

Math 115A — Discussion 03

Worksheet: Subspaces & Independence

Thursday, October 1, 2026

Instructions. These problems pick up where HW1 leaves off. Work in pairs or small groups. For each problem, justify every step using a subspace axiom (SS1–SS3), a span/independence definition, or a theorem from the notes by name — no unjustified claims.

Part A The Union Trap

Problem 1. Consider the following claim and “proof.”

Claim. For any two subspaces $W_1, W_2 \subseteq V$, the union $W_1 \cup W_2$ is a subspace of V .

“Proof.” Let $\mathbf{x}, \mathbf{y} \in W_1 \cup W_2$. Then $\mathbf{x} \in W_1$ or $\mathbf{x} \in W_2$, and $\mathbf{y} \in W_1$ or $\mathbf{y} \in W_2$. Since each W_i is closed under addition, $\mathbf{x} + \mathbf{y} \in W_1 \cup W_2$. Similarly for scalar multiples. Hence $W_1 \cup W_2$ is a subspace. $\square$

(a) Find the flaw in this argument. What exactly does the proof assume about $\mathbf{x}$ and $\mathbf{y}$ that need not be true?

(b) Give an explicit counterexample: two subspaces W_1, W_2 of a vector space V for which $W_1 \cup W_2$ is *not* a subspace. Exhibit specific vectors witnessing the failure.

(c) What *is* the smallest subspace of V containing both W_1 and W_2 ? Write down its definition.

Part B Induction and the Steinitz Exchange Lemma

Problem 2 (Induction warm-up). Before the real theorem, practice the *shape* of an induction proof on something simple: for all $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

- (a) **Base case.** Verify the formula at $n = 1$.

- (b) **Inductive step.** Assume the formula holds for $n = k$ (write down exactly what you're assuming). Using that assumption, show it holds for $n = k + 1$.

Problem 3 (Concrete exchange in $\mathbb{R}^3$). Let $G = \{e_1, e_2, e_3\}$ (the standard basis, spans $\mathbb{R}^3$) and let $L = \{l_1, l_2\}$ with $l_1 = (1, 1, 1)$, $l_2 = (1, -1, 0)$.

- (a) Write l_1 as a combination of e_1, e_2, e_3 . Which coefficient can you solve for to express e_1 in terms of l_1, e_2, e_3 ? Conclude that $\{l_1, e_2, e_3\}$ still spans $\mathbb{R}^3$.

- (b) Now write l_2 as a combination of l_1, e_2, e_3 (the spanning set from part (a)). Find a coefficient on e_2 or e_3 you can solve for, and conclude $\{l_1, l_2, e_3\}$ (or $\{l_1, l_2, e_2\}$) still spans $\mathbb{R}^3$.

Problem 4 (The general theorem — fill in the induction). Theorem (Steinitz Exchange / Replacement Theorem). Let $L = \{l_1, \dots, l_m\}$ be linearly independent in V and let $G = \{g_1, \dots, g_n\}$ span V . Then $m \leq n$, and (after reordering G) $\{l_1, \dots, l_m, g_{m+1}, \dots, g_n\}$ spans V .

Proof by induction on $k = 0, \dots, m$. Let $P(k)$ be: “ $k \leq n$, and after suitable reordering, $S_k = \{l_1, \dots, l_k, g_{k+1}, \dots, g_n\}$ spans V .”

- (a) **Base case.** Prove $P(0)$.

- (b) **Inductive step, setup.** Assume $P(k-1)$ for some $1 \leq k \leq m$, i.e. $S_{k-1} = \{l_1, \dots, l_{k-1}, g_k, \dots, g_n\}$ spans V . Since $l_k \in V$, write l_k as a combination of the vectors in S_{k-1} . (What does this combination look like, in terms of unknown coefficients?)

- (c) **Why some coefficient on a g_j is nonzero.** Suppose, for contradiction, that every coefficient on $g_k, \dots, g_n$ in your combination above were 0. What would that say about l_k ? Why does this contradict the independence of L ? Conclude also that $k \leq n$.

- (d) **Finishing the swap.** Let $b_j \neq 0$ be one such nonzero coefficient (reorder so $j = k$). Solve your equation from (b) for g_k . Explain why this shows $S_k = \{l_1, \dots, l_k, g_{k+1}, \dots, g_n\}$ spans V (hint: S_{k-1} already spans V , and you’ve just shown g_k is in the span of S_k).

- (e) **Conclude.** What does $P(m)$ say, and why does that prove the theorem?

Part C Why Does Dimension Make Sense?

Problem 5. Two facts from the notes:

- Any finite generating (spanning) set can be trimmed down to a basis.
 - Any linearly independent set can be extended to a basis.
- (a) Suppose B_1 and B_2 are both bases of the same finite-dimensional vector space V . Using the Replacement Theorem, explain why $|B_1| \leq |B_2|$.
- (b) By symmetry, what can you also conclude about $|B_2|$ versus $|B_1|$? What does this tell you about $|B_1|$ and $|B_2|$?
- (c) In your own words: why is it important — for the definition of $\dim(V)$ to even make sense — that this is true?

Part D Beyond HW1: A Subspace of $\mathbb{F}[X]$

Problem 6. In $\mathbb{F}[X]$, define

$$W = \{p \in \mathbb{F}[X] \mid p(1) = 0\}.$$

- (a) Verify that W is a subspace of $\mathbb{F}[X]$ (check SS1–SS3 explicitly).
- (b) Fix $n \geq 1$, and let $W_n = \{p \in W : \deg p \leq n\}$. Find a set of polynomials that spans W_n , and show it is linearly independent. (Hint: if $p(1) = 0$, what does that tell you about $(X - 1)$ and $p(X)$?)
- (c) Using part (b), what is $\dim W_n$?

Part E Independence with a Parameter

Problem 7. For which values of $t \in \mathbb{R}$ are

$$v_1 = (1, 1, t), \quad v_2 = (1, t, 1), \quad v_3 = (t, 1, 1)$$

linearly independent in $\mathbb{R}^3$? Justify your answer.

Part F Challenge

Problem 8 (Challenge). Let I be an infinite set and recall $\mathbb{F}^{(I)} = \{f \in \mathbb{F}^I : f(i) = 0 \text{ for all but finitely many } i\}$ (HW1 Problem 4). Show $\mathbb{F}^{(I)}$ is *not* finite-dimensional. (Hint: if a finite set $\{f_1, \dots, f_n\}$ spanned $\mathbb{F}^{(I)}$, consider the finite set $U = \bigcup_j \text{supp}(f_j) \subseteq I$. Since I is infinite, pick $i_0 \in I \setminus U$ and build a vector that no combination of the f_j can reach.)

Problem 9 (Harder Challenge). Let V be finite-dimensional and let $L = \{l_1, \dots, l_m\}$ be linearly independent in V . Using an argument in the spirit of Part B, show L can be extended to a basis of V (Cor. 1.6.31). (Hint: if L doesn't already span V , there's a vector outside $\text{Span}(L)$ — add it, and argue the resulting set is still independent. Why can't you repeat this process forever?)

Problem 10 (Challenge: Find the Flaw in This Induction). **Claim.** For every finite set $S \subseteq V \setminus \{0\}$ of nonzero vectors, any two vectors in S are scalar multiples of each other.

“Proof” (induction on $n = |S|$).

Base case $n = 1$: A one-element set trivially satisfies the claim.

Inductive step: Assume the claim holds for every set of size k . Let $S = \{v_1, \dots, v_{k+1}\}$, and consider $S_1 = \{v_1, \dots, v_k\}$ and $S_2 = \{v_2, \dots, v_{k+1}\}$, each of size k . By the inductive hypothesis, every vector in S_1 is a scalar multiple of v_2 , and every vector in S_2 is a scalar multiple of v_2 . Hence every vector in S is a scalar multiple of v_2 . By induction, the claim holds for all $n \geq 1$. $\square$

(a) This claim is false — give a two-vector counterexample.

- (b) Find the flaw in the “proof.” (Hint: the chaining argument needs v_2 to actually belong to *both* S_1 and S_2 . For which value of k does this first fail?)