufficient* for the construction of a model where Lebesgue measure has a countably additive extension to all subsets of the real line. Solovay showed it was also *necessary* for the construction, in the following sense: Starting with a model where Lebesgue measure had a countably additive extension to all subsets of the real line, Solovay constructed a model with a measurable cardinal. Thus the three theories:

- ZFC + "Lebesgue measure has a countably additive extension to all sets of reals"
- ZFC + "$2^{\aleph_0}$ is a real valued measurable cardinal"
- ZFC + "there exists a measurable cardinal"

were shown to be equi-consistent.[16]

The above results show that the measurable cardinal is essentially connected to the measure problem. This is no mere coincidence: Solovay exploited the structural similarities between the different notions of measures: On the one hand, a total, countably additive extension of Lebesgue measure on the reals and on the other, an atomic, κ-additive, non-trivial measure on a cardinal κ. Solovay showed how to transform one into the other. The size of the large cardinals may be very different from that of sets of reals; however, the measures on them bring out their structural similarities.[17,18]

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assigned their length as their measure. Since m^* extends Lebesgue measure, finite intervals are assigned their length by m^*. But m^* is not translation invariant, for if it were, it would contradict Vitali's impossibility result (VI) (since m^* is total and countably additive).

[16] Solovay's result is an equi-consistency result. Thus one has to be careful when stating the "necessity" of the existence of a measurable cardinal in the initial model. What is necessary is that the initial model be a model of a theory of the same "consistency strength" as that of ZFC + "a measurable cardinal exists."

[17] What is the route from a total, countably additive, non-trivial measure on the real line to a measurable cardinal? First, by Ulam's result quoted earlier in the paper, if there exists a non-trivial, countably additive, total measure on the real line then there exists a non-trivial, κ-additive measure on some cardinal κ which is less than or equal to $2^{\aleph_0}$. In other words, a real valued measurable cardinal exists (that the measure has no atoms follows from the fact

§3. Large cardinals and Lebesgue measure: The truth of the matter. We have seen the impact of large cardinals on Vitali's impossibility result (VI) in the realm of "consistency results." It is consistent that total measure can be regained *if* the existence of some large cardinal is consistent. But do large cardinals have any impact on the *truth* of the matter, consistency results notwithstanding?

Recall that Gödel constructed a model where there was a non-measurable set which was not much more complicated than the analytic sets. If we are to prove that more sets than the analytic sets are measurable, we had better see what it is, exactly, in Gödel's model that we are to avoid.

Gödel's model of constructible sets, L, is constructed by recursion on the ordinals, where at each stage α one adds a "minimal amount" of sets: only those definable over the part of the model already constructed. The result is a very structured but minimal model of set theory (ZFC). One indication of the "structure" of L is that it has a definable well-ordering: There is a formula of set theory which determines a well-ordering of all the constructible sets. In particular, there is a definable well-ordering of the constructible reals (i.e., the real numbers which are present in Gödel's model). It is this definable well-ordering which enabled Gödel to construct his definable, non-measurable set.

The minimality of Gödel's model is manifested in the fact that it is contained in any transitive (class) model of set theory. One can carry out

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that κ has size less than or equal to the continuum). The remaining question therefore is how to construct a model with a measurable cardinal provided there exists a model with a real valued measurable cardinal (I assume all models are models of ZFC). This can be done by relativizing Gödel's model of the constructible sets so that it also includes sets "constructed from" the family of those sets assigned measure zero by the real valued measure in question. See Section 3 for more on Gödel's model. Note that real valued measurable cardinals and non-trivial, countably complete, total measures on the real line are very closely related: One exists if and only if the other exists. This shows that when we decided to dispense with the length of finite intervals as their measures in addition to dispensing with translation invariance, we were not abstracting "too much": as long as the measures in question are non-atomic (and countably complete), given that one exists, a measure will exist on the real line which does assign finite intervals their lengths.

[18]Let me justify my earlier claim that the escape route mentioned in the text (constructing countably additive total measures which are not translation invariant) necessitates investigation of consistency results. That the existence of a countably additive total measure (which is not translation invariant) cannot be proven in ZFC follows from the fact, mentioned in the text, that the consistency strength of such a claim is stronger than that of ZFC (assuming ZFC is consistent). Might we hope to prove the existence of such a measure if we assume the existence of some large cardinals? The answer is still negative. The existence of such a (total) measure on the real line violates the continuum hypothesis, i.e., it implies that the cardinality of the set of reals is not the first uncountable cardinal. However, the existence of large cardinals does not contradict the continuum hypothesis, and therefore, one cannot hope that their existence will imply the existence of measures described above. See [5, pp. 298–303].

Gödel's construction in any such model and, since the construction itself is absolute, i.e., does not depend on the specifics of the model it is being carried out in, the result is always the same, namely, Gödel's model of the constructible sets. Thus no matter which model we set out to work in, Gödel's model will always be present as a submodel. If we are to succeed in constructing a model of set theory where more sets (more than the analytic sets) are Lebesgue measurable, we have to guarantee that our model is "sufficiently different from," i.e., "sufficiently larger than," Gödel's (minimal) model.

Large cardinals provide the means of achieving this. For example, if a measurable cardinal exists, then the first uncountable cardinal ($\aleph_1$) is itself a large cardinal in L: It is inaccessible in L. Thus small uncountable cardinals of the universe may become large in the minimal model L: It lacks the sets needed to demonstrate that the sets in question are really "small." Turning this fact around, we see that cardinals which are "declared" large by L may really be small. E.g., we just saw that an inaccessible cardinal of L could be the first uncountable cardinal of the universe. Referring still to the same example, all the many cardinals of L which are smaller than the above mentioned inaccessible cardinal are smaller than the "real" first uncountable cardinal, i.e., are "merely" countable ordinals in the universe. Thus in the presence of a measurable cardinal, there is indeed a big gulf between the universe of sets and Gödel's model of constructible sets, L.

It was once again Solovay who first showed how one could use this gulf in order to establish: If a measurable cardinal existed, more sets than the analytic sets were Lebesgue measurable. Solovay showed that if a measurable cardinal existed, sets belonging to the next level of the projective hierarchy beyond the analytic sets (i.e., Σ^1_2 and Π^1_2 sets) had to be Lebesgue measurable. This truth of the matter result is related to a consistency result proved by Solovay: Assuming an inaccessible cardinal existed (to be precise: Assuming the consistency of ZFC plus the assertion that an inaccessible cardinal exists), Solovay constructed a model of ZFC where *all* projective sets were Lebesgue measurable. Now truth in a model of set theory does not, in general, imply truth in the universe of sets. However, in this instance, Solovay was able to show that the existence of a *measurable* cardinal in the universe created certain similarities between the universe and his consistency-verifying model. Solovay exploited the gulf between the universe and L (in the presence of a measurable cardinal) alluded to above together with Shoenfield's absoluteness result ([14]; see Footnote 20) to prove that if a measurable cardinal existed then the sets directly beyond the analytic sets in the projective hierarchy, the Σ^1_2 and Π^1_2 sets, were Lebesgue measurable in the universe of sets.

What light do Solovay's results shed on Lusin's barrier? First, Solovay's consistency result: Solovay constructed a model of ZFC where all projective sets were Lebesgue measurable (assuming the consistency of an inaccessible cardinal). This complements Gödel's result which says that there is a model of ZFC where a set belonging to the level beyond the analytic sets in the projective hierarchy is *not* Lebesgue measurable. Thus Lusin's barrier can really be seen as a ZFC imposed barrier—this last cannot settle the measurability status of sets in the projective hierarchy, one way or another.

But next comes Solovay's *truth of the matter* result:

(STM) If a measurable cardinal exists, then all Σ^1_2 (and Π^1_2) sets are Lebesgue measurable, i.e., all sets of reals belonging to the second level of the projective hierarchy are Lebesgue measurable.

This result actually topples Lusin's barrier. The presence of a large cardinal guarantees that sets which were lying *outside* the barrier are now in fact Lebesgue measurable. The barrier has been passed.[19]

Is the barrier removed, or was it just placed one level higher in the projective hierarchy? Can we show *more* sets (e.g., all projective sets) are Lebesgue measurable, if we assume *more* large cardinals exist? This was unknown for many years, until Woodin and Shelah came up with an important extension of Solovay's result: If one assumes the existence of sufficiently many large cardinals then one can show that all sets in the projective hierarchy are Lebesgue measurable. The large cardinals in question are called "Woodin cardinals." Shelah and Woodin's *truth of the matter result* is:

(SWTM) If infinitely many Woodin cardinals exist, then all projective sets are Lebesgue measurable.

We have now come full circle trying to come to terms with Vitali's impossibility result, (VI). Whereas in ZFC, one cannot even show that sets belonging to the second level of the projective hierarchy are Lebesgue measurable, in the presence of large cardinals one can show that sets belonging to the entire projective hierarchy are Lebesgue measurable. And this result is no mere consistency result, it is a truth of the matter result, concerning the universe of sets and the "real" sets of reals.

This last result is interesting in its own right, describing modern developments following Vitali's impossibility result, (VI). But are the large cardinals

[19]See [15] for Solovay's "truth of the matter result." In the text Solovay's result was presented as "If a measurable cardinal exists then all Σ^1_2 and Π^1_2 sets are Lebesgue measurable." It is possible to weaken the hypothesis slightly (by assuming the existence of a smaller large cardinal). The assumption that an inaccessible cardinal exists is *not* sufficient for Solovay's truth of the matter result. Solovay's model where all projective sets are Lebesgue measurable is presented in [16]. That an inaccessible cardinal was unavoidable for constructing that model was shown by Shelah. See [12] and [10]. Solovay's proof also establishes the necessity of relying on the axiom of choice when constructing non-measurable sets.

involved in (SWTM) also relevant to the present essay's theme, viz., are they connected to the original measure problem for sets of reals? The answer is: Yes. Woodin cardinals are a generalization of measurable cardinals. The exact definition is rather elaborate, but the essence of a Woodin cardinal is still the concept of measure. This time it is not a measure measuring the subsets of the Woodin cardinal itself, but rather a whole system of measures, measuring subsets of smaller cardinals. Rather than having one measure on one cardinal, we now have many cardinals, each with many measures on it, where all these measures, viewed as one system, obey certain technical "coherence" criteria. Woodin cardinals are a *generalization* of measurable cardinals but they are a generalization of *measurable* cardinals. Since the measure on measurable cardinals is, itself, a generalization of Lebesgue measure, the measures associated with the Woodin cardinals are still linked to Lebesgue measure. Once again, the large cardinals may be far away from the sets of reals they affect; their structure, however, is quite similar.[20,21]

§4. Epilogue. In Shelah and Woodin's "truth of the matter" theorem we find an overall synthesis: The reunion of the real universe of sets and the original notion of Lebesgue measure, a sort of pre-1905 harmony restored. And with things thus restored, we have come round to our starting point. Conflicting dramatic philosophical presentations notwithstanding, point sets and large sets do not lead a segregated existence in the real universe of sets. Recall that the origin of our tale was the naive question regarding how many point sets are Lebesgue measurable. Catalyzed by the Vitali-Gödel result, the idea

[20] One way of viewing Shelah and Woodin's proof is the following: Solovay had already provided us with a model of ZFC where all projective sets were Lebesgue measurable. Our problem was that we could not transfer his result to the universe of sets, except for the case where the sets in question were defined in a very simple way, i.e., in a Σ^1_2 (or Π^1_2) way. The key to Solovay's success with these sets (but not with sets appearing higher up in the projective hierarchy) was Shoenfield's *absoluteness* result (in [14]; see also [1], p.163) which concerns transitive models of ZFC and Σ^1_2 formulas: Such a formula is true in a transitive model of ZFC if and only if it is true in the real universe. In general, one cannot improve this absoluteness result to more complicated formulas (i.e., to Σ^1_n formulas, where n is greater than 2). However, unpublished results of Woodin show how to construct models with "more absoluteness": The more large cardinals present in the model, the more complicated the formulas which the model is correct about. Woodin's absoluteness results enable us to show that if Solovay's techniques are applied in the presence of many large cardinals, then one can indeed deduce the Lebesgue measurability of all projective sets in the universe from their measurability in Solovay's model.

[21] For a definition of Woodin cardinals see [8]. For Shelah and Woodin's result, see [13]. Regarding the hypotheses used by Shelah and Woodin: It is known that one cannot prove all projective sets are Lebesgue measurable if one only assumes the existence of finitely many Woodin cardinals. Results showing the Lebesgue measurability of even more sets, assuming the existence of more large cardinals, are also known. See [21].

of a measure on point sets quickly led to generalized measures and in its wake measurable cardinals, large sets indeed. Soon enough results flowed in the reverse direction—just how many point sets are Lebesgue measurable turned out to depend on the existence of sets as large as the measurable cardinal. By (SWTM), yet larger sets, still bred by measures, are part of the full and on-going tale.[22]

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