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# HERMANN WEYL'S INTUITIONISTIC MATHEMATICS

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*Dedicated to Dana Scott on his sixtieth birthday.*

It is common knowledge that for a short while Hermann Weyl joined Brouwer in his pursuit of a revision of mathematics according to intuitionistic principles. There is, however, little in the literature that sheds light on Weyl's role and in particular on Brouwer's reaction to Weyl's allegiance to the cause of intuitionism. This short episode certainly raises a number of questions: what made Weyl give up his own program, spelled out in "*Das Kontinuum*", how did Weyl come to be so well-informed about Brouwer's new intuitionism, in what respect did Weyl's intuitionism differ from Brouwer's intuitionism, what did Brouwer think of Weyl's views, ... ? To some of these questions at least partial answers can be put forward on the basis of some of the available correspondence and notes. The present paper will concentrate mostly on the historical issues of the intuitionistic episode in Weyl's career.

Weyl entered the foundational controversy with a bang in 1920 with his sensational paper "On the new foundational crisis in mathematics".[1] He had already made a name for himself in the foundations of mathematics in 1918 with his monograph "The Continuum" [18] ; this contained in addition to a technical logical-mathematical construction of the continuum, a fairly extensive discussion of the shortcomings of the traditional construction of the continuum on the basis of arbitrary—and hence also impredicative—Dedekind cuts.

This book did not cause much of a stir in mathematics; that is to say, it was ritually quoted in the literature but, probably, little understood. It could not be properly appreciated until the phenomenon of impredicativity was better understood.[2] The paper "On the new foundational crisis in mathematics" had a totally different effect; it was the proverbial stone thrown into the quiet pond of mathematics. Weyl characterized it in retrospect with the somewhat apologetic words:

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[1]The page numbers quoted below are from the original 1921 paper [19]. Additional remarks and explanations are inserted within square brackets.

[2]Cf. [5].


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> Only with some hesitation do I acknowledge these lectures, which reflect in their style, which was here and there really bombastic, the mood of excited times—the times immediately following the First World War.[3]

Indeed, Weyl's "New Crisis" reads like a manifesto to the mathematical community; it uses an evocative language with a good many explicit references to the political and economic turmoil of the post-war period. The opening sentences castigate the mathematical community for its complacency, for its failure to pay any serious attention to the potential dangers of the various paradoxes of Cantor, Russell, Richard, . . . :

> The antinomies of set theory are usually regarded as border skirmishes that concern only the remotest provinces of the mathematical empire and that can in no way imperil the inner solidity and security of the empire itself or of its genuine central areas. Almost all the explanations given by highly placed sources for these disturbances (with the intention of denying them or smoothing them over), however, lack the character of a clear, self-evident conviction, born of totally transparent evidence, but belong to that sort of half to three-quarters honest attempts at self-deception that one so frequently encounters in political and philosophical thought. Indeed, every earnest and honest reflection must lead to the realization that the troubles in the borderland of mathematics must be judged as symptoms, in which what lies hidden at the center of the superficially glittering and smooth activity comes to light—namely the inner instability of the foundations upon which the structure of the empire rests.

The readers of 1920 were all too familiar with the phenomenon; the German State with its skirmishes in the Baltic and its political instability was a painful reminder of the self-deception of politics. Weyl used the political metaphor with great dexterity, comparing for example the classical use of existential statements with the use of paper money:

> The point of view sketched above[4] only expresses the meaning which the general and existential propositions in fact have for us. In its light mathematics appears as a tremendous "paper economy". Real value, comparable to that of food products in the national economy, attaches only to the direct, simple singular; general and existential statements participate only indirectly. And yet we mathematicians seldom think of cashing in this "paper money"! The existence theorem is not the valuable thing, but the construction carried out in the

[3] See [24].

[4] [i.e., the construction of a specific object versus its non-effective existence.]

> proof. Mathematics is, as Brouwer sometimes says, more activity than theory (*mehr ein Tun als eine Lehre* ).

And the final clarion blast of Weyl, one that fired the imagination and fed the wrath of many a practicing mathematician, rang through the next decade: "For this order can not be maintained in itself, as I have now convinced myself, *and Brouwer—that is the revolution!*"

Before analyzing Weyl's views on intuitionistic mathematics, let us see how Weyl got involved with this particular branch of constructive mathematics. Weyl was one of the outstanding mathematicians of his time. He was Brouwer's junior by 4 years, but he was already making a name for himself when Brouwer was still struggling for international recognition. In 1913 Weyl published one of the all-time successes of mathematics, his "The idea of Riemann surfaces" which established his name for good. The relation between Brouwer and Weyl dated back to at the latest 1912, as is documented in a letter from Weyl to Klein (16.5.1912).[5] With Brouwer attending conferences in Germany and regularly visiting Göttingen, they could hardly have missed each other. It seems most likely that the two, who shared a vivid interest in the foundations of mathematics, discussed the various issues in private, probably already before the war. The first concrete evidence of Brouwer's influence is to be found in a letter from Brouwer to Fraenkel, in which he mentioned that during a stay in the summer vacation of 1919 in Engadin in Switzerland, he had a number of conversations with Weyl. This piece of information solves much of the riddle of Weyl's early expertise in Brouwer's new intuitionism that went in an essential way beyond Brouwer's early foundational program, and which was the result of Brouwer's wartime research, carried out in isolation. Indeed, the intuitionism of Brouwer's dissertation, or of the inaugural address differs in essential points from the post-war intuitionism, as expounded to Weyl in the Swiss mountains. Brouwer gave a one-man course to Weyl on his new insights that went back to 1916 and that were published in communications of the Royal Academy in Amsterdam from 1918 onward. Weyl quickly grasped the main points of Brouwer's arguments and started to work out the consequences for himself. In May 1920 the paper on "the New Crisis" was finished and a copy was sent to Brouwer:

Zürich, 6.5.'20

> Dear Brouwer,
>
> Finally I have sent what I have long promised you. It should not be viewed as a scientific publication, but as a propaganda pamphlet; thence the size. I hope that you will find it suitable for this purpose, and moreover suited to rouse the sleepers; that is why I want to

[5] In this letter Weyl sums up the various points of the Brouwer-Koebe controversy on the theory of automorphic functions.

> publish it. I would be grateful for your criticism and comments. Did I enclose everything that you let me have only as a loan? If not, please reclaim it; the lecture on Formalism and Intuitionism[6] was already in my possession in the old days; at that time I did not pay attention to it or understand it ...

One can imagine how pleased Brouwer was with Weyl's conversion; here was one of the foremost members of the new generation of mathematicians, who was not only a master in almost all traditional topics in mathematics, but who had an exceptional insight in philosophical and foundational matters. The fact that Weyl was Hilbert's favorite son, and was viewed as the crown prince of mathematics in Göttingen, may have contributed to the importance of this first convert, but one should keep in mind that the Grundlagenstreit had not yet started and that the relations between Brouwer and Hilbert were still free from personal and scientific antagonism. As the enclosed manuscript showed, Weyl wholeheartedly gave up his own foundational program, trading his arithmetically definable sequences for Brouwer's choice sequences, and rejecting the principle of the excluded middle ($PEM$). Only a rough draft of Brouwer's reaction, full of crossed-out parts, has survived. Nonetheless, one can fairly well guess Brouwer's views from this draft and from a few remarks, inserted in pencil in the margin of Weyl's manuscript. The draft contains a number of comments on Weyl's manuscript; we will quote the various parts when discussing Weyl's paper. Here we translate the first few lines:

> Your wholehearted scientific support has given me infinite pleasure. The reading of your ms. was a continuous delight and your explanation, it seems to me, will also be clear and convincing for the public .... That we judge some side issues differently will only stimulate the reader. However, you are completely right in the formulation of these differences of opinion; in the restriction of the objects of mathematics you are in fact more radical than I am; however, one cannot argue about this, these matters can only be decided by individual concentration.

Brouwer's notes in the margin of the manuscript are mostly short comments on specific claims or assertions of Weyl concerning technical matters. There is one spot, however, where Brouwer thought his new disciple guilty of a strategic blunder: after introducing the functio continua, Weyl, somewhat apologetically, remarked that "It should be stressed once more that certain individual functions of that kind occur occasionally in mathematics, that general theorems are, however, never asserted about them. The general formulation of these notions is therefore required only if one is giving a

[6] See [1], [2].

justification of the meaning and methods of mathematics; for mathematics itself, the subject matter of its theorems, it is never considered at all". Weyl's view can be paraphrased as "in mathematical practice one does not quantify over functiones continuae". We will return to the matter after the discussion of the functio continua and just give Brouwer's comments from the draft:

> It seems to me that the whole purpose of your paper is endangered by the end of the second paragraph of page 34.[7] After you have roused the sleeper, he will say to himself: "So the author admits that the real mathematical theorems are not affected by his considerations? Then he should no longer disturb me!" and turns away and sleeps on. Thereby you do our cause injustice, for with the existence theorem of the accumulation point of an infinite point set, many a classical existence theorem of a minimal function, as well as the existence theorems for geodesics without the second differentiability condition, loses its justification!

In this respect Weyl showed himself to be less radical than Brouwer, that is to say Brouwer did not fear the confrontation with the traditional practice of mathematics, where Weyl considered his fundamental considerations as somewhat less relevant for everyday mathematics. It is not implausible that Weyl's position was dictated by his views on quantification in general (cf. below).

There are a few striking aspects to Weyl's version of intuitionism. One of them is the nature of the continuum and another the role of logic. Weyl distinguished two distinct views of the continuum, the atomistic one and the continuous one. In the first version the continuum is made up of individual real numbers which can be sharply distinguished. Weyl, in his program of "The Continuum", allowed only specific (arithmetic) relations on natural numbers, and as a consequence, also a restricted class of Dedekind cuts. For this particular class of reals Weyl accepted existence problems involving reals as meaningful:

> Only if we conceive the notion [of the continuum] in this way, which fixes and demarcates its extent, do questions of existence concerning real numbers become meaningful . By this restriction of the notion, a bunch of individual points is picked from the flowing mash. The continuum is smashed to isolated elements, and the blending of all its parts is replaced by certain conceptual relations, based on "greater – smaller", between these elements. Therefore I speak of an atomic conception of the continuum.

The arithmetically definable continuum of "The Continuum" has this atomistic character.

[7] See [19, p. 66].

Almost immediately Weyl added, however, that "It has never been my opinion that the continuum given by intuition is a number system of Weyl; rather that analysis needs only such a system for its constructions and does not have to worry about the "continuum" poured in between". (p. 47). Indeed, already in "*Das Kontinuum*, Chapter II, §6, Intuitive and mathematical continuum", Weyl pointed out that his atomic version of analysis represented "a *theory of the continuum* which has (over and above its logical correctness) to prove itself correct, just as a physical theory does".

After the discussion of "Weyl's continuum", the "New Crisis"-paper continues with an exposition of Brouwer's choice sequences as a basis for the continuum. Real numbers are given by Weyl as infinite sequences of shrinking intervals, and thus the need arises to further specify the notion of sequence. Weyl allowed two sorts of sequences: those given by a law—standing for the individual points of the continuum, and the free choice sequences—which determine a continuum of "becoming", or emerging, sequences (*werdende Folgen*,[8]) i.e., sequences of intervals that are freely chosen and hence cannot in a predetermined way point at an element of the continuum. In the most literal sense of the word, these sequences remain 'in statu nascendi' forever. In Weyl's words:

> It is a first basic insight of Brouwer, that the sequence which is emerging (*werdend*) through free choice acts, is a possible object of mathematical concept formation. Where the law $\varphi$, which determines a sequence out to infinity, represents the single real number, the choice sequence, restricted by no law in its freedom of development, represents the continuum.
>
> Brouwer's remark is simple but deep: here a "continuum" arises in which, indeed, the single real numbers fit, but which itself does not by any means resolve into a set of completed existing (*fertig seiender*) real numbers; rather it is a medium of free becoming. (p. 50.)

It should be mentioned here that Weyl had briefly mentioned choices in "*Das Kontinuum*", but only to dismiss them out of hand. He regarded as nonsense the conception of a set as a collection of infinitely many arbitrary choices, which is recognized as a whole. Here one hears an echo of the discussion of the axiom of choice after Zermelo's proof of the wellordering theorem.

Weyl's evocative description of the "flowing" continuum is an almost artistic act; he dared to put into writing what Brouwer would only describe in discussions and lectures. The image of a continuum in a continuous state of creation (becoming) was a decisive step forward in the quest for something

[8]The translation of the German "*werdende Folge*" is somewhat problematic. Various adjectives, such as 'becoming', 'developing', 'emerging', 'in statu nascendi' have been used in the literature.

truly "continuous", i.e., something that cannot be cut or broken into pieces, let alone be built up from the atomic particles called points!

There are basically two fundamental issues in Weyl's exposition of intuitionism:

- the properties of choice sequences,
- the consequences for logic.

As to the first issue, let us compare Brouwer's notion and Weyl's side by side:

> A spread (*Menge*) is a law, according to which, if over and over again a numeral of the sequence $\zeta$ [i.e., $N$] is chosen, each of these choices generates either a certain sign or nothing, or, on the other hand, the stopping of the process and the definite destruction of its result, where for each $n > 1$ after each non-stopped sequence of $n-1$ choices at least one numeral can be indicated, which, when chosen as the $n$-th numeral does not stop the process. Each sequence of signs generated in this way by an unlimited choice sequence, (which thus in general cannot be presented as finished) is called an *element of the spread* [3].[9]

In the next line the phrase "choice sequence" is introduced for the numerical sequences that generate the elements.

Weyl did not *give* a definition of 'choice sequence' but rather introduced the notion "in passing". A few quotations are listed below:

(1) The difficulty is located in the notion of sequence. If any viewpoint at all is at the basis of contemporary analysis, from which its propositions and proofs can be understood, then it is this: a sequence results from the successive arbitrary choices of its individual numbers; the result of these infinitely many choice acts is available in finished form, and with respect to the finished infinite sequence I can ask e.g., if 1 occurs among its numbers. But this standpoint is absurd and untenable; for inexhaustibility is inherent to the essence of the infinite. A single *determined* (and determined out to infinity) sequence can only be defined by a *law*. When on the contrary a sequence is created step by step by free acts of choice, then it should be considered as "becoming" (growing), and only those properties can be meaningfully attributed to a becoming choice sequence for which the decision "yes or no" ( ... ) is already reached when the sequence has reached a certain point .... (p. 50).

(2) *a sequence in which each choice is totally free* (p. 52).

(3) a law which generates a natural number $n$ from an emerging sequence, depending on the result of the choices, is necessarily such that the

[9]The correction of 1919 has been inserted.

> number $n$ is determined as soon as a certain finite segment of the choice sequence is given in finished form, and it remains the same, no matter how the choice sequence may develop further. (p. 51.)

A methodological difference between the approaches of Brouwer and Weyl is that Brouwer does not elaborate the choice process and concentrates on the concept of spread, whereas Weyl does not consider spread at all, but tries to get the choice process right. One gets the impression that Weyl hesitates to attribute to choice sequences the sort of individuality that a natural number or a lawlike sequence possesses. E.g., "where a law $\varphi$ which determines a sequence all the way to infinity, represents a single *real number*, a choice sequence, restricted by no law in its freedom of development represents the continuum" (p. 50). In particular, Weyl allows the passage from universal statements to particular ones only for lawlike sequences (objects), so from $\forall\alpha A(\alpha)$ one can infer that $A(a)$ holds for a lawlike $a$.

One should note that general statements of the form $\forall\alpha A(\alpha)$ may be true on the grounds that "it is part of the essence of emerging sequences that a sequence, in which every single choice is totally free", has a certain property $A$ (p. 52). Whereas for Brouwer a choice sequence is an individual (think of "the sequence that I am creating now"), Weyl apparently views only lawlike sequences as such: "a single particular (and determined out to infinity) sequence can only be defined by a law".

At the end of Part II, Section 1, another argument for his view is given: "Every application of mathematics must depart from certain objects, intended for mathematical treatment, which can be distinguished by a number of characters; the characters are natural numbers. By means of the symbolic process, which replaces objects by their characters, the connection with pure mathematics and its constructions is obtained". One has to keep in mind that Weyl uses the name "application of mathematics" rather generously, e.g., the point geometry on a line is an application of mathematics, where the coded basic objects are rational segments. Weyl's view is certainly compatible with Brouwer's early philosophical views, and—in a technical sense—with Brouwer's spread definition, where the "generated signs" can be thought of as objects coded in natural numbers.

At the time that Brouwer and Weyl were discussing the matter, the ideas concerning choice sequences were rather implicit, and in view of Brouwer's use of (and examples of) spreads we may conclude that already at the level of choice sequences both parties diverged. Brouwer allowed choice sequences, given by a law (e.g., in the spread of the natural numbers [3], p. 3, or finite spreads), whereas Weyl excluded lawlike sequences from the domain of choice sequences. At a first reading (after [12]) one might think that Weyl had lawless sequences in mind, with an absolute freedom of choice; both (2) and (3) seem to point that way. However, he also allowed choice sequences built from other choice sequences, e.g., the sequence $m_1 + m_2 + \cdots + m_n$ obtained from the sequence $m_n$ (p. 50). The reader will note that (3) looks very much like the principle of open data (for a function value), since it discusses the output of a function for a single choice sequence, and hence it points towards a strong degree of lawlessness. We know now that this would yield the wrong class of choice sequences, but it would be highly unjust to demand the present day insights in a pioneering paper in 1920. In comparison, Brouwer too would allow all choice sequences over the universal spread (tree), but he would not restrict the notion of "choice", i.e., for him all paths, no matter how chosen, were eligible. It would be tempting to ascribe a similar view to Weyl, but at various places in the paper it is made clear that Weyl actually means something close to our present day 'lawless', e.g., in his summing up of the differences with Brouwer's views. It seems most plausible that the finer distinctions were not quite fathomed by Weyl. It even is hard to guess to what extent Brouwer knew exactly which pitfalls were to be found on his path; nonetheless it is remarkable that he (almost) always formulated the right concepts and principles. The miracle therefore is not that Weyl's conception of choice sequence was somewhat awkward and cumbersome, but that Brouwer got his notion right! One is tempted to conclude from Weyl's paper that after Brouwer's tutorial, he set out to reduce the Brouwerian notion of choice sequence to something more basic.

Note that in the case of (3) above, Brouwer had formulated the correct principle already in his 1918 paper:

> (4) A law that assigns to each element $g$ of $C$ [the universal spread] an element $h$ of $A$ [the natural numbers], must have determined the element $h$ completely after a certain initial segment $\alpha$ of the sequence of numerals of $g$. (p. 13.)

The crux is that Brouwer considered *all* choice sequences of $C$, instead of a single choice sequence, as in (3). It should be added that Brouwer did not further justify this principle, but it shows that he realized that the universe of choice sequences was the home of a rich variety of sequences, ranging from lawlike to lawless, with all kinds of versions in between.

Brouwer's spreads may be thought of as trees with natural numbers at their nodes (a subset of the set of all finite sequences of natural numbers), where some particular nodes were forbidden (and hence also all nodes below them). The choice sequences could be visualized as infinite paths through the tree. In fact the choices made were in practice from a given domain, thus real numbers appeared as choice sequences of rational numbers, or of rational intervals. But the basic function of a spread was to regulate the possible choices, a matter not left till later. In Weyl's approach, all natural numbers were eligible, and only afterwards other sequences were generated by suitable mappings. From a topological point of view one could say that

Weyl got the desired choice sequences by means of continuous mappings acting on Baire space. Hence there was no need for Weyl to restrict the possible choices in advance, the mappings would take care of the need for specific sequences. Hence also Weyl's preference for unrestricted choices.

Given the natural numbers and the choice sequences, one wants to have functions on these domains in order to carry out a reasonable amount of mathematics. Weyl introduced for that purpose three classes of functions:

(1) *functio discreta*: a law assigning numbers to numbers (a lawlike function from $N$ to $N$)
(2) *functio mixta*: a lawlike function from numbers to lawlike functions $(N \to (N \to N))$, or a law assigning numbers to choice sequences, $((N \to N) \to N)$.
(3) *functio continua*: a lawlike function from choice sequences to choice sequences $((N \to N) \to (N \to N))$.

The functio discreta is introduced by means of the un-analyzed notion of a law. The functio mixta for the first case is given by a double sequence, which is essentially a functio discreta, and for the second case it is regulated by the following *essential insight* (*Wesenseinsicht*) (p. 64):

> *According to this law there always is a moment for an emerging sequence, no matter how it develops, when it gives birth to a number.*

As a matter of fact, Weyl listed two generating principles for the functio mixta, but he explicitly warned the reader that he did not claim these principles to yield all cases of functio mixta.

The clauses for Weyl's functio mixta are in modern formulation:

(1) $F$, with $F(\alpha) = f(\overline{\alpha}k)$ is a functio mixta, where $f : N \to N$ is a lawlike function, $\overline{\alpha}k$ is the (coded) initial segment of length $k$ of the choice sequence $\alpha$.
(2) if $H$ and $G$ are functiones mixtae then $F$ with $F(\alpha) = H(G(\alpha), \alpha)$ is a functio mixta.

The class of functions given by the generating principles is evidently a proper subclass of the class $K$ of Brouwer operations.[10] In the margin of the manuscript Brouwer noted: "this rule is indeed included in my notion of spread".

As in Brouwer's spread, Weyl allows partial functions, i.e., the function may be undefined for certain choice sequences, but from the functio mixta law it must follow for each $\alpha$ that "at a certain argument (depending on $\alpha$) either the generated number is available or the certainty exists that the sequence under consideration is a deaf sequence,[11] which will be barren for

[10] cf. [14], Ch. 4, Section 8.
[11] i.e., a sequence not in the domain of the functio mixta.

all eternity". Brouwer's comments on the concept of 'functio mixta' are a bit cryptic. With respect to the functions given by the two generating principles, he remarked "this rule is indeed covered by my concept of spread", and with respect to the partial functio mixta, "this is not necessary from my point of view".

What is the relation between 'functio mixta' and a spread? Let us note that Weyl used the functio mixta and functio continua to obtain suitable choice sequences; so a functio mixta operating on "all basic choice sequences" produces natural numbers as outputs. Brouwer's remark should be understood as saying that the range of a functio mixta can also be obtained by a spread law. This is easily seen as follows, consider the choice sequence $a_0, a_1, a_2, a_3, \ldots$ assign nothing to the initial segments until the functio mixta $F$ gives an output $n$, then assign $n$, and assign nothing to all longer initial segments. A similar trick works for the partial functio mixta. Conversely, of course, a spread of natural numbers can be mimicked by a suitably chosen functio mixta.

The functio continua takes choice sequences to sequences, and it is defined by Weyl as a continuous operation:

> A functio continua is a law, according to which every step which adds another term to a sequence, emerging through acts of free choice, generates a certain number or nothing. (p. 65.)

Since this definition does not guarantee that the resulting sequence will proceed indefinitely, Weyl explicitly stipulated that eventually a new number will be generated. Here Brouwer commented, "superfluous from my point of view, it excludes unnecessarily discontinuous functions" (p. 65). This seems curious in view of Brouwer's later theorem that functions on the continuum (and also Baire space, i.e., the set of choice sequences) are continuous; the condition of Weyl, however, excluded partial functions, and they could easily be discontinuous.

Weyl obtained discontinuities by mimicking the destruction clause from Brouwer's spread definition. He had, however, to add a complicated clause to ensure closure under substitution. Brouwer again declared this clause superfluous for spreads. So much for Weyl's view of choice sequences.

Weyl seemed to consider first choice sequences over all possible number-choices (all lawless paths through the universal tree), and only afterwards to use continuous operations in order to get other families of choice sequences. This is in a way a plausible and even modern view; in the recent past these techniques have been used by Troelstra–van Dalen and Van der Hoeven to obtain suitable choice universes as "projections of lawless sequences" [15], [16]. Needless to say that in 1920 one would not even dream of such matters.

Weyl's second personal innovation in intuitionistic mathematics attracted a certain amount of attention. It was his reconsideration of the role of existential and general statements. In Part II Section 1, Weyl described his own "conversion" to Brouwer's rejection of the principle of the excluded third. He started by pointing out that for propositions involving choice sequences he had no difficulty accepting Brouwer's view, but even here his arguments do not follow the plausible pattern of the Brouwerian counterexamples. Brouwer's simple counterexamples are based on such existential statements as "the decimal expansion of $\pi$ contains a sequence 0123456789", and hence do not even make use of the choice concept! Weyl's reason for rejecting the principle of the excluded third rested on an analysis of the meaning of $\exists\alpha A(\alpha)$ and $\forall\alpha A(\alpha)$ (where $\alpha$ from now on ranges over choice sequences). $\exists\alpha A(\alpha)$ meant for him that there is an individual sequence with the property, but on his interpretation of "individual sequence" this had to be a lawlike sequence. In the system of "The Continuum", the laws determining sequences were inductively given, and—according to Weyl's pre-intuitionistic interpretation—either there was such a law or not, but under the present circumstances

> the notion of law is no longer put on the Procrustes bed of the construction principles. Rather, if in any non-circular way, no matter how, a construction of a law of the desired kind has been successful, then we are justified in asserting that such a law *exists*. Thus, here it is definitely no longer a matter of the *possibility* of the construction, but rather we make such an existential statement in reference to the *successful construction*, the *proof actually carried out*. The negative statement that such a proof does not exist remains, of course, devoid of any meaning.

Here Weyl argues that with respect to an open-ended notion, such as 'law,' the negation makes no sense. Or in his terminology, the negation of an "*umfangs-definite*" property need not be "*umfangs-definit*"—the analogue of 'the complement of a set in an open-ended universe need not be a set.' Weyl then goes on to consider the positive version of the above negation:

> We can, however, interpret it positively: every sequence has the property $\neg A$, and now it [i.e., the negation of the existential statement] gets a meaning, provided we mean by 'sequence' not the law but, . . . , the sequence emerging through free choice acts.

Weyl paraphrased the above as " "there exists" ties us to Being and Law, "every" puts us into Becoming and Freedom".[12] Since the interpretation

[12] *Das "es gibt" verhaftet uns dem Sein und dem Gesetz, das "jeder" stellt uns ins Werden und die Freiheit.*

of the bound variable in $\exists\alpha A\alpha$ and $\forall\alpha\neg A\alpha$ was completely different, "it would be absurd to think here of a complete disjunction". Summing up: on Weyl's interpretation of the quantification over sequences, there was not a shred of evidence for the principle of the excluded third. The matter was not so simple for quantification over natural numbers; for such statements $A(n), \exists nA(n)$ is established if one finds a number $n$, during the consecutive testing of $A(0), A(1), A(2), \ldots$. The negation would then come to "after running through all natural numbers no instance $A(n)$ has been established", but Weyl correctly rejects this as nonsense, $\forall n \neg A(n)$ can only be true because it is part of the essence of the notion natural number. He concluded that these statements do not act as each other's negation. So this pleads against *PEM*. But, he continued:

> Although this speaks in favor of Brouwer, I was always thrown back to my old standpoint by the thought: if I run through the sequence of natural numbers and break off when I find a number with the property $A$, then this eventual breaking off does or does not occur; it is or is not the case, without wavering and without a third possibility [ ... ] Finally I found for myself the magic word. An existential statement—e.g., "there is an even number"—is not a judgment[13] in the proper sense at all, which states a state of affairs; existential states of affairs are an empty invention of logicians. (p. 54.)

Weyl termed such pseudo-statements "judgement abstracts", to be compared to "a piece of paper which announces the presence of a treasure, without divulging its location". Similarly, general statements are only hints at judgments. For universal statements Weyl uses the following metaphor: "To use another picture: if I compare knowledge with a fruit and the discerning realization of the knowledge with the consumption of the fruit, then a universal statement is to be compared to a hard shell filled with fruit. It certainly has its value, but not the shell by itself, rather because of its contents consisting of fruit; it is no use to me as long as I do not open it and actually take out a fruit and eat it". Here Brouwer jotted in the margin: "The Invariant theorem of Hilbert without a method of computation was only the certainty that this shell does not contain a fruit, this kind of existence theorem does not seem to be discussed here at all."[14]

Since, in Weyl's opinion, existential statements and general statements are not proper statements, they cannot be negated, and "the possibility to formulate an 'axiom of the excluded third' with respect to them has

[13]Weyl's use of judgment is not in agreement with present day use, which follows Frege. It is rather a proposition, cf. [18], p. 1.

[14]The invariant theorem referred to is the famous finiteness theorem of the theory of invariants, nowadays known as Hilbert's basis theorem. Brouwer views it as a $\neg\forall\neg$– statement.

disappeared". Brouwer's view of logic, later made explicit in the proof interpretation, had the advantage of allowing full iteration of all logical connectives, Weyl stopped at the quantifiers. Both approaches found a systematic treatment—Brouwer's logic in [7] and [11]—Weyl's logic in Kleene's realizability. Kleene went even further than Weyl (and Hilbert-Bernays) by considering all, say, arithmetical statements as incomplete, and hence in need of completion, cf. [10].

Readers of the foundational controversy between intuitionism and formalism will have noted that Hilbert lumped Brouwer and Weyl together; in his early papers (after [19]) Weyl is mentioned explicitly, but in the later ones only Brouwer is mentioned, and some of Weyl's views are conveniently attributed to Brouwer. E.g., in [8], Brouwer is charged: "To prohibit existence statements and *PEM* is tantamount to relinquishing the science of mathematics altogether". The charge perfectly fits Weyl's position, but does not do full justice to Brouwer. After all Brouwer had no quarrel with existential statements, he just interpreted them in a constructive manner! Hilbert's example shows that even the leading proof theorist was not able to distinguish properly between the brands of intuitionism of Weyl and Brouwer (although Weyl explicitly enumerated the points of difference!); how much less could one expect the general reader to spot the distinctions.

Weyl's conception of existential and general statements has influenced the treatment of the finitary viewpoint in arithmetic.[15] The finitary meaning of $\exists x A(x)$ is given as a "partial assertion",— i.e., "an incomplete communication of a more precisely determined statement, which either consists of the direct indication of a natural number with the property $A(n)$, or the indication of a procedure for the extraction of such a number ... " A general statement has a hypothetical finitary meaning, "such an assertion states a law which must verify itself in each particular instance". Hilbert and Bernays follow Weyl fairly faithfully (intentionally or not—Weyl is not quoted), in the passages concerning finitary consequence, including the *PEM*, "The complicated situation, which is found here with respect to the negation of judgments from the finitary viewpoint, corresponds to Brouwer's thesis that *PEM* is not valid for infinite collections. This non-validity consists from the finitary standpoint in this: that one does not succeed in finding a negation with finitary content which satisfies *PEM*".[16]

There is a certain temporal aspect to Weyl's rejection of *PEM*, but it is rather indirect; the undeterminedness of choice sequences with respect to future choices is of course at the basis of Weyl's considerations, but he added another major aspect. Namely that of the meaning of statements with choice quantifiers; it was on the basis of this difference in meaning of $\exists\alpha$ and $\forall\alpha$

[15] See [9], Section 2b.
[16] See [9], p. 34.

that he came to the conclusion that the unrestricted 'excluded middle' made no sense.

The combination of criticism of quantified statements and insistence on "free choice sequences" brought considerable complications. In II, Section 3, Weyl looked at the consequences for the simplest kind of statements and properties. Since free choice sequences could not satisfy mild conditions, such as "all values are even", he was faced with the problem what to make of relativized statements (or implications). In modern notation, what does "$\forall\alpha(A(\alpha) \rightarrow B(\alpha))$ mean? Let us, for convenience, follow Weyl's example: $A(\alpha) := \forall nE(\alpha(n))$, where $E(x)$ stands for "$x$ is even". Consider $\forall\alpha(\forall nE(a(n)) \rightarrow B(\alpha))$; we can strengthen the statement (constructively) by pulling out the $\forall n$, $\forall\alpha\exists n(E(\alpha(n)) \rightarrow B(\alpha))$. By continuous choice we get $\forall\alpha(E(\alpha(f(\alpha)) \rightarrow B(\alpha))$ for a functio mixta $f$. Now we have succeeded in giving a constructive meaning to the problematic relativized general quantifier. This is roughly the escape route Weyl indicated.

Surprisingly, Weyl denied the possibility to interpret a relativized statement of the same sort, only with choice quantifiers instead of numerical quantifiers (p. 71). Brouwer, in the draft of the letter, pointed out that this conflicted with his views:

> Your assertion of p. 37, lines 3–6, which contradicts my view (as you, by the way, know) should be explained a bit further. It seems to me that the reader who has followed you so far will have trouble with this passage.

It is not hard to see that indeed the job done by the functio mixta, could just as well be done by a functio continua.

The above gives an impression of the complications inherent in Weyl's rejection of quantified statements in general.[17]

At the end of Section 3, "Mathematical propositions, properties, and sets", Weyl summed up the main foundational points:

> I adopt from Brouwer 1. the basis, which is in all respects the essential thing, namely the idea of emerging sequences and the doubt of the principium tertii exclusi, 2. the idea of the functio continua. For the functio mixta and the viewpoint, which I summarize in the following three theses, I am responsible:
>
> (1). The notion of sequence oscillates, depending on the logical context in which it occurs, between "law" and "choice", "being" and "becoming".
>
> (2). The general and existential statements are no judgments in the proper sense, they do not state states of affairs, but are judgment indications, resp. judgment abstracts.

[17]Note the similarity with the position of finitism with respect to quantifiers.

> (3). Arithmetic and analysis only contain general statements about numbers and freely emerging sequences; no general function and set theory of independent content.

There are two comments of Brouwer in the margin:—with respect to (1),

> for me "emerging sequence" is neither one; one considers the sequences from the stand-point of a helpless spectator, who does not know at all in how far the completion has been free

and with respect to (2) and (3),

> I do, however, not agree with (2) and (3).

Brouwer's comment on Weyl's notion of choice sequence is consistent with the few remarks one finds in his publications. Apparently he wished to allow for choice sequences also higher order restrictions of choices, so that during the choice process one might decide at any moment that all future choices (of the particular sequence that one is generating) will be made according to a law. Hence the powerless spectator.

The "New Crisis" paper is extraordinarily rich in ideas and claims; the last section "The Continuum" contains a few gems that anticipate some developments of the twenties. One of them explicitly states the connectedness of the continuum:

> A true continuum is simply something connected in itself and cannot be split into separated pieces; that contradicts its nature.

This is the first enunciation of the unsplittability of the continuum. However, Weyl does not prove the principle, but rather asserts its plausibility on conceptual grounds. Similarly, the prize-winner—the continuity of real functions—is announced without a proof:

> Above all, however, there can be no other functions at all on a continuum than continuous functions (p. 76).

Did Weyl thus establish the continuity theorem years before Brouwer? The answer is 'no' for a number of reasons. In the first place Weyl defined real functions via mappings of the intervals of the choice sequence determining the argument to intervals of the image sequence. Hence such a function is continuous by definition. Weyl's argument for this definition is on the whole plausible: one should be able to find effectively approximations to the output from approximations to the input. Brouwer's continuity theorem, however, goes one step further, it establishes the continuity of a function from choice sequences to choice sequences or from reals to reals. So, where Weyl reduced the type of a function from $R$ to $R$ in advance to that of a function from $N$ to $N$ (initial segments to initial segments), Brouwer showed that this by necessity followed from the intuitionistic principles and the functional character (i.e., the $\forall\exists!$-combination). It should be pointed out that Weyl's assertion of the continuity of real functions was more or less dictated by his refusal to grant choice sequences individually, i.e., the combination $\forall\alpha\exists!\beta$ could not be handled on the basis of the principle "existence entails lawlikeness and generality entails choice"; the only way to handle such combinations was to invoke a functio continua. $\forall\alpha\exists!\beta A(\alpha, \beta)$ could not even be viewed as an "instruction for a statement", but $\forall\alpha A(\alpha, f(\alpha))$ for a continuous $f$, would do nicely.

Furthermore, there is ample evidence that Weyl was, also in this case, inspired by Brouwer. Brouwer had already quite early reached the conclusion that there could not be any full (i.e., total) discontinuous functions on $R$ [18]: "Through this above mentioned theorem, which is an immediate consequence of the intuitionistic stand-point, and which has since 1918 often been mentioned by me in lectures and in conversation, ..."; and he was already looking out for a natural notion of a discontinuous function on (a subset of) $R$ in 1920. This is substantiated by the draft mentioned above:

> With reference to your considerations on the notion of continuous function I would like to draw your attention to my notion of the fully defined (*volldefinierte*) function of the continuum. I mean by that a law which assigns to every point of a set equivalent to the continuum (*örtlich übereinstimmenden Punktspezies*) a further point of the continuum. Such a function can very well be discontinuous without being generated by a conjoining of continuous functions on separated continua; one can, by the way, operate with them in many ways (one can e.g., in certain cases integrate them without having information on their continuity or discontinuity).

After pointing out the absence of discontinuous functions on $R$, Weyl went on to say that:

> When the old analysis allowed the formation of discontinuous functions, it thereby showed most clearly how far it is from grasping the essence of the continuum. What one calls nowadays a discontinuous function, consists in fact (and this also is basically a return to older intuitions) of a number of functions on separated continua.

Brouwer added in the margin the remark:

> better: is a not everywhere defined function. The definition of the function does not have to be split in cases, as the pointwise discontinuous function shows.

Weyl illustrated his statement by the piecewise defined function $f(x)$ which is $x$ for $x > 0$ and $-x$ for $x < 0$. This partial function, he remarked can be extended to the total function $|x|$, but the function $g$ with $g(x) = 1$ for $x > 0$ and $-1$ for $x < 0$ cannot be extended to a total function. Brouwer

[18] See [4], p. 62.

noted, with respect to the latter part "very true! To be underlined, for this is what matters".

The conjecture of the continuity of all real functions was, according to Brouwer's letter to Fraenkel (28.1.1927), brought up in 1919: "In the summer of 1919 I have in the course of private discussions with Weyl in Engadin, as a result of which he was converted to my views, once in connection with the definition of continuous functions in Section 1 of the first part of my "Founding set theory independent from the logical proposition of the excluded third", expressed and motivated the conjecture that these functions are the only ones existing in the full continuum [...]. It can be only on the basis on this utterance of a conjecture (and others, which were half-understood by Weyl) by me, that since then the legend has been spread by Weyl, that it is self-evident that in Brouwer's analysis none but continuous functions can exist".

In [4] there is a reference to the conjecture: "By this theorem,[19] which is an immediate consequence of the intuitionistic position, and which I repeatedly mentioned in discussions and in lectures, theorem 3 below[20] which asserts much more, and which I managed to prove only much later, is suggested".

The above comment of Brouwer has a somewhat bitter ring, for reasons that have nothing to do with Weyl or his papers,—at the time of writing Brouwer had the (not wholly unjustified) impression that the credits in the foundations of mathematics were distributed in a rather off-handed way. But at least it makes it quite clear that Weyl had not quite grasped Brouwer's arguments about the continuous functions.

Not surprisingly, the available information on Brouwer's reaction raises as many questions as it provides answers. It makes clear that Brouwer did not share Weyl's radical rejection of existential and general statements, and it also formulates the objections against Weyl's notion of choice sequence. The question remains, however, why did Brouwer not give in the letter to Weyl an exposition of his own views on these topics? Since we only have the draft of the letter, it cannot be excluded that the letter itself contained more extensive arguments. It seems more likely, however, that Brouwer did not wish to repeat his arguments of the Engadin discussion in 1919, and that he wanted to respect Weyl's version, as appears from the first paragraph of the draft:

> these matters can only be decided by individual concentration.

This probably explains why Brouwer did not provide more detailed comments on Weyl's wording of the continuity principle; for in Weyl's version of choice sequence, his "open data"-like formulation—"for a single choice sequence" rather than "for all choice sequences"—was plausible indeed. It

[19] the negative continuity of all real functions.

[20] the positive locally uniform continuity of all real functions.

would also have been interesting to have Brouwer's view on the meaning of the quantifiers in writing; here also he respected Weyl's view and was content to just register his disagreement.

Apart from the "propaganda pamphlet" of 1920, Weyl figures only twice in the history of intuitionism. After this short episode Weyl practiced intuitionism only once more; in 1924 he gave a constructive proof of the fundamental theorem of algebra.[21] One should not get the impression that Weyl's commitment to intuitionism was a mere infatuation; in a letter to the president of the *Schulrat* in Switzerland[22] Weyl reported the offer of a chair in Amsterdam, which rather appealed to him, as " ... the close relation I have to Brouwer—we cooperate in revolutionizing and re-founding analysis—, disposes me to consider the offer seriously". As a matter of fact, the offer from Amsterdam was eventually turned down, to the bitter disappointment of Brouwer. Weyl remained a lifelong admirer of Brouwer and his intuitionism, but he dissociated himself from the actual practice and also from the foundational debate. His influence on the ideas of Hilbert and Bernays should by no means be underestimated. We have already seen that the treatment of finitary statements and of quantifiers by Hilbert and Bernays was very similar to Weyl's treatment of meaningful statements (in particular the quantifier combinations, e.g., $\forall\exists$, are handled very much in the same manner). As a matter of fact, there had been discussions of foundational matters between Weyl and Bernays, of which at least one is documented: Bernays reported in a letter to Hilbert (25.10.25)that he met Weyl in Zürich, where "in particular we discussed your foundational theory, to which, as you know, Weyl is now no longer opposed". Even more important, it is mentioned in the same letter that Bernays just had observed a certain distinction between the finitary viewpoint and that of Brouwer—at that time a far from common insight! It is not far fetched to assume that this insight was a result of the above mentioned discussion with Weyl .

When the Grundlagenstreit, which ensued from his action in 1920, was well under way, Weyl remained rather aloof. Only in 1928, when Hilbert attacked intuitionism during a seminar talk in Hamburg, Weyl came once more to the defense of Brouwer.[23] Following Hilbert's talk, Weyl took it upon himself to straighten out a number of points, in particular the role of Brouwer's intuitionism in connection with Hilbert's program. He pointed out that Brouwer had systematically pursued the investigation of the extent of the "contentual thought".[24]

[21] See [20].
[22] 27.9.1920, [6] p. 49.
[23] [22]
[24] *inhaltliches Denken*.

> Brouwer, like everybody else, required of mathematics that its theorems be (in Hilbert's terminology) "real propositions", meaningful truths. But he was the first to see exactly and to the full extent how it had in fact everywhere far transcended the limits of contentual thought. I think that we owe him gratitude for this recognition of the limits of contentual thought.

Hilbert, he said, had respected as a matter of course in his metamathematical arguments these limits; contrary to earlier remarks of Hilbert,[25] Brouwer's insistence on constructive content could thus not be construed as "artificial prohibitions". Weyl went on to express his admiration for Hilbert's "radical re-interpretation" of mathematics, and to point out that even an intuitionist could not object to the passage through ideal statements, i.e., the conservative extension property of, say, full arithmetic with respect to finitary mathematics.[26] Brouwer's view on conservative extension results (e.g., as resulting from Gödel's translation) is not known, but from a constructive viewpoint, there is little one can object to. Hence, in principle, Weyl was quite right in his assessment. He fully acknowledged the situation in his conclusion. "Also in the epistemological evaluation of the new situation thus created, nothing separates me any longer now from Hilbert". From the following passages, it appears that Weyl had Hilbert's conservative extension program (or "real interpretation of ideal statements") in mind. He compared the situation with that in physics, where the (mathematical) theories were no longer in immediate contact with the phenomenal physical world, and quoted Hilbert's justification of the theory with an appeal to its success. Although Weyl recognized the potential effects of Hilbert's program, which was still a real option at the time, he somewhat sadly reflected that "if a triumph of formalism over intuitionism were to be the case – as it seems to all appearances , then I see in that a decisive defeat of the philosophical approach of a phenomenology, which proves itself insufficient for the understanding of creative science, on the most primitive and most accessible area of knowledge, mathematics". It is a bit hard to draw the right conclusion from this contribution. On the one hand Weyl felt the need to stand up for Brouwer, on the other hand he no longer felt compelled to stick to the intuitionist views on the role of formalism. One is tempted to view Weyl's statement as rather ahead of his time, i.e., his acceptance of the results of proof theory as a theoretical enterprise based on intuitionistic principles. This view certainly is compatible with the paper and also with later ones.

In his influential book "Philosophy of Mathematics and Natural Science" (1949), which grew out of an early essay "The present state of knowledge in

[25] cf. [8], p. 80

[26] which had not yet been settled in the negative by Gödel's theorems

mathematics" ([21]) and its subsequent extension *"Philosophie der Mathematik und Naturwissenschaft"* (1927), Weyl presented a penetrating survey of the foundational situation in mathematics, in which he discussed both intuitionism and formalism. The 1925 version still witnesses the enormous relief of having found a faithful theory of the continuum. The section on "Brouwer's intuitive mathematics" opens with the magisterial sentence

> The ice field was broken into floes, and now the element of the flowing will soon completely be master over the rigid. L. E. J. Brouwer designs—and this is an accomplishment of the greatest epistemological importance—an exact mathematical theory of the continuum, which conceives it not as a rigid being, but as a medium of free becoming.[27]

The same paper contains also the first sign of his despair in view of the ultimate success:

> Mathematics attains with Brouwer the highest intuitive clarity; his doctrine is idealism in mathematics thought through to the end. But with pain the mathematician sees the larger part of his towering theories fall apart.

Two years later Weyl expressed himself even more pessimistically:

> Mathematics attains with Brouwer the highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural way, retaining much closer contact with intuition than before. But one cannot deny that, when one progresses to higher and more general theories, the fact that the simple principles of classical logic are not applicable finally results in a hardly bearable awkwardness. And with pain the mathematician sees that the larger part of his tower, which he thought to be built from strong blocks, dissolves in smoke.[28]

These statements are in curious contrast to the—almost dual—one in the "New Crisis"-paper, where Weyl drew a diametrically opposed conclusion:

> The new conception, as one sees, brings very far-reaching restrictions in comparison with the generality that fades into vagueness, to which the analysis developed so far has made us accustomed in the last few decades. We must again learn modesty. We wanted to storm the heavens and we have only piled fog on fog that cannot support anybody who tries in earnest to stand on them. What remains tenable could at first sight appear so meager that even the possibility

[27] *Die Eisdecke war in Schollen zerborsten, und jetzt ward das Element des Fließenden bald vollendes Herr über das Feste. L. E. J. Brouwer entwirft—und das ist eine Leistung von der größten Erkenntnistheoretischen Tragweite—eine strenge mathematische Theorie des Kontinuums, die es nicht als starres Sein, sondern als Medium freien Werdens faßt* [21].

[28] See [22].

> of analysis is questionable; this pessimism is, however, unfounded, as the next section will show.

We now, of course, know that Weyl's later pessimism was somewhat exaggerated; things were not as bad as they seemed. In a sense Brouwer was himself to be blamed for the lack of popularity of his intuitionistic practice. Instead of polishing up the presentation of important parts of mathematics, say differential and integral calculus, he preferred to concentrate on the basic phenomena of intuitionistic mathematics. This strategy was not designed to win over the young practicing mathematicians but at best the isolated devotee of foundational research. It was not before Bishop's book in 1967, that a convincing basic text became available. Weyl, in his postscript of 1955 ([24]), had already somewhat modified his view; he hailed the (just published) book of Lorenzen, [13], as a big step forward: "The operations of the formal calculus are here intertwined in a fruitful and relaxed way with conceptual considerations on their products; the discovery of Gödel thereby loses all of its disturbing aspects". An additional factor for Weyl's high opinion of Lorenzen's book may have been Lorenzen's predicative rather than intuitionistic concerns.

Weyl never could quite part with his earlier intuitionistic past. He remained fair with respect to Brouwer's intuitionism in a period where intuitionism had become a minor curiosity in the larger context of mathematics. In a larger review paper in 1946, he summed up his feelings:

> This history should make one thing clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics, like everybody and everything in the world to-day, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on the enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world. [23]

Weyl may have been right, maybe he was not alone in taking the "crisis in mathematics" seriously in a personal way. He certainly was the most prominent mathematician with a conscience sensitive to the cracks in the foundations of the building of mathematics. The story had a sequel,— Brouwer visited Weyl at the end of his life in Zürich and the two had a long talk. At the parting of the two old friends, Weyl sadly remarked: "*Brouwer, es ist alles wieder schwankend geworden*".

**Appendix. Draft Brouwer to Weyl – 1920 (undated).** Ihre rückhaltslose wissenschaftliche Handreichung hat mir eine unendliche Freude bereitet. Die Lektüre Ihres M.S. war mir ein fortwährender Genuss und Ihre Auseinandersetzung scheint mir - auch für das Publ. klar und überzeugend wird sein [ ... ] Dasz wir beide über einige Nebensachen verschieden urteilen, wird auf die Leser nur anregend wirken. Allerdings haben Sie mit Ihrer Formulierung dieser Meinungsverschiedenheiten volkommen recht; in der Einschränkung des Objektes der Mathematik sind Sie tatsächlich radikaler als ich; es läszt sich aber darüber nicht diskutieren, diese Sachen sind nur durch individuelle Konzentration zu entscheiden. Im Anschlusz an Ihre Darlegungen über den Begriff der stetigen Funktion möchte ich Sie auf meinen Begriff einer volldefinierten Funktion des Kontinuums aufmerksam machen. Ich verstehe darunter ein Gesetz, das jedem Punkte einer mit den Kontinuum örtlich übereinstimmenden Punktspezies einen weiteren Punkt des Kontinuums zuordnet. Eine solche Funktion kann sehr gut unstetig sein, ohne irgendwie durch Zusammensetzung von stetigen Funktionen in getrennten Kontinuen erzeugt zu werden; man kann übrigens mit ihnen in mannigfacher Weise operieren. [man kann sie z.B. in gewissen Fallen integrieren ohne hinsichtlich ihrer Stetigkeit oder Unstetigkeit Aufschlusz zu haben]. Abgesehen von unseren Differenzpunkten hatte ich noch folgende Bemerkungen: Den Non-existential Sätzen (wozu z.B. die Mächtigkeitssätze auf S. 134 u. 43 des ersten Teiles meiner Abhandlung und auch der Hilbertschen Endlichkeitssatz des vollen Invariantensystems in seiner ersten Herleitung gehören) räumen Sie bei Ihrer Aufzählung der mathematischen Urteile gar keinen Platz ein. S. 3, Z. 8 (ebenso wie an der analogen Stelle auf S. 13 von "Das Kontinuum") ist mir der Sinn des Wortes "Sachkenntnissen" dunkel. Durch den Schluss des zweiten Absatzes von S. 34 scheint mir der ganze Zweck Ihrer Schrift gefährdet zu werden. Nachdem sie eben den Schlafenden aufgerüttelt haben, sagt er sich hier: "Also der Verf. gibt zu, dass die wirklichen mathematischen Sätze von seinen Darlegungen nicht berührt werden? Dann soll er mich weiter nicht stören!" und wendet sich ab und schläft weiter. Damit tun Sie aber unserer Sache Unrecht, denn mit dem Existenzsatz des Verdichtungspunktes einer unendlichen Punktmenge wird doch gleichzeitig manchem klassischen Existenz-theorem einer Minimalfunktion wie auch dem Existenzialsatze der geodätischen Linie bei Fortlassung der zweiten Differenzierbarkeitsbedingungen der Boden entzogen! Ihre S. 37 Z.3–6 aufgestellte, übrigens wie Sie wissen, meiner Auffassung widersprechende Behauptung sollte, wie ich glaube, etwas näher erklärt werden. Es scheint mir, dass auch der Leser, der Ihnen bis dahin genau gefolgt ist, mit diesem Passus Mühe haben wird. Ihr Funktio diskreta und Funktio mixta scheinen mir eben so gut wie Ihre Funktio kontinua in meinem Mengenbegriff enthalten zu sein. Mein Mengengesetz kann ja sehr gut von vornherein für jede Wahlfolge Sicherheit geben, dass nachdem einmal im

Zeichen erzeugt ist, weiterhin immer wieder nichts erzeugt wird. Auf Ihre Entscheidung zwischen Göttingen, Berlin und Zürich bin ich ungeheuer gespannt. Mögen Sie klar sehen, und die richtige Wahl treffen! Leicht wird Ihnen das nicht fallen! Den gesandten Abzug Ihres Manuskripts darf ich wohl behalten? Sie brauchen mir nichts mehr zurückzusenden. Weil inzwischen einige meiner älteren Separata neu abgedruckt worden sind, so möchte ich Sie um Mitteilung bitten, welche von meinen folgenden Publikationen Sie augenblicklich besitzen:

(1) Intuitionisme en formalisme (Holländisch)
(2) Intuitionism and formalism (Englisch)
(3) De onbetrouwbaarheid der logische principes (Holländisch)
(4) Het wezen der meetkunde (Holländisch)

Ich kann Ihnen das eventuell Fehlende jetzt ergänzen. Nochmals aufrichtigen Dank für die Freude und die Satisfaktion, die Ihre Schrift mir bereitet hat, herzliche Grüsse auch an Ihre Frau und auf Wiedersehen!

Ihr Egbertus Brouwer.

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