

Math 115A — Homework 1 Solutions

TA Reference

Axiom labels follow the course notes: F1–F10 (field), VS1–VS8 (vector space), SS1–SS3 (subspace: $0 \in W$; closed under $+$; closed under scalar multiplication).

Problem 1.

(a) Define, for $f, g \in \mathbb{F}^I$ and $\alpha \in \mathbb{F}$, the functions $(f + g)(i) = f(i) + g(i)$ and $(\alpha f)(i) = \alpha f(i)$ for all $i \in I$. These are again functions $I \rightarrow \mathbb{F}$, so $+$ and scalar multiplication are well defined. Let $f, g, h \in \mathbb{F}^I$, $\alpha, \beta \in \mathbb{F}$. Two functions are equal iff they agree at every $i \in I$, so each axiom is checked pointwise, using the field axioms in $\mathbb{F}$.

- *Addition is associative and commutative.* $((f + g) + h)(i) = (f(i) + g(i)) + h(i) = f(i) + (g(i) + h(i)) = (f + (g + h))(i)$, and $(f + g)(i) = f(i) + g(i) = g(i) + f(i) = (g + f)(i)$.
- *Zero.* Let $0_{\mathbb{F}^I}$ be the constant function $i \mapsto 0$. Then $(f + 0_{\mathbb{F}^I})(i) = f(i) + 0 = f(i)$, so $f + 0_{\mathbb{F}^I} = f$.
- *Additive inverses.* Let $(-f)(i) = -f(i)$. Then $(f + (-f))(i) = 0$, so $f + (-f) = 0_{\mathbb{F}^I}$.
- *Unit.* $(1 \cdot f)(i) = 1 \cdot f(i) = f(i)$.
- *Compatibility.* $((\alpha\beta)f)(i) = (\alpha\beta)f(i) = \alpha(\beta f(i)) = (\alpha(\beta f))(i)$.
- *Distributivity over vector addition.* $(\alpha(f + g))(i) = \alpha(f(i) + g(i)) = \alpha f(i) + \alpha g(i) = (\alpha f + \alpha g)(i)$.
- *Distributivity over scalar addition.* $((\alpha + \beta)f)(i) = \alpha f(i) + \beta f(i) = (\alpha f + \beta f)(i)$.

This is all eight axioms VS1–VS8, so $\mathbb{F}^I$ is an $\mathbb{F}$ -vector space.

(b) Let V be an $\mathbb{F}$ -vector space. For $f, g \in V^I$ and $\alpha \in \mathbb{F}$ define

$$(f + g)(i) = f(i) + g(i), \quad (\alpha f)(i) = \alpha f(i),$$

where the operations on the right are those of V . The zero is the constant function $i \mapsto 0_V$ and $(-f)(i) = -f(i)$. The verification of VS1–VS8 is word for word the same as in (a), with the field axioms replaced by the corresponding vector space axioms of V (e.g. distributivity $\alpha(f(i) + g(i)) = \alpha f(i) + \alpha g(i)$ is now VS for V). Note that (a) is the case $V = \mathbb{F}$.

Problem 2. Write $p = \sum_{k \geq 0} a_k X^k$ with $a_k \in \mathbb{F}$ and $a_k = 0$ for all but finitely many k . Then $\mathbb{F}[X]$ is the set of such sequences, and the operations are

$$\left(\sum a_k X^k \right) + \left(\sum b_k X^k \right) = \sum (a_k + b_k) X^k, \quad \alpha \sum a_k X^k = \sum (\alpha a_k) X^k.$$

The results are again finitely supported (if $a_k = b_k = 0$ for $k > N$ then $a_k + b_k = 0$ for $k > N$; and $\alpha a_k = 0$ whenever $a_k = 0$), so the operations are well defined. Under the identification of p with the function $k \mapsto a_k$ in $\mathbb{F}^{\mathbb{N}}$, these are exactly the pointwise operations of Problem 1(a). The axioms VS1–VS8 are therefore verified coefficientwise exactly as there, with zero the polynomial 0 (all $a_k = 0$) and $-p = \sum (-a_k) X^k$. (Equivalently, $\mathbb{F}[X]$ is a subspace of $\mathbb{F}^{\mathbb{N}}$ by the same argument as Problem 4.)

Problem 3. The operations are well defined: $(x_1 + y_1, x_2 + y_2) \in V_1 \times V_2$ and $(\alpha x_1, \alpha x_2) \in V_1 \times V_2$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ and $\alpha, \beta \in \mathbb{F}$. Each axiom holds in each coordinate because V_1 and V_2 are vector spaces:

- $(x + y) + z = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2) = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)) = x + (y + z)$, and $x + y = y + x$ similarly.
- Zero is $(0_{V_1}, 0_{V_2})$: $x + 0 = (x_1 + 0, x_2 + 0) = x$. The inverse of x is $(-x_1, -x_2)$.
- $1x = (1x_1, 1x_2) = x$ and $(\alpha\beta)x = (\alpha(\beta x_1), \alpha(\beta x_2)) = \alpha(\beta x)$.
- $\alpha(x + y) = (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) = \alpha x + \alpha y$ and $(\alpha + \beta)x = (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) = \alpha x + \beta x$.

Hence $V_1 \oplus V_2$ is an $\mathbb{F}$ -vector space.

Problem 4. Let $W = \mathbb{F}^{(I)}$. We check SS1–SS3 in $\mathbb{F}^I$.

- *SS1.* The zero function vanishes at every i , so it has empty support, which is finite. Thus $0 \in W$.
- *SS2.* Let $f, g \in W$ and set $S_f = \{i : f(i) \neq 0\}$, $S_g = \{i : g(i) \neq 0\}$, both finite. If $i \notin S_f \cup S_g$ then $(f + g)(i) = 0 + 0 = 0$. Hence $\{i : (f + g)(i) \neq 0\} \subseteq S_f \cup S_g$, which is finite (a union of two finite sets). So $f + g \in W$.
- *SS3.* Let $f \in W$, $\alpha \in \mathbb{F}$. If $f(i) = 0$ then $(\alpha f)(i) = \alpha \cdot 0 = 0$, so $\{i : (\alpha f)(i) \neq 0\} \subseteq S_f$ is finite. So $\alpha f \in W$.

Hence $\mathbb{F}^{(I)}$ is a subspace of $\mathbb{F}^I$. (Note the containment $\subseteq$, not equality, in SS2: e.g. f and $-f$ both have support S_f , but $f + (-f) = 0$ has empty support.)

Problem 5. Throughout $n \geq 1$ and we use $(A + B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$.

(a) Symmetric: subspace. $0^T = 0$. If $A^T = A$, $B^T = B$ then $(A + B)^T = A^T + B^T = A + B$ and $(\alpha A)^T = \alpha A^T = \alpha A$.

(b) Skew-symmetric: subspace. $0^T = 0 = -0$. If $A^T = -A$, $B^T = -B$ then $(A + B)^T = -A - B = -(A + B)$ and $(\alpha A)^T = \alpha A^T = -(\alpha A)$.

(c) Upper-triangular: subspace. The zero matrix has all entries 0. If $A_{ij} = B_{ij} = 0$ for all $i > j$, then $(A + B)_{ij} = 0$ and $(\alpha A)_{ij} = \alpha \cdot 0 = 0$ for all $i > j$.

(d) Diagonal: subspace. Identical to (c), with the condition “ $i \neq j$ ” in place of “ $i > j$ ”.

(e) Strictly lower-triangular: subspace. The condition is $A_{ij} = 0$ for $i \leq j$, i.e. the entries on and above the diagonal vanish. The zero matrix qualifies, and the entrywise computation of (c) shows closure under $+$ and scalars.

(f) Orthogonal: not a subspace. The zero matrix 0 is not invertible (as $n \geq 1$, $0 \cdot B = 0 \neq I_n$ for every B), and indeed $0^T 0 = 0 \neq I_n$. So $0 \notin \{A : A^T A = I_n\}$ and SS1 fails. (Closure also fails: I_n is orthogonal, but $2I_n$ is not, since $(2I_n)^T(2I_n) = 4I_n \neq I_n$ unless $4 = 1$ in $\mathbb{F}$; and $I_n + (-I_n) = 0$ is not orthogonal even though $-I_n$ is.)

Problem 6.

(a) Let $W = \bigcap_{i \in I} W_i$.

- *SS1.* Each W_i is a subspace, so $0 \in W_i$ for every i ; hence $0 \in W$.
- *SS2.* Let $x, y \in W$. Then for every $i \in I$, $x, y \in W_i$, so $x + y \in W_i$ by SS2 for W_i . Hence $x + y \in W$.
- *SS3.* Let $x \in W$, $\alpha \in \mathbb{F}$. For each i , $x \in W_i$, so $\alpha x \in W_i$. Hence $\alpha x \in W$.

(If $I = \emptyset$, the intersection is conventionally V , which is a subspace.)

(b) A single counterexample suffices. Let $V = \mathbb{R}^2$, $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ (the x -axis), $W_2 = \{(0, y) : y \in \mathbb{R}\}$ (the y -axis); both are subspaces. Then $(1, 0) \in W_1 \subseteq W_1 \cup W_2$ and $(0, 1) \in W_2 \subseteq W_1 \cup W_2$, but $(1, 0) + (0, 1) = (1, 1)$ lies on neither axis, so $(1, 1) \notin W_1 \cup W_2$. SS2 fails, so the union is not a subspace.

(c) Since $W_1 \cup W_2$ fails to be closed under addition, we enlarge it by adding all the missing sums:

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Claim: $W_1 + W_2$ is a subspace of V , it contains $W_1 \cup W_2$, and it is the smallest subspace containing $W_1 \cup W_2$.

Proof. Taking $w_2 = 0$ shows $W_1 \subseteq W_1 + W_2$ and taking $w_1 = 0$ shows $W_2 \subseteq W_1 + W_2$.

- SS1: $0 = 0 + 0 \in W_1 + W_2$.
- SS2: $(w_1 + w_2) + (w'_1 + w'_2) = (w_1 + w'_1) + (w_2 + w'_2)$ with $w_1 + w'_1 \in W_1, w_2 + w'_2 \in W_2$.
- SS3: $\alpha(w_1 + w_2) = \alpha w_1 + \alpha w_2 \in W_1 + W_2$.

Minimality: if U is any subspace with $W_1 \cup W_2 \subseteq U$, then for $w_1 \in W_1, w_2 \in W_2$ we have $w_1, w_2 \in U$, so $w_1 + w_2 \in U$ by SS2. Hence $W_1 + W_2 \subseteq U$. ■

In the example of (b), $W_1 + W_2 = \mathbb{R}^2$. More generally $\bigcap \{U : U \text{ subspace}, W_1 \cup W_2 \subseteq U\} = W_1 + W_2$, which is a subspace by (a).